Existence of Periodic Solutions for Delay Evolution Integrodifferential Equations

JONG-YEOUL PARK  
Department of Mathematics, Pusan National University  
Pusan 609-739, Korea  
jyepark@pusan.ac.kr  

YOUNG-CHEL KWUN  
Department of Mathematics, Dong-A University  
Pusan 604-714, Korea  

JIN-MUN JEONG  
Division of Mathematical Sciences, Pukyong National University  
Pusan 608-737, Korea  
jmjeong@pknu.ac.kr  

(Received April 2003; revised and accepted September 2003)

Abstract—In this paper, we will study the existence of periodic solutions for the delay evolution integrodifferential equations in a general Banach space with unbounded operator. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Existence of periodic solutions, Delay evolution, Integrodifferential equations, Unbounded operator.

1. INTRODUCTION

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $r > 0$ be a constant. We denote $C([-r,0], X)$, the space of continuous functions from $[-r,0]$ to $X$ with the sup-norm, $\| \phi \|_C = \max_{s \in [-r,0]} \| \phi(s) \|$, and define $u_t \in C([-r,0], X)$ by $u_t(s) = u(t+s)$, $s \in [-r,0]$, for a function $u$.

We consider the following finite delay evolution integrodifferential equation

$$ u'(t) + A(t)u(t) = f(t, u_t, \int_0^t g(t,s,u_s) \, ds), \quad t > 0, $$

$$ u(t) = \phi(t), \quad t \in [-r,0], $$

(1.1)
in a general Banach space $(X, \| \cdot \|)$, with $A(t)$, an unbounded operator, and $f : R^+ \times C \times X \to X$, $g : R^+ \times R^+ \times C \to X$ are continuous functions. Equation (1.1) has received some attention. For example, Xiang and Ahmed [1] studied the existence of periodic solutions of the finite delay evolution equations, and Oliveira [2] studied the instability for the finite delay evolution equations.

This work was supported by Grant No. R01-2002-000-00491-0.

In this paper, we will study the existence of periodic solutions for the delay evolution integrodifferential equation (1.1). For this purpose, we define \( P \) along the solution in such a way that, for \( u(\cdot, \phi) \) a solution of equation (1.1) with the initial function \( \phi \),

\[
P\phi = u_T(\cdot, \phi), \quad \phi \in C([-r, 0], X),
\]

i.e., \( (P\phi)(s) = u_T(s, \phi) = u(T + s, \phi), \ s \in [-r, 0] \), where \( r \) is a positive constant, and then, examine when the map \( P \) has a fixed point.

We will obtain the existence of solutions for the delay evolution integrodifferential equation in Section 2 and study the existence of periodic solutions for the delay evolution integrodifferential equation in Section 3.

2. EXISTENCE OF SOLUTIONS

In this section, we study the existence and uniqueness of solutions for equation (1.1). We make the following assumptions.

ASSUMPTION 2.1. For a constant \( T > r, f(t + T, x, y) = f(t, x, y) \) with \( f(t, x, 0) = 0, A(t + T) = A(t), g(t + T, s + T, x) = g(t, s, x) \) with \( \int_0^T g(t, s, x) \, ds = 0, t \geq s \geq 0, f \) is continuous in its variables and is locally Lipschitz in the second and third variables, and maps a bounded set into a bounded set and \( g \) is continuous in its variables and is locally Lipschitz in the third variables, and maps a bounded set into a bounded set.

DEFINITION 2.1. (See [6].) A two parameter family of bounded linear operators \( U(t, s), 0 < s < t < T \) on \( X \) is called an evolution system if the following two conditions are satisfied.

(i) \( U(s, s) = I, U(t, r)U(r, s) = U(t, s), \) for \( 0 < s < t < T \).

(ii) \( (t, s) \rightarrow U(t, s) \) is strongly continuous, for \( 0 < s < t < T \).

ASSUMPTION 2.2.

(H1) For \( t \in [0, T] \), the domain \( D(A(t)) = D \) is independent of \( t \) and is dense in \( X \).

(H2) For \( t \geq 0 \), the resolvent \( R(\lambda, A(t)) = (\lambda I - A(t))^{-1} \) exists for all \( \lambda \) with \( \text{Re} \lambda \leq 0 \) and is compact, and there is a constant \( M \) independent of \( \lambda \) and \( t \), such that

\[
\|R(\lambda, A(t))\| \leq M (|\lambda| + 1)^{-1}, \quad \text{Re} \lambda \leq 0.
\]

(H3) There exist constants \( L \) and \( 0 < a < 1 \), such that

\[
\| (A(t) - A(s)) A(r)^{-1} \| \leq L |t - s|^a, \quad s, t, r \in [0, T].
\]

Under these assumptions, Pazy [6] implies the existence of a unique evolution system \( U(t, s), 0 \leq s \leq t \leq T, \) for equation (1.1).

THEOREM 2.1. Let Assumptions 2.1 and 2.2 be satisfied and let \( \phi \in C([-r, 0], X) \). Then, there exists a constant \( t_1 > 0 \) and a unique continuous function \( u : [-r, t_1] \to X \), such that \( u_0 = \phi \) and

\[
u(t) = U(t, 0) \phi(0) + \int_0^t U(t, s) f\left(s, u_s, \int_0^s g(s, r, u_r) \, dr\right) \, ds, \quad t \in [0, t_1]. \tag{2.1}
\]

PROOF. We only need to set up the framework for the use of the contraction mapping theorem. With \( \phi \in C([-r, 0], X) \) being fixed and with \( t_1 > 0 \) to be determined, we define a map \( Q \) on \( C([-r, t_1], X) \), such that, for \( Qu(s) = \phi(s) \) for \( s \in [-r, 0] \) and

\[
(Qu)(t) = U(t, 0) \phi(0) + \int_0^t U(t, s) f\left(s, u_s, \int_0^s g(s, r, u_r) \, dr\right) \, ds, \quad t \in [0, t_1]. \tag{2.2}
\]
Using the property of the evolution system \( U \), we have

\[
Q : C([-r, t_1], X) \to C([-r, t_1], X).
\]

Next, for \( u, v \in C([-r, t_1], X) \) with \( u_0 = v_0 = \phi \) and \( t \in [0, \Omega] \), we have

\[
(Qu)(t) - (Qv)(t) = \int_0^t U(t, s) \left[ f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) - f \left( s, v_s, \int_0^s g(s, \tau, v_\tau) \, d\tau \right) \right] \, ds
\]

\[
= \int_0^t U(t, s) \left[ f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) - f \left( s, v_s, \int_0^s g(s, \tau, v_\tau) \, d\tau \right) \right] \, ds.
\]

Now, \( f \) is locally Lipschitzian in the second and third variables, \( g \) is locally Lipschitzian in the third variables, and \( U(t, s) \) is a bounded operator, it is clear that we can obtain the result by using the contraction mapping theorem.

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section, we study the periodic solutions for the finite delay evolution integrodifferential equation,

\[
\begin{align*}
 u'(t) + A(t)u(t) &= f \left( t, ut, \int_0^t g(t, s, u_s) \, ds \right), \quad t > 0, \\
 u(t) &= \phi(t), \quad t \in [-r, 0],
\end{align*}
\]

in a general Banach space \((X, \| \cdot \|)\).

For this purpose, we need some definitions and lemmas (see [7]).

DEFINITION 3.1. Solutions of equation (3.1) are bounded if, for each \( B_1 > 0 \), there is a \( B_2 > 0 \), such that \( \| \phi \| \leq B_1 \) and \( t \geq 0 \) imply \( \| u(t, \phi) \| < B_2 \).

DEFINITION 3.2. Solutions of equation (3.1) are ultimate bounded if there is a bounded \( B > 0 \), such that, for each \( B_3 > 0 \), there is a \( k > 0 \), such that \( \| \phi \| \leq B_3 \) and \( t \geq K \) imply \( \| u(t, \phi) \| < B \).

DEFINITION 3.3. An operator \( P : Z \to Z \) is called compact on \( Z \), if \( P \) maps bounded sequence into precompact sequence.

LEMMA 3.1. (See [8].) Let \( E_0 \subset E_1 \subset E_2 \) be convex subsets of Banach space \( Z \), with \( E_0 \) and \( E_2 \) compact subsets and \( E_1 \) open relative to \( E_2 \). Let \( P : E_2 \to Z \) be a continuous map, such that for some positive integer \( m \),

\[
\begin{align*}
P^j(E_1) &\subset E_2, \quad 1 \leq j \leq m - 1, \\
P^j(E_1) &\subset E_0, \quad m \leq j \leq 2m - 1.
\end{align*}
\]

Then, \( P \) has a fixed point in \( E_0 \).

To establish the existence of periodic solutions for the integrodifferential equation, we define a map \( P \) along the solution in such a way that, for \( u(\cdot, \phi) \) a solution of equation (3.1) with the initial function \( \phi \).

\[
P\phi = u_T (\cdot, \phi), \quad \phi \in C([-r, 0], X),
\]

and then, examine whether map \( P \) has a fixed point. We note that a fixed point of \( P \) gives rise to a periodic solution, because if \( P\phi = \phi \), then, for the solution \( u(0) = u(\cdot, \phi) \) with \( u_0(\cdot, \phi) = \phi \), we can
define \( y(t) = y(t + T) \). Now, for \( t \geq 0 \), we can use the known formula [6], \( U(t, s) = U(t, r)U(r, s) \) and hypotheses to obtain

\[
y(t) = u(t + T) = U(t + T, 0) \phi(0) + \int_0^{t+T} U(t + T, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds
\]

\[
= U(t + T, T) U(T, 0) \phi(0) + \int_0^T U(t + T, T) U(T, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds
\]

\[
+ \int_0^T U(t + T, s + T) g(s + T, \tau, u_{\tau+T}) \, d\tau \, ds
\]

\[
= U(t, 0) \left[ U(T, 0) \phi(0) + \int_0^T U(T, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds \right]
\]

\[ (3.3) \]

\[
+ \int_0^T U(t, s) f \left( s, u_s, \int_0^s g(s, \tau, y_\tau) \, d\tau \right) \, ds
\]

\[
= U(t, 0) \left[ U(T, 0) \phi(0) + \int_0^T U(T, s) f \left( s, u_s, \int_0^s g(s, \tau, y_\tau) \, d\tau \right) \, ds \right]
\]

\[ (3.4) \]

\[
+ \int_0^T U(t, s) g(s) \, ds
\]

This implies that \( y \) is also a solution and \( y_0 = u_T(\phi) = P(\phi) = \phi \). Then, the uniqueness implies that \( (u(t + T))y(t) = u(t) \), so that \( u(\phi) \) is a \( T \)-periodic solution.

The following lemma will be needed here to show that \( P \) is a compact operator. Recall that in the usual way (see, e.g., [6,7]), we define fractional power operator \( A^\alpha \) and Banach space \( X_\alpha \) for \( 0 \leq \alpha \leq 1 \), where \( A = A(0) \) and \( X_\alpha = D(A^\alpha), \| \cdot \|_\alpha \). We also write the norm in \( L(X_\alpha, X_\beta) \) (space of bounded linear operator from \( X_\alpha \) to \( X_\beta \)) as \( \| \cdot \|_{\alpha, \beta} \).

**Lemma 3.2.** (See [7].)

(i) Suppose that \( 0 \leq \alpha \leq \beta < 1 \). Then, for \( \beta - \alpha < \gamma < 1 \), there is a constant \( C(\alpha, \beta, \gamma) \), such that

\[ \|U(t, h)\| \leq C(\alpha, \beta, \gamma) (t - h)^{-\gamma}, \quad 0 \leq h < t \leq T. \]

(ii) For \( 0 \leq \gamma < 1 \), there is a constant \( C(\gamma) \), such that, for \( g \in C([0, L], X) \) \( (L > 0 \) is a constant), one has for \( 0 \leq s, t \leq L, \)

\[ \left\| \int_0^t U(t, h) g(h) \, dh - \int_0^s U(s, h) g(h) \, dh \right\| \leq C(\gamma) |t - s|^\gamma \max_{0 \leq h \leq L} \|g(h)\|. \]

(iii) Let \( 0 \leq \alpha < \beta \leq 1 \). Then,

\[ K(x, g)(t) \equiv U(t, 0) x + \int_0^t U(t, h) g(h) \, dh, \quad 0 \leq t \leq T, \]

defines a continuous linear operator from \( X_\beta \times C([0, T], X) \) into \( C^\gamma([0, T], X_\alpha) \), for every \( \gamma \in [0, \beta - \alpha) \).
**THEOREM 3.1.** Let Assumptions 2.1 and 2.2 be satisfied and assume that solutions of equation (3.1) are bounded. Then, \( P : C([-r, 0], X) \rightarrow C([-r, 0], X) \) defined by (3.2) is a compact operator.

**PROOF.** Let \( H \subset C([-r, 0], X) \) be bounded. Since solutions of equation (3.1) are bounded, it follows that \( E = P(H) \subset C([-r, 0], X) \) is bounded. In the following, we will use the Arzela theorem to show that \( E \) is precompact.

For \( s \in [-r, 0] \), a function in \( E \) can be expressed as,

\[
(P \phi)(s) = u_T(h, \phi) = y(T + h, \phi) = U(T + h, 0) \phi(0) + \int_0^{T+h} U(T + h, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds, \quad \phi \in H. 
\]  

(3.4)

Since \( T - r > 0 \), there is \( k > 0 \), such that \( T + h > k \), for \( h \in [-r, 0] \). From [6, p. 164], one has, for \( h \in [-r, 0] \),

\[
U(T + h, s) f(s) = V(T + h, k) U(k, 0) f(0), \quad \phi \in H. 
\]

(3.5)

Fix \( \eta \in (0, 1) \). Then, from Lemma 3.2(i), \( U(k, 0) : X \rightarrow X_\eta \) is bounded. Next, the embedding \( X_\eta \rightarrow X \) is compact under Assumption 2.2(H2), (see, e.g., [9]), thus, \( \{U(k, 0) \phi(0) : \phi \in H\} \) is precompact in \( X \) since \( \{\phi(0) : \phi \in H\} \) is bounded in \( X \). Therefore, the closure of \( \{U(k, 0) \phi(0) : \phi \in H\} \) is compact in \( X \). One can verify that as function on \( \cdot \in [-r, 0] \),

\[
\{U(T + \cdot, 0) \phi(0) : \phi \in H\} = \{U(T + \cdot, k) [U(k, 0) \phi(0)] : \phi \in H\} 
\]

(3.6)

is equicontinuous. Next, from Lemma 3.2(ii), for \( 0 \leq \gamma < 1 \), there is a constant \( C(\gamma) \), such that

\[
\left\| \int_0^{T+h_2} U(T + h_2, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds 
- \int_0^{T+h_1} U(T + h_1, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds \right\| 
\leq C(\gamma) |h_1 - h_2|^\gamma \max_{0 \leq s \leq T, 0 \leq \tau \leq s} \left\| f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \right\|. 
\]

Because the solutions of equation (3.1) are bounded and \( f, g \) map any bounded set into a bounded set, we see that there exists \( M_1 = M_1(H) > 0 \), such that

\[
\left\| f \left( t, u_t, \int_0^t g(t, s, u_s) \, ds \right) \right\| \leq M_1, \quad t \in [0, T]. 
\]

Thus, as functions on \( \cdot \in [-r, 0] \),

\[
\left\{ \int_0^{T+} U(T + \cdot, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) \, d\tau \right) \, ds : \phi \in H \right\} 
\]

is also equicontinuous. Therefore, functions in \( E \) are equicontinuous. Next, fix \( h_0 \in [-r, 0] \). From the above argument, we also know that

\[
\{U(T + h_0, 0) \phi(0) : \phi \in H\} 
\]

(3.7)

is precompact in \( X \). We also note that by Lemma 3.2(i), there are constants \( \gamma \in (0, 1) \) and \( M_2 > 0 \), such that

\[
\|U(T + h_0, s)\|_{0, \eta} \leq M_2 (T + h_0 - s)^{-\gamma}, \quad 0 \leq s < T + h_0. 
\]
Thus,
\[
\left\| \int_0^{T+h_0} U(T+h_0, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) d\tau \right) ds \right\| \leq \frac{M_1 M_2 T^{1-\gamma}}{1-\gamma}, \quad \phi \in H.
\]

Therefore,
\[
\left\{ \int_0^{T+h_0} U(T+h_0, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) d\tau \right) ds : \phi \in H \right\}
\]
is bounded in \( X_\eta \). Then, use the fact that the embedding \( X_\eta \to X \) is compact again, we see that the set defined by (3.8) is precompact in \( X \). Now, the Arzela theorem implies that \( P \) is a compact operator.

**THEOREM 3.2.** Let Assumptions 2.1 and 2.2 be satisfied. If the solutions of equation (3.1) are bounded and ultimate bounded, then, equation (3.1) has a \( T \)-periodic solution.

**PROOF.** Let map \( P \) be defined by (3.2). Using an argument similar to (3.3), we see that
\[
P^m(\phi) = u_mT(\phi), \quad \phi \in C([-r, 0]).
\]

Next, let \( B > 0 \) be bounded in the definition of ultimate boundedness. Using boundedness, there is \( B_2 > B \), such that \( \{ \| \phi \|_C \leq B, \ t \geq 0 \} \) implies \( \| u(t, \phi) \| < B_2 \). And also, there is \( B_4 > 2B_2 \), such that \( \{ \| \phi \|_C \leq 2B_2, \ t \geq 0 \} \) implies \( \| u(t, \phi) \| < B_4 \). Next, using ultimate boundedness, there is a positive integer \( m \) such that, \( \{ \| \phi \|_C \leq 2B_2, \ t \geq (m-1)T \} \) implies \( \| u(t, \phi) \| < B \). These imply
\[
\| P^{i-1}\phi \|_C = \| u((i-1)T + \cdot, \phi) \|_C < B_4,
\]
for \( i = 1, 2, 3, \ldots \), and \( \| \phi \|_C \leq 2B_2,
\]
\[
\| P^{i-1}\phi \|_C = \| u((i-1)T + \cdot, \phi) \|_C < B,
\]
for \( i = 1, 2, 3, \ldots \), and \( \| \phi \|_C \leq 2B_2 \).

Now, let
\[
H = \{ \phi \in C([-r, 0], X) : \| \phi \|_C < B_4 \}, \quad E_2 = \text{cl}(\text{cov}(P(H))),
\]
\[
K = \{ \phi \in C([-r, 0], X) : \| \phi \|_C < B_2 \}, \quad E_1 = K \cap E_2,
\]
\[
G = \{ \phi \in C([-r, 0], X) : \| \phi \|_C < B \}, \quad E_0 = \text{cl}(\text{cov}(P(G))),
\]
where \( \text{cov}(F) \) is the convex hull of set \( F \) defined by \( \text{cov}(F) = \{ \sum_{i=1}^n \lambda_i f_i : n \geq 1, f_i \in F, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \} \), and
\[
\| u(t) - v(t) \|
\]
\[
\leq \| U(t, 0) [\phi_1(0) - \phi_2(0)] \|
\]
\[
+ \left\| \int_0^t \left( U(t, s) f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) d\tau \right) - U(t, s) f \left( s, v_s, \int_0^s g(s, \tau, v_\tau) d\tau \right) \right) ds \right\|
\]
\[
\leq \| U(t, 0) [\phi_1(0) - \phi_2(0)] \|
\]
\[
+ \int_0^t \| U(t, s) \| \left\| f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) d\tau \right) - f \left( s, v_s, \int_0^s g(s, \tau, v_\tau) d\tau \right) \right\| ds
\]
\[
+ \int_0^t \| U(t, s) \| \left\| f \left( s, u_s, \int_0^s g(s, \tau, u_\tau) d\tau \right) - f \left( s, v_s, \int_0^s g(s, \tau, v_\tau) d\tau \right) \right\| ds
\]
\[
\leq c_1 \| \phi_1(0) - \phi_2(0) \| + c_2 \int_0^t \| u_\tau - v_\tau \|_C \| \tau \| ds + c_3 \int_0^t \| u_s - v_s \|_C \| s \| ds.
\]
By Gronwall's lemma, we get
\[
\|u(t) - v(t)\|_C \\
\leq c_1 \|\phi_1(0) - \phi_2(0)\|_C \left[ 1 + \int_0^T c_3 \exp \left( \int_0^s (c_3 + c_2) \, ds \right) \, ds \right] \\
\leq c_1 \|\phi_1 - \phi_2\|_C + \left[ 1 + \frac{1}{c} \exp (ct - 1) \right].
\]

Thus, mapping $P$ is continuous. Now, it is clear that Horn's fixed point theorem can be used to get a $T$-periodic solution of equation (3.1).

REFERENCES