The Upper Interval Topology, Property $\mathcal{M}$, and Compactness

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1 Introduction

A topology on a lattice $L$, or more generally on a partially ordered set $P$, is called an intrinsic topology if it is defined directly from the order. Early examples of such topologies were the interval topology of O. Frink [Fr42] and the order topology introduced by G. Birkhoff [Bi42]. The early intrinsic topologies were typically symmetric, i.e., the topology defined on $L$ agreed with the topology defined on $L^d$, the dual lattice with the order reversed. The theory of continuous lattices, however, provided strong motivation for the consideration of topologies such as the Scott topology or the hull/ker topology which were not symmetric in this sense [COMP], indeed were not even $T_1$.

Indeed, from hindsight, it is very natural to consider intrinsic topologies that are not symmetric. Given any $T_0$-topology on a set, there results a partial order, the order of specialization, defined by $x \leq y$ if and only if $x \in \{y\}$. It is then natural to consider “order-consistent” topologies on a partially ordered set $P$, topologies for which the order of specialization agrees with the original given partial order. These typically satisfy the $T_0$-separation axiom, but nothing stronger. There is a weakest order-consistent topology on a partially ordered set, which we call the lower interval topology, and which has as a subsbasis for the closed sets all principal ideals $\downstream{x}$, $x \in P$, where $\downstream{x} := \{y: y \leq x\}$. This topology was called the upper topology in [COMP] and has also been called the weak topology.

We quickly recall basic notions of ordered sets arising in continuous domain theory (see, for example, [AJ95] or [COMP]). Let $(P, \leq)$ be a partially ordered set (or poset). A non-empty subset $D$ of $P$ is directed if $x, y \in D$ implies there exists $z \in D$ with $x \leq z$ and $y \leq z$. A set $A$ is a lower set if

$$A = \downstream{A} := \{y \in P: y \leq x \text{ for some } x \in A\},$$

and an ideal of $P$ is a directed lower set. The poset $P$ is a directed complete partially ordered set (DCPO) if every directed subset of $P$ has a supremum. A useful order-consistent topology for a DCPO $P$ is the Scott topology, which

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has as its topology of closed sets all directed complete lower sets [Sc70].

If \( x, y \) belong to a partially ordered set \( P \), we write \( x \ll y \) and say that \( x \) approximates \( y \) if for every directed set \( D \) with \( y \leq \sup D \), we have \( x \leq d \) for some \( d \in D \). A partially ordered set is called a continuous poset if every element is the directed supremum of all elements which approximate it, \( y = \bigvee \{ x : x \ll y \} \) for all \( y \in P \). The partially ordered set is a continuous domain if it is both a DCPO and a continuous poset. It is well-known that for a continuous domain the Scott topology has as a basis of open sets all sets of the form
\[
\uparrow x := \{ y : x \ll y \}.
\]

Another approach to the study of topology on ordered structures is to begin with a set \( X \) equipped both with a partial order \( \leq \) and a topology. One needs to assume that these are related to one another in some appropriate fashion. One common assumption dating back to the pioneering work of L. Nachbin [Na65] is that the partial order \( \{(x, y) : x \leq y\} \) is a closed subset of \( X \times X \), where the latter is given the product topology. When this assumption is satisfied we call the order a closed order and the resulting ordered topological space a pospace.

\section{Compact Pospaces}

Compactness alone is a rather weak hypothesis for \( T_0 \)-spaces; for example, compact spaces need not be locally compact. However, there is a particularly well-behaved and important class of spaces that has emerged which does appear to be a suitable generalization of compact Hausdorff spaces to the \( T_0 \)-setting. Since these spaces are closely tied to compact pospaces, we recall some basic theory of the latter. Some of the material dates back to the early sources on the subject and other is of more recent origin. As a general reference we refer the reader to [COMP].

Let \( (X, \tau, \leq) \) be a compact pospace, a compact space equipped with a closed order. It is then standard that \( X \) is Hausdorff [Na65] since the diagonal \( \Delta = \leq \cap \geq \) is closed.

We define two new topologies on \( X \) which arise from \( \tau \). Let \( \tau^* \) denote the collection of all open upper sets \( U = \uparrow U := \{ x : u \leq x \text{ for some } u \in U \} \) and \( \tau_* \) denote the collection of all open lower sets. Then the collections \( \tau_* \) and \( \tau^* \) are both topologies, called the \( \tau \)-lower and \( \tau \)-upper topologies respectively. From the compactness one deduces the following lemma (see [Na65] or Chapter VI.1 of [COMP]).

\begin{lemma}
Let \( (P, \tau, \leq) \) be a compact pospace.
\begin{enumerate}
  \item Sets of the form \( U \cap V, U \in \tau_*, V \in \tau^* \), form a basis for the topology \( \tau \).
  \item If \( K \subseteq P \) is compact, then \( \downarrow K \) and \( \uparrow K \) are compact.
  \item A directed set possesses a supremum, to which it converges.
\end{enumerate}
\end{lemma}
Definition 2.2 An element $x$ is a minimal upper bound of a subset $A$ of a partially ordered set $P$ if $x$ is minimal in the set of all upper bounds of $A$. The set of all minimal upper bounds of $A$ is denoted $\text{mub}(A)$. The poset $P$ is said to be $\text{mub}$-complete if given a finite subset $F$ and an upper bound $u$ of $F$, there exists $x \in \text{mub}(F)$ such that $x \leq u$.

Lemma 2.3 A compact pospace is $\text{mub}$-complete.

Proof. Let $F$ be a finite set. Then the set of upper bounds $U = \bigcap \{y : y \in F\}$ is a finite intersection of closed sets, hence compact. Let $y$ be an upper bound for $F$; then $y \in U$. Let $C$ be a maximal chain in $U$ containing $y$. By the dual of the third item in Lemma 2.1, the totally ordered set $C$ has a greatest lower bound $x$ in $U$. It follows directly from the maximality of $C$ that $x \in \text{mub}F$. $\square$

A topological space is called strongly sober if every ultrafilter has a nonempty set of limit points consisting of the closure of a unique singleton set. Strongly sober spaces, as expected, are sober (i.e., every irreducible closed set is the closure of a unique singleton subset). A space is said to be locally compact if it has as a basis of (not necessarily open) compact neighborhoods at each point. A sober space is stably locally compact if it is locally compact, compact, and has the property that a finite intersection of compact upper sets (with respect to the order of specialization) is again compact (it readily follows that arbitrary intersections of compact upper sets are compact).

The cocompact topology $\tau^k$ on a $T_0$ space $X$ has as a subbasis for the closed sets all compact upper (with respect to the order of specialization) sets in $(X, \tau)$. We note that a subset of $(X, \tau)$ is an upper set if and only if it is a saturated set, i.e., an intersection of open sets. The $\tau^k$-topology always has as order of specialization the dual order $\geq$ of the order of specialization for $\tau$. The patch topology is the join $\tau \vee \tau^k$ of the original and the cocompact topologies.

Locally compact strongly sober spaces are intimately connected with compact pospaces as the next proposition shows. Indeed they are alternate ways of looking at the same objects.

Theorem 2.4 Let $(X, \tau)$ be a $T_0$ space. The following are equivalent:

(i) $(X, \tau)$ is locally compact and strongly sober.
(ii) $(X, \tau)$ is stably locally compact and sober.
(iii) The patch space $(X, \tau \vee \tau^k, \leq)$ equipped with the order of specialization is a compact pospace.

In these cases, $\tau = \pi^*$ and $\tau^k = \pi_*$, where $\pi$ is the patch topology $\tau \vee \tau^k$.

Conversely, given a compact pospace $(X, \pi, \leq)$, the space $(X, \pi^*)$ is a stably locally sober space and $\pi_* = (\pi^*)^k$. The two constructions of passing from a stably locally compact sober space to the compactly ordered space which is the patch of the original space and the cocompact topology and of passing from
a compact ordered space to the stably locally compact sober space whose open sets are the open upper sets are mutually inverse constructions.

This result is quoted from [La91]. A comprehensive treatment is difficult to find, but the proof may be pieced together from exercises in Section 7.1 of [COMP], from [Jo82], and from [HM81]. Actually the equivalence of the second and third items follows rather directly from the spectral theory of continuous lattices and can be found implicitly as early as [HL78]. These matters have been alternately treated by R. Kopperman in [Ko95], where the cocompact topology is called the de Groot dual topology. He also calls spaces which satisfy condition 3) skew compact; we occasionally employ the latter terminology in what follows.

3 The Upper Interval Topology

Let \((P, \leq)\) be a partially ordered set. We define the upper interval topology by taking as a subbase for the closed sets all principal filters \(\uparrow x\) for \(x \in P\). Thus the upper interval topology is the order dual of the lower interval topology. A finitely generated upper set is one of the form \(\uparrow F\), where \(F\) is finite. Note that the complements of finitely generated upper sets form a basis of open sets for the upper interval topology.

The join of the upper and lower interval topologies yields the interval topology of Frink (a motivation for our terminology). However, as we shall see, there is strong mathematical motivation for taking the join of the upper interval and Scott topologies, rather than the upper and lower interval topologies. Thus even for the “two-sided” intrinsic topologies, it is frequently asymmetric ones that are more natural and useful.

Observe that the order of specialization for the upper interval topology is \((P, \geq)\), the order dual of \(P\). The topology arising as the join of the upper interval topology and the Scott topology is generally referred to as the Lawson topology or \(L\)-topology.

Let \(P\) be a partially ordered set. For non-empty \(F, G \subseteq P\), we write \(F \ll G\) and say \(F\) approximates \(G\) if whenever a directed set \(D\) satisfies \(\sup D \in \uparrow G\), then \(d \in \uparrow F\) for some \(d \in D\). The poset \(P\) is called a quasicontinuous poset if for each \(x \in P\), \(\uparrow x\) is the directed (with respect to reverse inclusion) intersection of sets of the form \(\uparrow F\) where \(F \ll x\) (short for \(F \ll \{x\}\)) and \(F\) is finite; if \(P\) is also a DCPO, then it is called a quasicontinuous domain. If in addition it is possible to choose the finite sets \(F\) such that \(F \ll F\) for each \(F\), then we can \(P\) a quasialgebraic poset resp. domain.

For the basic theory of quasicontinuous domains we refer the reader to [GLS83], where they were introduced. There it is shown that quasicontinuous domains typically satisfy analogues of standard properties of continuous domains (with singletons typically replaced by finite sets). For example, sets of the form \(\uparrow F := \{y : F \ll y\}\) for finite sets \(F\) are Scott open; furthermore,
given any Scott open set $U$ and any $x \in U$, there exists a finite set $F \subseteq U$ such that $x \in \uparrow F$.

We recall the following result of M. E. Rudin (see [Ru80] or [GLS83]), which we refer to as Rudin’s Lemma.

**Lemma 3.1** If $\{\uparrow F_i : i \in I\}$ is a descending family of finitely generated upper sets in a partially ordered set $P$, then there is a directed subset $D$ of $\bigcup_{i \in I} F_i$ that intersects each $F_i$ nontrivially.

**Lemma 3.2** In a DCPO $P$, if an upper set $A$ is a directed intersection of finitely generated upper sets, then any Scott open set containing $A$ contains one of the finitely generated upper sets. Furthermore, $A$ is compact in the Scott topology.

**Proof.** Let $\{\uparrow F_i : i \in I\}$ be a descending family of finitely generated upper sets with intersection $A$. Let $W$ be a Scott open set containing $A$. If $F_i \setminus W \neq \emptyset$ for all $i$, then the family $\{\uparrow (F_i \setminus W) : i \in I\}$ satisfies the hypothesis of Rudin’s Lemma. Hence there exists a directed subset $D$ of $\bigcup_{i \in I} F_i \setminus W$ which intersects each $F_i \setminus W$. Let $d$ be the supremum of $D$. Then $d \in \uparrow F_i$ for each $i$, and hence $d \in A$. But $D \subseteq P \setminus W$, a Scott closed set, implies $d$ is not in $W$, a contradiction. Thus $\uparrow F_i \subseteq W$ for some $i$.

Let $\{U : U \in \mathcal{U}\}$ be a cover of $A$ with Scott open sets and let $W$ denote their union. Then by the preceding paragraph some $F_i$ is contained in $W$. Since finitely many of the $U \in \mathcal{U}$ contain $F_i$, they contain $\uparrow F_i$, and hence $A$.

**Lemma 3.3** Let $P$ be a quasicontinuous domain. If $A = \uparrow A$ is compact in the Scott topology, then every Scott open neighborhood $U$ of $A$ contains a finite set $F$ such that $A \subseteq \uparrow F \subseteq \uparrow F \subseteq U$. Furthermore, $A$ is a directed intersection of all finitely generated upper sets which contain $A$ in their Scott interior.

**Proof.** Suppose $A = \uparrow A$ is compact in the Scott topology and has $U$ for a Scott open neighborhood. For each $x \in A$, there exists (from the preceding lemma and the definition of quasicontinuous) a finite set $F_x \subseteq U$ such that $x \in \uparrow F_x$. Finitely many of the $\uparrow F_x$ cover $A$, and the union $F$ of the finitely many $F_x$ satisfies $\uparrow F \subseteq U$ and $A \subseteq \uparrow F$.

Consider the family $\mathcal{F}$ of all finite sets $F$ such that $A$ is contained in the Scott interior of $\uparrow F$. Given $F_1, F_2 \in \mathcal{F}$, then $U$, the intersection of the Scott interiors of $F_1$ and $F_2$, is a Scott open set containing $A$. By the preceding paragraph there exists a finite set $F \subseteq U$ such that $A \subseteq \uparrow F$, and thus $A$ is contained in the Scott interior of $\uparrow F$. Hence the family $\mathcal{F}$ is directed.

For $z \notin A$, the set $V = P \setminus \downarrow z$ is a Scott open set containing $A$, and one can repeat the preceding argument to find $F \subseteq U$ such that $F \in \mathcal{F}$. Thus the intersection of $\mathcal{F}$ is $A$. 


Definition 3.4 A quasicontinuous domain $P$ is said to satisfy property $\mathcal{M}$ if

(i) $P$ is mub-complete,
(ii) $P = \uparrow Z$ for some finite subset $Z$, and
(iii) given $F_i \ll x_i$ for finite subsets $F_i, i = 1, \ldots, n$, there exists a finite subset $F \subseteq \text{mub}(F_1, \ldots, F_n)$ such that $F \ll \uparrow x_1 \cap \ldots \cap \uparrow x_n$, where

\[
\text{mub}(F_1, \ldots, F_n) := \bigcup \{ \text{mub}(y_1, \ldots, y_n) : \forall i = 1, \ldots, n, y_i \in F_i \}.
\]

We may think of condition 2 as asserting property $\mathcal{M}_0$ and condition 3 as asserting property $\mathcal{M}_n$ for all $n \geq 2$. Although we hypothesize property $\mathcal{M}$ in the next lemma, we observe that in the proof we need only $\mathcal{M}_2$.

Lemma 3.5 Let $P$ be a quasicontinuous domain which satisfies property $\mathcal{M}$. Then the intersection of two Scott compact upper sets is again Scott compact.

Proof. Let $A = \uparrow A$ and $B = \uparrow B$ be compact in the Scott topology. We consider the collection $\mathcal{F}$ of all finite sets $F$ defined by

\[
F \in \mathcal{F} \iff \exists F_A \ll A, F_B \ll B, |F_A|, |F_B| < \infty, \text{ with } \uparrow F_A \cap \uparrow F_B \subseteq \uparrow F.
\]

We claim that the family $\{\uparrow F : F \in \mathcal{F}\}$ is directed (with respect to reverse inclusion). Suppose that $F, G \in \mathcal{F}$. Let $F_A, F_B, G_A, G_B$ be finite sets satisfying the appropriate conditions for $F$ and $G$ resp. for membership in $\mathcal{F}$. Since $\uparrow F_A$ and $\uparrow G_A$ are Scott open sets containing $A$, there exists a finite set $H_A \subseteq \uparrow F_A \cap \uparrow G_A$ with $H_A \ll A$. Similarly we find a finite set $H_B \ll B$ with $H_B \subseteq \uparrow F_B \cap \uparrow G_B$. Finally pick finite sets $E_A$ and $E_B$ with $E_A \ll A, E_B \ll B$, $E_A \subseteq \uparrow H_A$, and $E_B \subseteq \uparrow H_B$.

By property $\mathcal{M}$ for each $x \in E_A$ and each $y \in E_B$, there exists a finite subset $H(x, y)$ of $\text{mub}(H_A, H_B)$ such that $\uparrow x \cap \uparrow y \subseteq \uparrow H(x, y)$. Set $H := \bigcup \{ H(x, y) : x \in E_A, y \in E_B \}$. Then $H$ is finite and $\uparrow E_A \cap \uparrow E_B \subseteq \uparrow H$, so $H \in \mathcal{F}$. Furthermore

\[
H \subseteq \uparrow H_A \cap \uparrow H_B \subseteq \uparrow F_A \cap \uparrow F_B \subseteq \uparrow F \cap \uparrow G,
\]

which establishes the directedness.

We have seen in the previous lemma that each Scott compact upper set $A$ is the intersection of all $\uparrow F_A$ such that $F_A$ is finite and $F_A \ll A$. Similarly $B$ is the intersection of all such $\uparrow F_B$. In the construction of the previous paragraph we found $H \subseteq \uparrow F_A \cap \uparrow F_B$ such that $H \in \mathcal{F}$. It then follows directly that $\bigcap \{ \uparrow F : F \in \mathcal{F} \}$ is equal to $A \cap B$. Thus by the preceding lemma $A \cap B$ is Scott compact.

Lemma 3.6 If a quasicontinuous domain $Q$ is mub-complete, a finitely generated upper set, and satisfies $\bigcap_{i=1}^n \uparrow x_i$ is Scott compact for any finite set $\{x_1, \ldots, x_n\}$, then it satisfies property $\mathcal{M}$.

Proof. Suppose that we are given $F_i \ll x_i$ for finite subsets $F_i, i = 1, \ldots, n$. By hypothesis the set $A := \bigcap_i \uparrow x_i$ is compact in the Scott topology. Since
the set $\uparrow F_i$ is Scott-open for each $i$, $A$ is in the Scott open set $W := \bigcap_i \uparrow F_i$. By Lemma 3.3, we can obtain a finite set $G$ such that $G \ll A$ and $G \subseteq W$. For $u \in G$, $u \in \uparrow F_i$ for $i = 1, \ldots, n$. Pick $v_i \in F_i$ for each $i$ such that $v_i \leq u$. Since $Q$ is mub-complete, there exists $x_u \in \text{mub} \{v_1, \ldots, v_n\}$ such that $x_u \leq u$. Choosing $x_u$ in this fashion for each $u \in G$, we obtain a finite set $F := \{x_u : u \in G\} \subseteq \text{mub}(F_1 \uplus \ldots \uplus F_n)$ such that $\uparrow G \subseteq \uparrow F$. Since $G \ll A$, we conclude that $F \ll A$. $\square$

We come now to our main theorems on the upper interval topology in quasicontinuous domains.

**Theorem 3.7** Let $P$ be a quasicontinuous domain. The following statements are equivalent.

(i) The Lawson topology on $P$ is compact.

(ii) The domain $P$ has property $M$.

(iii) The domain $P$ is a finitely generated upper set in which the intersection of two Scott compact upper sets is again Scott compact.

(iv) The sets closed in the upper interval topology are compact in the Scott topology.

(v) The Scott compact upper sets are precisely the sets closed in the upper interval topology.

(vi) Every set closed in the upper interval topology is a directed intersection of finitely generated upper sets.

(vii) The Scott topology on $P$ satisfies any of the equivalent conditions of Theorem 2.4.

(viii) $P$ is a compact pospace in the Lawson topology for which the open upper resp. lower sets in the Lawson topology are precisely the sets open in the Scott resp. upper interval topology.

**Proof.** 1.$\Rightarrow$2.: Since $P$ is quasicontinuous, if $x \nleq y$, then there exists a finite set $F$ such that $F \ll x$ and $y \notin \uparrow F$. Then $\uparrow F \times (P \setminus \uparrow F)$ is a product of Lawson open sets which contains $(x, y)$ and misses the order relation $\leq$. Thus $P$ is a pospace in the Lawson topology. Hence $P$ is mub-complete by Lemma 2.3. Since (by compactness) $P$ is covered by finitely many $\uparrow F$ for $F$ finite, it follows that the finite union $Z$ of the finite sets $F$ must satisfy $P = \uparrow Z$. For $x_1, \ldots, x_n \in P$, $\uparrow x_1 \cap \ldots \cap \uparrow x_n$ is closed, hence Lawson compact, hence Scott compact. Thus by Lemma 3.6, $P$ has property $M$.

2.$\Rightarrow$3.: This implication follows directly from Lemma 3.5.

3.$\Rightarrow$4.: It is immediate that the finitely generated upper sets are Scott compact. Hence $P$ is Scott compact. Every other non-empty set closed in the upper interval topology is an intersection of finitely generated upper sets. It follows from the hypothesis that any finite intersection of finitely generated upper sets is again Scott compact. Thus any set closed in the upper interval topology is the intersection of a descending family of Scott compact upper
Using Lemma 3.3, one can proceed precisely as in the case of continuous domains to show that the intersection of a descending family of compact sets is again compact (see, for example, Lemmas 4.12 and 4.17 of [Ju89]). Alternately one can use the fact that $P$ endowed with the Scott topology is sober [GLS83] and the standard fact (a version of the Hofmann-Mislove Theorem) that the intersection of a descending family of compact upper sets in a sober space is again compact (see, e.g., [HM81]).

4.$\Rightarrow$5.: It follows from Lemma 3.3 that every Scott compact upper set is closed in the upper interval topology, and assertion 5 then follows immediately from assertion 4.

5.$\Rightarrow$6.: This follows immediately from Lemma 3.3.

6.$\Rightarrow$4.: By Lemma 3.2 every set closed in the upper interval topology is compact in the Scott topology.

5.$\Rightarrow$8.: That $P$ is locally compact in the Scott topology follows from Lemma 3.3, since each $\uparrow F$ for $F$ finite is compact in the Scott topology. It is immediate from the hypothesis that the intersection of Scott compact upper sets is again compact. By [GLS83] $P$ equipped with the Scott topology is sober. Thus by Theorem 2.4 the Scott topology and its corresponding cocompact topology, the upper interval topology, are the upper and lower topologies for a compact pospace topology on $P$. Since the compact pospace topology is the join of the two topologies, the Lawson topology is compact. Furthermore, the last assertion also now follows from Theorem 2.4.

8.$\Rightarrow$1.: Immediate.

8.$\Rightarrow$7.: Immediate.

7.$\Rightarrow$3.: It follows from condition 2 of Theorem 2.4 that finite intersections of Scott compact upper sets are again Scott compact. Since $P$ itself is also Scott compact and quasicontinuous, it is covered by finitely many sets $\uparrow F_i$ and hence is of the form $\uparrow(\bigcup F_i)$. \hfill \Box 

It is not immediately clear how one should best generalize property $\mathcal{M}$ to general DCPOs, since the approximation relation is intimately tied with the definition. However, the previous theorem offers one possibility. One can take the equivalent condition 6, since it can be easily considered in more general settings.

**Definition 3.8** A partially ordered set $P$ is said to have property DINT (the "directed intersection" property) if every set closed in the upper interval topology is a directed intersection of finitely generated upper sets.

In the study of topologies on ordered sets, one can pose the following general problem: given a topology $\tau_1$ consisting of open upper sets and a topology $\tau_2$ consisting of open lower sets, under what conditions does the join $\tau_1 \vee \tau_2$ have the property that its open upper (resp. lower) sets are precisely the sets open in $\tau_1$ (resp. $\tau_2$)? The Scott topology is extremely robust in this regard. It reappears as the set of open upper sets whenever the join of it
and any topology consisting of lower sets is taken. To see this note (i) that all subsets with the property that whenever the supremum of a directed set is in the subset then the directed set residually belongs to the subset form a topology, (ii) that this topology contains all Scott open sets and all lower sets and hence the join of the Scott topology with any topology of lower sets, and (iii) that the open upper sets in this topology are precisely the Scott open sets. Little is known, however, about how the upper interval topology behaves in the context of this problem. The next lemma provides one result along this line.

**Lemma 3.9** Let \((P, \leq)\) be a partially ordered set satisfying property \(\text{DINT}\). Let \(\tau\) denote the upper interval topology, and suppose there exists a topology \(\nu\) containing the lower interval topology with all open sets being upper sets such that \(P\) endowed with the join topology \(\sigma := \tau \vee \nu\) is a compact Hausdorff space. Then the topology \(\sigma_*\) of \(\sigma\)-open lower sets is equal to \(\tau\).

**Proof.** We first note that we can without loss of generality replace \(\nu\) with the topology \(\sigma_*\) of all \(\sigma\)-open upper sets, since this topology contains \(\nu\) and is contained in \(\sigma\), and hence satisfies \(\sigma = \tau \vee \sigma_*\). We thus henceforth assume that \(\nu = \sigma_*\).

Since \(\tau \subseteq \sigma\) and each open set in \(\tau\) is a lower set, it is clear \(\tau \subseteq \sigma_*\). If they are not equal, then there exists a non-empty \(\sigma\)-closed upper set \(A\) such that \(A\) is not \(\tau\)-closed. Let \(B\) denote the \(\tau\)-closure of \(A\). By property \(\text{DINT}\) there exists a descending family \(\{\uparrow F_i; i \in I\}\) (where we assume \(I\) is directed and \(i \leq j\) implies \(\uparrow F_j \subseteq \uparrow F_i\)) of finitely generated upper sets such that \(B = \bigcap_{i \in I} \uparrow F_i\). Pick \(b \in B \setminus A\). Let \(C\) be a maximal chain in \(B\) containing \(b\). Then \(\{y \cap B; y \in C\}\) is a family of non-empty compact (with respect to \(\sigma\)) sets totally ordered with respect to inclusion, and hence has non-empty intersection. If \(x\) is in the intersection, then \(x\) is in \(B\), is a lower bound for \(C\), and hence is in \(C\) by the maximality of \(C\). From the maximality of \(C\) it follows that \(x\) is a minimal element in \(B\). Note that \(x \notin A\), for otherwise we would have \(b \in A\), since \(A\) is an upper set. We break the remainder of the proof into a series of claims. We always work in the topology \(\sigma\) unless explicitly stated otherwise. We also note that the following arguments remain valid if \(\uparrow x \cap A\) is empty, although they could be simplified some in this case.

**Step 1.** There exists a finite set \(G\) such that \(x \notin \uparrow G\) and \(\uparrow x \cap A\) is contained in the interior of \(\uparrow G\). Let \(E := \uparrow x \cap A\), a closed and hence compact set. Since sets of the form \(U \setminus \uparrow F\), \(U\) an open upper set and \(F\) finite, form a basis for the Hausdorff topology \(\sigma\), there exists for each \(y \in E\) an open upper set \(U_y\) a finite set \(F_y\) and an open set \(V_y\) such that \(x \in U_y \setminus \uparrow F_y\), \(y \in V_y\) and \((U_y \setminus \uparrow F_y) \cap V_y = \emptyset\). Since \(U_y\) is an open upper set and \(x \leq y\), we have \(U_y \cap V_y\) is an open set containing \(y\). It follows from the disjointness of neighborhoods that \(U_y \cap V_y \subseteq \uparrow F_y\). Since \(\uparrow x \cap A\) is compact, finitely many of the \(U_y \cap V_y\) cover \(\uparrow x \cap A\), say \(\{U_{y_k} \cap V_{y_k}; k = 1, \ldots, n\}\). Set \(G := \bigcup_{k=1}^n F_{y_k}\). Then \(\uparrow G\) contains \(V := \bigcup\{U_{y_k} \cap V_{y_k}; k = 1, \ldots, n\}\), an open set containing \(E\), and
Step 2. There exists \( i \in I \) such that if \( z \in F_i \) and \( z \leq x \), then \( \uparrow z \cap A \) is contained in \( \uparrow G \). For suppose that this claim is false. The the set \( E_i := \{ z \in F_i : z \leq x, \uparrow z \cap (A \setminus \uparrow G) \neq \emptyset \} \) is non-empty for every \( i \). Since the \( \uparrow F_i \) form a descending family, it follows readily that the \( \uparrow E_i \) do also. By Rudin’s Lemma there exists a directed set \( D \subseteq \bigcup_i E_i \) such that \( D \cap E_i \) is nonempty for each \( i \). It is straightforward to verify that the directed set \( D \) converges to its supremum \( y \) (see Proposition VI.1.3 of [COMP]). Since \( x \) is an upper bound for \( D \), \( y \leq x \). But \( y \in \bigcap_i \uparrow F_i = B \) and \( x \) is minimal in \( B \), so \( y = x \). Thus \( x \) is the supremum of \( D \) and \( \uparrow x = \bigcap \{ \uparrow d : d \in D \} \).

For each \( d \in D \), pick \( y_d \in \uparrow d \cap (A \setminus \uparrow G) \). Some subnet of the \( \{ y_d \} \) converges to some \( y \in A \) which is not in the interior of \( \uparrow G \). Since for any \( d_0 \in D \) the net \( \{ y_d \} \) is eventually in the closed set \( \uparrow d_0 \), we conclude that \( y \in \uparrow d_0 \). Thus \( y \in \bigcap_{d \in D} \uparrow d = \uparrow x \). But this contradicts the first claim that \( \uparrow x \cap A \) is contained in the interior of \( \uparrow G \).

Step 3. There exists a finite set \( F \) such that \( A \subseteq \uparrow F \), but \( x \notin \uparrow F \). We pick \( G \) finite as in Step 1. By the second step there exists \( F_i \) such that \( z \in F_i \) and \( z \leq x \) imply \( \uparrow z \cap A \) is contained in \( \uparrow G \). Set \( F := G \cup (F_i \setminus \{ x \}) \). Since \( A \subseteq \uparrow F_i \), it follows directly from Step 1 and the preceding assertions that \( A \subseteq \uparrow F \). Also \( x \notin \uparrow F \) since \( x \notin \uparrow G \).

The lemma now follows immediately from the last step. Since \( \uparrow F \) contains \( A \) and is closed in the upper interval topology, it contains \( B \). However \( x \in B \) and \( x \notin \uparrow F \), a contradiction. \( \square \)

**Theorem 3.10** Let \( (P, \leq) \) be a partially ordered set satisfying property DINT. Let \( \tau \) denote the upper interval topology, and suppose there exists a topology \( \nu \) containing the lower interval topology with all open sets being upper sets such that \( P \) endowed with the join topology \( \sigma := \tau \vee \nu \) is a compact pospace. Then \( P \) is a quasicontinuous domain, the topology \( \sigma \) is the Lawson topology, the upper interval topology is \( \sigma_* \), and the Scott topology is equal to \( \sigma^* \) and is the cocompact topology for the upper interval topology. The Lawson topology is the unique topology \( \sigma \) making \( P \) a compact pospace with \( \sigma_* \) equal to the upper interval topology.

**Proof.** By Lemma 2.1 every directed subset has a supremum, so \( P \) is a DCPO. By the preceding lemma \( \sigma_* = \tau \). Thus \( \tau \) satisfies the three equivalent conditions of Theorem 2.4.

Let \( x \in P \). We consider all finite sets \( F \) such that \( \uparrow x \subseteq \text{int}(\uparrow F) \), where the interior is taken with respect to the topology \( \sigma \). Since a directed set converges to its supremum in the topology \( \sigma \) (Lemma 2.1), it follows that \( F \ll x \) for all such finite sets \( F \). To show that \( P \) is a quasicontinuous domain, we show that \( \uparrow x \) is the directed intersection of such finitely generated upper sets.

Suppose that \( \uparrow x \) is contained in the interior of both \( \uparrow F_1 \) and \( \uparrow F_2 \). Let \( U \) be the intersection of the interiors. By Lemma 2.1, \( \uparrow (P \setminus U) \) is compact, and it misses \( \uparrow x \); then \( W := P \setminus (\uparrow (P \setminus U)) \) is an open upper set such that
By the normality of $P$, we can pick an open set $V$ such that $\uparrow x \subseteq V \subseteq \uparrow W \subseteq U$. Since $W$ is an upper set, we have $\uparrow V \subseteq W$ and by Lemma 2.1 $\uparrow V$ is compact. Since $\uparrow V$ is closed in $\sigma$, which is the upper interval topology, by property DINT it is the directed intersection of finitely generated upper sets. Since each of these sets is closed, hence compact, we conclude that there exists some finite set $F$ such that $\uparrow V \subseteq \uparrow F \subseteq W$. It then follows that $\uparrow x$ is contained in the interior of $\uparrow F$ and that $\uparrow F \subseteq \uparrow F_1 \cap \uparrow F_2$. Thus the intersection is a directed intersection.

Suppose that $x \not\leq y$. Then $P \setminus \downarrow y$ is an open increasing set containing $x$. By an argument analogous to that of the preceding paragraph, we can find $F \subseteq P \setminus \downarrow y$ such that $\uparrow x$ is contained in the interior of $\uparrow F$. It follows that $\uparrow x$ is the directed intersection of all finitely generated upper sets $\uparrow F$ such that $\uparrow x$ is contained in the interior of $\uparrow F$, and thus that $P$ is a quasicontinuous domain.

It now follows from the preceding theorem that the Lawson topology makes $P$ a compact pospace. Thus each of the Scott topology and $\sigma^*$ is the cocompact topology for the upper interval topology, and hence the two agree. Hence the topology $\sigma$ is equal to the Lawson topology.

There remains an interesting open problem of whether the conclusions of Theorem 3.10 remain true under the weaker hypotheses of the lemma preceding it.

The following corollary can be deduced directly from Theorems 3.7 and 3.10.

**Corollary 3.11** Let $P$ be a partially ordered set and consider the following conditions.

(i) $P$ is a quasicontinuous domain.

(ii) $P$ satisfies property DINT.

(iii) $P$ is a compact pospace in the Lawson topology.

Then any two of these conditions implies the third.

Recall that a Priestley space is a compact pospace $P$ with the property that given $x \not\leq y$ there exists a clopen upper set containing $x$ that misses $y$. Such spaces arise in the Stone duality of distributive lattices. A topology of open lower sets $\tau$ is said to generate the topology of the Priestley space $P$ if there exists a topology $\nu$ of upper sets which contains the lower interval topology such that the join topology $\tau \lor \nu$ is the topology of the Priestley space. Note that if $\tau$ is the topology of open lower sets of the Priestley space, then $\tau \lor \tau^k$ is the Priestley topology (by Theorem 2.4), and hence $\tau$ generates the topology of the Priestley space.

**Corollary 3.12** Let $P$ be a partially ordered set equipped with a topology $\sigma$. The following are equivalent.

(i) The triple $(P, \leq, \sigma)$ is a Priestley space for which the upper interval topol-
ogy satisfies DINT and generates $\sigma$.

(ii) The triple $(P, \leq, \sigma)$ is a Priestley space for which each clopen upper set is finitely generated.

(iii) The poset $P$ is a quasialgebraic domain and the topology $\sigma$ is the Lawson topology, which is compact.

(iv) The poset $P$ is a quasialgebraic domain satisfying property $\mathcal{M}$ and $\sigma$ is the Lawson topology.

If any of these equivalent conditions is satisfied, then the upper interval topology is $\sigma^*$, and the Scott topology is $\sigma^*$.

Proof. 1.⇒2.: Let $A$ be a clopen upper set. It follows from Theorem 3.10 that $A$ is closed in the upper interval topology. Hence by condition DINT it is the directed intersection of finitely generated upper sets. Since $A$ is also open and $P$ is compact, it follows that $A \subseteq \uparrow F \subseteq A$ for some $\uparrow F$ in the directed family. Thus $A = \uparrow F$.

2.⇒3.: It is straightforward to argue that in a Priestley space every closed upper set, in particular every $\uparrow x$, is an intersection, and hence a directed intersection, of clopen upper sets. By hypothesis such a clopen upper set has the form $\uparrow F$ for some finite $F$; thus $\sigma^*$ is equal to the upper interval topology, which satisfies condition DINT. Using the fact that in the compact pospace $P$ a directed set converges to its supremum and the openness of $\uparrow F$, we conclude that $F \ll x$ and $F \ll F$. It thus follows that $P$ is a quasialgebraic domain. That the Lawson topology is equal to $\sigma$ and that the conditions at the end of the corollary hold now follow from Theorem 3.10.

3.⇒1.: Since condition 1 of Theorem 3.7 is satisfied, it follows that property $\mathcal{M}$ holds (condition 2) and the $\sigma$ topology, which is the Lawson topology, gives $P$ the structure of a compact pospace with open lower sets the sets open in the upper interval topology and open upper sets the Scott open sets (condition 8). Thus the upper interval topology generates $\sigma$.

Let $x \not\in y$. Then there exists $F \ll F$ such that $x \in \uparrow F$ and $y \not\in \uparrow F$. Since $F \ll F$, it is a direct argument to show that $\uparrow F = \uparrow \uparrow F$, a Scott open set. But also $\uparrow F$ is closed in the upper interval topology, hence closed in $\sigma$. Thus $\uparrow F$ is a clopen set containing $x$ and missing $y$, so $P$ is a Priestley space.

3.⇔4.: This equivalence follows from Theorem 3.7.

The preceding generalizes results of H. A. Priestley [Pr94], who established the equivalence of conditions 1 and 4 in the more restrictive context of posets satisfying property $\mathcal{M}$, introduced in the next section.

4 Property $\mathcal{M}$

The first lemma of this section shows that the notion of property $\mathcal{M}$ that we have introduced for quasicontinuous domains is a true generalization of property $\mathcal{M}$ as it is usually defined for continuous domains (the conditions in
the latter part of the lemma). Thus the equivalence of Lawson compactness and property $\mathcal{M}$ in Theorem 3.7 for quasicontinuous domains generalizes the previously established equivalence for continuous domains (see e.g., Section 4.2 of [Ju89]).

**Lemma 4.1** Let $P$ be a continuous domain. Then $P$ has property $\mathcal{M}$ if and only if $P$ is mub-complete, a finitely generated upper set, and satisfies the property that if $y_i \ll x_i$ for $i = 1, \ldots, n$, then there exists a finite subset $F \subseteq \text{mub}\{y_1, \ldots, y_n\}$ such that $\bigcap_{i=1}^{n} \uparrow x_i \subseteq \bigcup\{\uparrow w : w \in F\}$.

**Proof.** Suppose the alternate conditions are assumed and let $F_i \ll x_i$ for $i = 1, \ldots, n$. Since $x_i$ is the directed sup of all $z \ll x_i$, it follows that there exists $z_i \ll x_i$ such that $z_i \in \uparrow F_i$. Pick $y_i \in F_i$ such that $y_i \leq z_i$. Then the finite set $F$ ensured by the hypotheses for the set $\{y_1, \ldots, y_n\}$ also works for the collection $\{F_i\}$. The implication that property $\mathcal{M}$ gives the other properties results immediately by taking $F_i := \{y_i\}$ (and noting by the argument just given that $\uparrow F = \bigcup\{\uparrow b : b \in F\}$ for a finite subset of a continuous domain). \(\square\)

A partially ordered set $P$ is said to have property $\mathcal{M}$ if it is mub-complete and every finite set has finitely many minimal upper bounds. An alternate characterization is to require that the set of upper bounds for any finite set is a (possibly empty) finitely generated upper set (the minimal elements of the finite generating set being the minimal upper bounds). Note that the hypothesis of property $\mathcal{M}$ applied to the empty set yields that $P$ is a finitely generated upper set.

Recall that an algebraic domain is a DCPO in which each element is a directed supremum of compact elements $k \ll k$. The next theorem appears as the “2/3 SFP Theorem” in Plotkin’s Pisa Lecture Notes [Pl78]; a proof also appears in Section 4 of [Ju89].

**Theorem 4.2** A algebraic domain $P$ is Lawson compact if and only if the poset $K(P)$ of compact elements satisfies property $\mathcal{M}$.

Since an algebraic domain is a special kind of continuous domain, the following corollary is immediate from Theorems 3.7 and 4.2. However, we should point out that it can also be deduced in a straightforward fashion from the definitions of properties $\mathcal{M}$ and $\mathcal{M}$, and then used to deduce Theorem 4.2 as a corollary via Theorem 3.7.

**Corollary 4.3** An algebraic domain $P$ satisfies property $\mathcal{M}$ if and only if the poset $K(P)$ of compact elements satisfies property $\mathcal{M}$.

In the previous section we saw that for quasicontinuous domains the properties $\mathcal{M}$ and DINT were equivalent. Property $\mathcal{M}$ is a stronger property in the sense that it always implies property DINT.

**Lemma 4.4** If a partially ordered set $P$ satisfies property $\mathcal{M}$, then it satisfies property DINT.
Proof. Let $A$ be a set closed in the upper interval topology. If $A = \emptyset$, then it is already a finitely generated upper set. We remarked previously that property $M$ implies $P$ is a finitely generated upper set. Otherwise $A$ is an intersection of non-empty finitely generated upper sets. Let $\uparrow F$ and $\uparrow G$ be non-empty finitely generated upper sets. For each $x \in F$, $y \in G$, define $H(x, y)$ to be the (finite) set of minimal upper bounds of $\{x, y\}$. Then $H := \bigcup \{H(x, y) : x \in F, y \in G\}$ is a finite set and $\uparrow F \cap \uparrow G = \uparrow H$. Thus the family of finitely generated upper sets is closed under finite intersection. It is then immediately that $A$ is a directed intersection of finitely generated upper sets.

If a quasicontinuous domain satisfies property $M$, then by the previous lemma it also satisfies property DINT, i.e., condition 6 of Theorem 3.7. We thus obtain directly the following corollary.

Corollary 4.5 If a quasicontinuous domain satisfies property $M$, then it satisfies all the various equivalent conditions of Theorem 3.7. Alternately if $P$ is a DCPO satisfying property $M$ for which the upper interval topology satisfies any of the equivalent conditions of Theorem 2.4, then the conclusions of Theorem 3.10 are satisfied.

A DCPO $P$ is said to be bounded complete if it has a bottom element $\bot$ and every pair of elements which are bounded above have a least upper bound. It is equivalent to require that $P$ be a complete semilattice, a partially ordered set in which every non-empty set has a greatest lower bound. It is immediate that a bounded complete DCPO has property $M$ since every finite set has either no upper bounds or a unique least upper bound.

Corollary 4.6 If $P$ is a bounded complete quasicontinuous domain, then $P$ satisfies the equivalent conditions of Theorem 3.7.

References


