Constrained proportional integral control of dynamical distribution networks with state constraints* 

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Abstract—This paper studies a basic model of a dynamical distribution network, where the network topology is given by a directed graph with storage variables corresponding to the vertices and flow inputs corresponding to the edges. We aim at regulating the system to consensus, while the storage variables remain greater or equal than a given lower bound. The problem is solved by using a distributed PI controller structure with constraints which vary in time. It is shown how the constraints can be obtained by solving an optimization problem.

I. INTRODUCTION

In this paper we continue our study of the dynamics of distribution networks. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to storage, and with every edge a control input variable corresponding to flow, possibly subject to constraints. In previous work [1], [2], [3] it has been shown under which conditions a constrained proportional-integral (PI) controller will regulate the system towards output agreement in the presence of unknown constant external disturbances (corresponding to constant in/outflows of the network).

In many cases of practical interest it is natural to require that the state variables of the distribution network will remain larger than a given minimal value, e.g. zero. A hydraulic network with state variables being the storage of fluid is a clear example of such a situation. On the other hand, the previously developed PI-controller can give rise to damped oscillatory behavior which may violate such state constraints. The aim of the current paper is to modify the PI-controller in such a way that the lower bounds for the state variables will be satisfied for all time while the system will still converge to output agreement. This is done by adapting the constraints of the PI controller.

The main related work can be summarized as follows. In [4] an alternative scheme is given in order that the state variables remain nonnegative. However, this scheme does not respect mass conservation. In [5], the authors consider a similar distribution network model but with a proportional controller instead of a PI controller. A discontinuous Lyapunov-based controller is given to stabilize the system without violating the storage and flow constraints; however it is not robust with respect to constant disturbances.) In [6], using the same model as in [5], the authors focus on a different problem of driving the state to a small neighborhood of the reference value and relate the control input value at equilibrium to an optimization problem. The structure of the paper is as follows. Preliminaries and notations are given in Section II. In Section III, we introduce the basic model which can be identified as a port-Hamiltonian system, in line with the general definition of port-Hamiltonian system on graphs [7], [8], [9], [10]; see also [11], [12]. In Section IV we briefly recall from our previous work [1], [2], [3] the necessary and sufficient graphical conditions in order that a constrained PI controller structure will solve the regulation problem.

In Section V we formulate the main problem of this paper, namely the adaptation of the constraints of the PI-controller such that the system will reach output agreement while the state variables will remain greater or equal than a given lower bound. In Section VI an optimal control protocol for the adaptation of the flow (control) constraints is developed, while stability analysis of the scheme is given in Section VII. The conclusions are in Section VIII.

II. PRELIMINARIES AND NOTATIONS

We recall some standard definitions regarding directed graphs, as can be found e.g. in [13]. A directed graph \( G \) consists of a finite set \( V \) of vertices and a finite set \( E \) of edges, together with a mapping from \( E \) to the set of ordered pairs of \( V \), where no self-loops are allowed. Thus to any edge \( e \in E \) there corresponds an ordered pair \( (v, w) \in V \times V \) (with \( v \neq w \)), representing the tail vertex \( v \) and the head vertex \( w \) of this edge.

A directed graph is specified by its incidence matrix \( B \), which is an \( n \times m \) matrix, \( n \) and \( m \) being the number of vertices and edges respectively, with \( (i, j) \)th element equal to 1 if the \( j \)th edge is towards vertex \( i \), and equal to \(-1\) if the \( j \)th edge is originating from vertex \( i \), and \( 0 \) otherwise. In this paper ‘graph’ will throughout mean ‘directed graph’ unless stated otherwise. A graph is strongly connected if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. It is called weakly connected if it is possible to reach any vertex from every other vertex using the edges not taking into account their direction. A graph is weakly connected if and only if \( \ker B^T = \text{span} \, \mathbb{I}_n \). Here \( \mathbb{I}_n \) denotes the \( n \)-dimensional vector with all elements equal to 1. A graph that is not weakly connected falls apart into a number of weakly connected components. The number of connected components is equal to \( \dim \ker B^T \). For each vertex, the

*The work of the first author is supported by the Chinese Science Council (CSC). The research of the second author leading to these results has received support from the EU 7th Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of Excellence.

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number of incoming and outgoing edges are called the in- and out-degree of the vertex respectively. A graph is called balanced if and only if the in-degree and out-degree of every vertex are equal to each other. A graph is balanced if and only if $1_n \in \ker B$.

Given a graph, we define its vertex space as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the rest of this paper we will take $\mathcal{R} = \mathbb{R}$, in which case the vertex space can be identified with $\mathbb{R}^n$. Similarly, we define its edge space as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R} = \mathbb{R}$, which can be identified with $\mathbb{R}^m$. In this way, the incidence matrix $B$ of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$. Extensions of the results to general linear spaces $\mathcal{R}$ are straightforward.

**Notation:** For $a, b \in \{-\infty, \infty\}^m$ the notation $a \leq b$ (resp. $a < b$) will denote element-wise inequality $a_i \leq b_i$ (resp. $a_i < b_i$), $i = 1, \ldots, m$. For $a < b$ the multidimensional saturation function $\text{sat}(x; a, b) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as

$$\text{sat}(x; a, b)_i = \begin{cases} a_i & \text{if } x_i \leq a_i, \\ x_i & \text{if } a_i < x_i < b_i, \\ b_i & \text{if } x_i \geq b_i, \end{cases} \quad (1)$$

### III. A Dynamical Network Model with PI Controller

Let us consider the following dynamical system defined on the vertices of graph $\mathcal{G}$ ([10], [8], [7])

$$\begin{align*}
\dot{x} &= u + E\Bar{d}, \quad x, u \in \mathbb{R}^n, \quad d \in \mathbb{R}^k \\
y &= \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^n,
\end{align*} \quad (2)$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of $H$. Here $x_i$ and $u_i$ are the state and input variables associated to the $i$th vertex of the graph respectively. And $E$ is an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while the rest of the elements is zero. Thus $E$ specifies the $k$ terminal vertices where flows can enter or leave the network ([14]). Finally, $d$ is a vector of constant disturbances. System (2) defines a port-Hamiltonian system ([14], [15]), satisfying the energy-balance

$$\frac{d}{dt} H = u^T y + \frac{\partial^2 H}{\partial x}(x) E\Bar{d}. \quad (3)$$

Here the state variables on vertices are controlled by the flow in the edges of network in the following manner

$$u = B\mu, \quad \mu \in \mathbb{R}^m \quad (4)$$

where $\mu_j$ is a flow variable associated to the $j$th edge of the graph. In this paper we consider the case when the controller is defined on the edges to provide the flow variables. As explained in [2], when $d \neq 0$, the proportional control will not be sufficient to reach load balancing. Hence we consider a proportional-integral (PI) controller given by the dynamic output feedback

$$\begin{align*}
\dot{\eta} &= \zeta, \quad \eta, \zeta \in \mathbb{R}^n, \\
\mu &= -R\zeta - \frac{\partial H}{\partial \eta}(\eta)
\end{align*} \quad (5)$$

where $\eta_i$ is state variable associated to $i$th edge, $R$ is a diagonal matrix with strictly positive diagonal elements $r_1, r_2, \ldots, r_m$, and $H_\ast$ the Hamiltonian function corresponding to the controller. Here the controller is driven by the relative output of the systems (2) on vertices, i.e.,

$$\zeta = B^T y \quad (6)$$

The closed-loop system of (2,4,5,6) is a port-Hamiltonian system

$$\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} = \begin{bmatrix}
-B\Bar{B}B^T & -B \\
B^T & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial H}{\partial x}(x) \\
\frac{\partial H}{\partial \eta}(\eta)
\end{bmatrix} + \begin{bmatrix}
E \end{bmatrix} d, \quad (7)$$

with total Hamiltonian

$$H_{\text{tot}}(x, \eta) := H(x) + H_\ast(\eta).$$

Suppose now that the constant disturbance $\Bar{d}$ satisfies the matching condition, i.e., there exists a controller state $\Bar{\eta}$ such that

$$E\Bar{d} = B\frac{\partial H_\ast}{\partial \eta}(\Bar{\eta}). \quad (8)$$

By modifying the total Hamiltonian $H_{\text{tot}}$ into

$$V_{\Bar{d}}(x, \eta) := H(x) + H_\ast(\eta) - \frac{\partial^2 H_\ast}{\partial \eta^2}(\eta)(\eta - \Bar{\eta}) - H_\ast(\Bar{\eta}) \quad (9)$$

which serves as a candidate Lyapunov function, we can obtain the following theorem.

**Theorem 1:** ([11], [2]) Consider the dynamical system (2,5,4,6) on the graph $\mathcal{G}$. Let the constant disturbance $\Bar{d}$ satisfies the matching condition (8) with a $\Bar{\eta}$. Assume $V_{\Bar{d}}(x, \eta)$ is radially unbounded. Then the trajectories of the closed-loop system (7) will converge to an element of the load balancing set

$$E_{\text{tot}} = \left\{(x, \eta) \mid \frac{\partial H}{\partial x}(x) = \alpha I, \alpha \in \mathbb{R}, B\frac{\partial H_\ast}{\partial \eta}(\eta) = E\Bar{d}\right\}. \quad (10)$$

if and only if $\mathcal{G}$ is weakly connected.

**Corollary 2:** If $\ker B = 0$, which is equivalent [13] to the graph having no cycles, then for every $\Bar{d}$ there exists a unique $\frac{\partial H_\ast}{\partial \eta}(\Bar{\eta})$ satisfying (8). Suppose $H_\ast$ has positive definite Hessian matrix, then in (8) $\Bar{\eta}$ is also unique and the convergence is towards the set $E_{\text{tot}} = \{(x, \eta) \mid \frac{\partial H}{\partial x}(x) = \alpha I, \alpha \in \mathbb{R}\}$.

**Corollary 3:** In case of the standard quadratic Hamiltonians $H(x) = \frac{1}{2}\|x\|^2, H_\ast(\eta) = \frac{1}{2}\|\eta\|^2$ there exists for every $\Bar{d}$ a controller state $\Bar{\eta}$ such that (8) holds if and only if

$$\text{im} E \subset \text{im} B. \quad (10)$$

Furthermore, in this case $V_{\Bar{d}}$ equals the radially unbounded function $\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\eta - \Bar{\eta}\|^2$, while convergence will be towards the load balancing set $E_{\text{tot}} = \{(x, \eta) \mid x = \alpha \Xi, \alpha \in \mathbb{R}, B\Xi = E\Bar{d}\}$.

A necessary (and in case the graph is weakly connected necessary and sufficient) condition for the inclusion $\text{im} E \subset \text{im} B$ is that $1^T E = 0$. In its turn $1^T E = 0$ is equivalent to the fact that for every $\Bar{d}$ the total inflow into the network equals to the total outflow. The condition $1^T E = 0$ also implies

$$1^T \dot{x} = -1^T B\mu + 1^T E\Bar{d} = 0,$$

implying (as in the case $d = 0$) that $1^T x$ is a conserved quantity for the closed-loop system (7).
IV. Results Review for the Case with Flow Constraints

In many cases of interest, the flows in the edges are constrained. Here we briefly review the main results in [2, 3] where we consider instead of (4) its constrained version

\[ u = B \text{sat}(\mu, \mu^-, \mu^+) \quad \text{(11)} \]

where \( \mu^-, \mu^+ \in \mathbb{R}^m \) are the lower and upper bounds for the flow constraints. For simplicity of exposition we only consider the PI controller (5) with the flow constraints. For simplicity of exposition we only consider the PI controller (5) with the flow constraints. For simplicity of exposition we only consider the PI controller (5) with the flow constraints.

As is shown in [2, 3], without loss of generality we can assume that the flow constraints satisfy \( u^+ \geq u^- \geq 0 \) and the two equality signs do not hold at the same time. We call this kind of the constraints are compatible with the orientation. Furthermore, adding the disturbance satisfying the constrained version of the matching condition, i.e.,

\[ E \dot{d} = B \text{sat}(\eta; \mu^-, \mu^+) \quad \text{(12)} \]

is equivalent to translation of the constraints. It follows that, without loss of generality, we can focus on the closed-loop system without disturbance

\[ \dot{x} = B \text{sat}(-B^T \frac{\partial H}{\partial x}(x) - \eta; \mu^-, \mu^+), \]
\[ \dot{\eta} = B^T \frac{\partial H}{\partial x}(x). \quad \text{(13)} \]

The main results about system (13) are summarized in

Theorem 4: ([2]) Consider the dynamical system (13) with compatible flow constraints. Then for any \( \mu^- < \mu^+ \in \mathbb{R}^m \) such that \( \bigcap_{i=1}^m [\mu_i^-, \mu_i^+] \) contains an open interval, the trajectories of (13) converge to

\[ \mathcal{E}_{\text{tot}} = \{ (x, \eta) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{I}_n, \alpha \in \mathbb{R}, \text{ sat}(\eta; \mu^-, \mu^+) = 0 \} \quad \text{(14)} \]

if and only if the graph is strongly connected and balanced.

For any network with given orientation and constraints on the edges, we can define the interior point condition.

Definition 5: (Interior Point Condition) Given a directed graph with arbitrary constraints \( [\mu^-, \mu^+] \), the network will be said to satisfy the interior point condition if there exists a vector \( z \in [\mu^-, \mu^+] \cap \ker B \) such that the subgraph \( G_0 = \{ V, E_0 \} \) is weakly connected where

\[ E_0(z; \mu^-, \mu^+) = \{ e_i \mid e_i \in E, z_i \in (\mu_i^-, \mu_i^+) \}. \]

Theorem 6: ([3]) Consider the dynamical system (13) defined on a weakly connected graph. Then the trajectories will converge to

\[ \mathcal{E}_{\text{tot}} = \{ (x, \eta) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{I}_n, \text{ sat}(\eta; u^-, u^+) = 0 \}. \]

V. Constrained PI-Controllers Maintaining a Lower Bound for the State Variables

Although, as summarized in the previous sections, the PI controller with both unconstrained (4) and constrained flow connection (11) is successful in obtaining output agreement for the plant, it introduces oscillatory behavior which may cause the state variables \( x \) become smaller than some given lower bounds. For certain applications this may be undesirable or infeasible, as is illustrated by the following example.

Example 5.1 (Hydraulic network): Consider a hydraulic network, modeled as a directed graph with vertices (nodes) corresponding to reservoirs, and edges (branches) corresponding to pipes. Let \( x_i \) be the volume of fluid stored at vertex \( i \), and \( \mu_j \) the flow through edge \( j \). Then the mass balance of the network is summarized as (2) and (4). Let \( H(x) \) denote the stored energy in the reservoirs (e.g., gravitational energy).

For the rest of the paper we focus on the closed-loop system (2,5,6,15), given as

\[ \dot{x} = B \text{sat}(-RB^T \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(\eta) \mu^-(t), \mu^+(t)), \]
\[ \dot{\eta} = B^T \frac{\partial H}{\partial x}(x). \quad \text{(16)} \]

with in/outflows are zero (i.e., \( \bar{d} = 0 \)). Furthermore, we assume \( H_\circledast(\cdot) \in C^1, H(x) = \sum_{i=1}^n H_i(x_i) \in \mathbb{R}^2 \) with \( H_i(\cdot) : \mathbb{R} \to \mathbb{R} \) be strictly convex and arg min \( H_i(x_i) = \gamma_i, i = 1, 2, \ldots, n \). \( u^- (t) \) and \( u^+(t) \) are parameters to be designed such that \( x(t) \geq \gamma_i \forall t \geq 0 \). Notice that this is equivalent to keeping the output of the plant \( \frac{\partial H}{\partial x} \) being non-negative.

Remark 7: Note that when \( H(x) = H_1(x_1) + \ldots + H_n(x_n) \) and \( H_\circledast \) are convex, there are many controllers which fulfill the control aim of driving output of the plant \( \frac{\partial H}{\partial x}(x) \) to agreement while keeping it non-negative. For example the proportional controller

\[ \mu = -R \zeta \quad \text{(17)} \]

with \( R \) a positive diagonal matrix has the property that the evolution of the closed-loop system (2,4,6,17) with \( \bar{d} = 0 \), i.e.

\[ \dot{x} = -RB^T \frac{\partial H}{\partial x}(x) \]
remains in the set \( \{ x \mid \frac{\partial H}{\partial x_i}(x(t)) \in \mathbb{R}^n_+ \forall t \geq 0 \} \) whenever \( \frac{\partial H}{\partial x_i}(x(0)) \geq 0 \). This directly follows from the properties of the weighted Laplacian matrix \( BRB^T \); whenever at a certain moment \( \frac{\partial H}{\partial x_i}(x(t)) = 0 \), then \( \dot{x_i}(t) = -\sum j \in e \; r_k \left( \frac{\partial H}{\partial x_i}(x(t)) - \frac{\partial H}{\partial x_j}(x(t)) \right) \geq 0 \) where \( r_k \) is the \( k \)-th diagonal element of \( R \) and \( e_k \sim (v_i, v_j) \). However for the second order system (7), in order to achieve the control aim the flows on the edges need to be regulated.

**Example 5.2 (Hydraulic network continued):** In this example we want to show that instead of keeping \( \frac{\partial H}{\partial x_i} > 0 \), \( i = 1, 2, \ldots, n \), they can be above any value \( y_i \) satisfying \( y_i = \frac{\partial H}{\partial x_i}(\bar{x}_i) \) for some \( \bar{x}_i \) for all time. In that case we just replace the Hamiltonian \( H \) in the system (16) by the Bregman distance with respect to \( \bar{x}_i \), i.e. \( H'_i(x_i) = H_i(x_i) - y_i(x_i - \bar{x}_i) - H(\bar{x}_i) \), then the design of parameters \( u^- \) and \( u^+ \) as done in the next section will keep \( \frac{\partial H}{\partial x_i} > y_i \). For instance with the same setting as in Example 5.1, suppose we want to keep the pressure of each reservoir greater or equal than a given value \( P_i = y_i \geq 0 \), \( i = 1, 2, \ldots, n \), then we can modify the Hamiltonian as \( H_i(x_i) = \frac{\partial H}{\partial x_i}(x_i - S_i) \), which is the extra stored energy in the \( i \)-th reservoir compared to the stored energy at height \( \bar{h}_i \). In this case \( \frac{\partial H}{\partial x_i} = y_i \), which is the relative pressure with respect to the height \( \bar{h}_i \).

**VI. THE DESIGN OF THE FLOW CONSTRAINTS**

In this section we will design the parameters \( \mu^- \) and \( \mu^+ \) in (16). The only situation in which a state variable \( x_i \) may become smaller than \( \gamma_i \) is that at a certain time instant \( t \), \( x_i(t) = \gamma_i \) and \( \dot{x}_i(t) < 0 \). The basic idea underlying the design of the time-varying flow constraints is to eliminate this situation by adding saturation on the flows in the edges in such a way that \( \dot{x}_i(t) \geq 0 \) whenever \( x_i(t) = \gamma_i \). For each time \( t \) and each vertex \( v_i \), the edges adjacent to it can be divided into two sets

\[
\begin{align*}
    f_{i}^{in}(t) & = \{ e_j \in E \mid B_{ij}(u_j) > 0 \} \\
    f_{i}^{out}(t) & = \{ e_j \in E \mid B_{ij}(u_j) < 0 \}.
\end{align*}
\]

For each time \( t \), the vertices of the network can be divided into the following subsets, referred to as **white**, **gray** and **black** (with the last category divided into two subsets)

\[
\begin{align*}
    V^{W}(t) & = \{ v_i \in V \mid x_i(t) > \gamma_i \} \\
    V^{G}(t) & = \{ v_i \in V \mid x_i(t) = \gamma_i \} \\
    V^{B1}(t) & = \{ v_i \in V^{G} \mid (i,j) \in \mu(t) < 0 \} \\
    V^{B2}(t) & = \{ v_i \in V^{G} \mid \exists v_j \in V^{B1} \text{ s.t. } f_{i}^{in}(t) \cap f_{i}^{out}(t) = \emptyset \}
\end{align*}
\]

where \( B(i,j) \) is the \( i \)-th row of \( B \). Furthermore, we denote \( V^{B} = V^{B1} \cup V^{B2} \).

**Example 6.1:** Let us consider a part of the network given as given in Fig. 1. This example shows that the states of the **black** nodes can become negative. Indeed suppose that at time \( t \) the state variable at \( v_2 \), i.e. \( x_2(t) \), decreases to \( \gamma_2 \), while \( \mu_2(t) + \mu_3(t) > \mu_1(t) \geq 0 \), then \( \dot{x}_2(t) < 0 \).

Let us denote the set of outgoing edges of all vertices in \( V^{B}(t) \), i.e. \( \cup_{v_i \in V^{B}(t)} f_{i}^{out}(t) \), as \( E_{out}(t) \). Along the edges \( e_l \in E_{out}(t) \), a saturation \( \min \left\{ -|\phi_t^l|, |\phi_t^l| \right\} \) is imposed on the flow, while along the rest of the edges there are no saturations where \( \phi^*(t) \in \mathbb{R}^m \) is the optimal solution of the following optimization problem

\[
\begin{align*}
    \min_{\phi} & \sum_{e_j \in E_{out}(t)} \frac{1}{2|v_j(t)|} \left( (\phi_j - \mu_j(t))^2 + \phi_j^2 \right) \\
    \text{s.t.} & \; B(i,j)\phi = 0, \forall v_i \in V^{B}(t), \\
    & \; \phi_k = \mu_k(t), \; \text{if } e_k \in E \setminus E_{out}(t).
\end{align*}
\]

Furthermore, let us denote

\[
\phi^*_j(t) = \begin{cases} 
|\phi^*_j(t)| & \text{if } e_j \in E_{out}(t) \quad j = 1, \ldots, m. \\
+\infty & \text{else,}
\end{cases}
\]

then the closed-loop (16) can be written as

\[
\begin{align*}
    \dot{x} & = B \text{ sat}(-RB^T \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(\eta), -\phi^+(t), \phi^+(t)), \\
    \dot{\eta} & = B^T \frac{\partial H}{\partial x}(x)
\end{align*}
\]

with the parameters \( \mu^- \) and \( \mu^+ \) in (16) being chosen as \( -\phi^+(t) \) and \( \phi^+(t) \) respectively.

**Example 6.2:** Continuing Example 6.1, suppose at time \( t \), the flows are subject to \( \mu_2(t) + \mu_3(t) > \mu_1(t) \geq 0 \) and \( x_2(t) = \gamma_2 \). Then the above control protocol will set \( x_2 = \mu_1(t) + \sum_{k=1,2,3} \text{ sat}(\mu_1(t), -\phi_1^-(t), \phi_1^+(t)) \) where \( \phi_1^+(t) = \frac{1}{\sum_{k=1,2,3} \mu_k(t)} | \mu_k(t) |, \; t = 2, 3 \).

Furthermore, the solution of the optimization problem (19) can be seen as the limit of the following algorithm.

**Algorithm: Initialization:** at time \( t \) when there are grey nodes in the network, set the initial value \( \phi^0 = \mu(t) \in \mathbb{R}^m \).

**Step k:** Let \( \phi^{k-1} \) be the value from the previous step \( k-1 \). If there exists a node \( i \) such that \( B(i,i) \phi^{k-1} < 0 \), then

\[
\phi^*_j(t) = \begin{cases} 
\sum_{e_j \in f_{i}^{in}(t)} |\phi_j^{k-1}| & \text{if } e_j \in f_{i}^{out}, \; j = 1, \ldots, m. \\
\phi_j^{k-1} & \text{else,}
\end{cases}
\]

This algorithm is converging since in each iteration the absolute values of the flows are non-increasing. It can be proved that \( \lim_{k \to \infty} \phi^k(t) = \phi^*(t) \). However the proof is omitted due to lack of space.

**Example 6.3:** In this example, we consider the structure of the network as given in Fig. 2. Suppose at time \( t \), \( x_1(t) > \gamma_1, x_2(t) = \gamma_2, x_3(t) = \gamma_3 \) and the output of controller (5) is \( \mu(t) = [1, 3, 1, 2]^T \). Then \( V^{B}(t) = \{ v_2, v_3 \} \). By using the algorithm, the flows on \( f_2^{out}(t) \cup f_3^{out}(t) = \{ e_2, e_3, e_4 \} \), are saturated to the values \( \frac{3}{2}, \frac{3}{2}, \text{ and } 1 \), respectively. Furthermore, it can be verified that \( \frac{3}{2}, \frac{3}{2}, 1 \) is the solution of optimization problem (19).

**VII. STABILITY ANALYSIS**

In this section we will prove the stability of the system (21), and its convergence to consensus.
Since the right-hand-side of the system (21) is discontinuous, we will consider Filippov solutions. The notations are taken from [16].

**Definition 8:** ([16]) Let $\mathcal{B}(\mathbb{R}^d)$ denote the collection of subsets of $\mathbb{R}^d$. For $X : \mathbb{R}^d \to \mathbb{R}^d$, define the Filippov set-valued map $F[X] : \mathbb{R}^d \to \mathcal{B}(\mathbb{R}^d)$ as

$$F[X](x) \triangleq \bigcap_{\delta > 0} \cap_{\mu(S) = 0} \mathcal{S}(X(B(x, \delta) \setminus S)) \quad (23)$$

**Definition 9:** A Filippov solution of $\dot{x}(t) = X(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous map $x : [0, t_1) \to \mathbb{R}^d$ that satisfies

$$\dot{x}(t) \in F[X](x) \quad (24)$$

for almost all $t \in [0, t_1]$.

Here are two useful facts about computing the Filippov set-valued map.

**Proposition 10:** ([16]) **Product Rule:** If $X_1 : \mathbb{R}^d \to \mathbb{R}^d$ and $X_2 : \mathbb{R}^d \to \mathbb{R}^n$ are locally bounded at $x \in \mathbb{R}^d$, then

$$F[X_1, X_2]^T(x) \subset F[X_1](x) \times F[X_2](x). \quad (25)$$

Moreover, if either $X_1$ or $X_2$ is continuous at $x$, then equality holds.

**Matrix Transformation Rule:** If $X : \mathbb{R}^d \to \mathbb{R}^m$ is locally bounded at $x \in \mathbb{R}^d$ and $Z : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is continuous at $x \in \mathbb{R}^d$, then

$$F[ZX](x) = Z(x)F[X](x). \quad (26)$$

**Theorem 11:** Consider the system (21) on the graph $\mathcal{G}$ in closed loop with the saturation bounds as given in (20). Assume that $H = \sum_{i=1}^{m} H_i(x_i) \in C^2$ and $H_i \in C^1$ are positive definite and radially unbounded. Furthermore $H_i$ are strictly convex with arg min$_{x \in \mathbb{R}^n} H(x) = \gamma \in \mathbb{R}^n$. Then (i) $x(t) \geq \gamma$ for all $t > 0$ if $x(0) \geq \gamma$; (ii) the trajectories of the closed-loop system (21) will converge to an element of the load balancing set

$$\mathcal{E}_{tot} = \{(x, \eta) | \frac{\partial H}{\partial \eta}(x) = \alpha \eta, \alpha \in \mathbb{R}^+, B \frac{\partial H}{\partial \eta}(\eta) = 0\}$$

if and only if $\mathcal{G}$ is weakly connected.

**Proof:** (i) It can be verified from the form of optimization problem (19) which grants that $\dot{x}_i(t) \geq 0$ when $x_i(t) = \gamma_i$.

(ii) **Sufficiency.** First by using Proposition 10, the differential equations (21) are replaced by the differential inclusion

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{bmatrix} \in F \begin{bmatrix} \begin{bmatrix} B \text{sat}(\mu(t), -\phi_i^+(t), \phi_i^+(t)) \\ B^T \frac{\partial H}{\partial x}(x) \end{bmatrix} \\ B^T \frac{\partial H}{\partial \eta}(\eta) \end{bmatrix} \triangleq F(x, \eta) \quad (27)$$

where the equality is implied by Proposition 10. Notice that the set-valued map $F(x, \eta)$ is locally bounded and its values are nonempty, compact and convex sets. Furthermore, for each $t \in \mathbb{R}$, $(x, \eta) \to F(x, \eta)$ is upper semi-continuous.

Take as Lyapunov function the Hamiltonian function

$$V(x, \eta) := H(x) + H_c(\eta), \quad (28)$$

which is differentiable. Then the set-valued Lie derivative $\mathcal{L}_F V : \mathbb{R}^{n+m} \to \mathcal{B}(\mathbb{R})$ of $V$ with respect to $F$ at $(x, \eta)$ is defined as

$$\mathcal{L}_F V = \{ (\nabla V)^T \omega | \omega \in F(x, \eta) \}$$

$$= \frac{\partial^T H}{\partial x}(x) B_F \left[ \text{sat}(\mu(t), -\phi_i^+(t), \phi_i^+(t)) \right] + \frac{\partial^T H}{\partial \eta}(x) B^T_c \frac{\partial H_c}{\partial \eta}(\eta) \quad (29)$$

For the $i$-th edge, the Filippov set-valued map is given as

$$F \left[ \text{sat}(\mu_i(t), -\phi_i^+(t), \phi_i^+(t)) \right]$$

$$\subset \begin{cases} [0, \mu_i(t)] & e_i \in \mathcal{E}_i^0(t) \land \mu_i(t) > 0, \\
[\mu_i(t), 0] & e_i \in \mathcal{E}_i^D(t) \land \mu_i(t) < 0, \quad i = 1, \ldots, m, \\
{\mu_i(t)} & e_i \in \mathcal{E}_i^1(t), \quad \text{else} \end{cases} \quad (30)$$

For the $i$-th edge of $\mathcal{G}$ on which $\phi_i^+(t) = +\infty$, i.e. $e_i \in \mathcal{E}_i^D(t)$, we have

$$\frac{\partial^T H}{\partial x}(x(t)) B_i \text{sat}(\mu_i(t), -\phi_i^+(t), \phi_i^+(t)) + \frac{\partial^T H}{\partial \eta}(x(t)) B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t))$$

$$= -\frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t)) \quad (31)$$

where $B_i$ is the $i$-th column of $B$.

For the $i$-th edge on which $\phi_i^+(t) < +\infty$, i.e. $e_i \in \mathcal{E}_i^0(t)$, we have that $\forall \omega_i \in F[\text{sat}(\mu_i(t), -\phi_i^+(t), \phi_i^+(t))]$ which can be written as

$$\omega_i = (1 - \kappa_i) 0 + \kappa_i (\mu_i(t)), \text{ for some } \kappa_i \in [0, 1]$$

This implies that

$$\frac{\partial^T H}{\partial x}(x(t)) B_i \text{sat}(\mu_i(t), -\phi_i^+(t), \phi_i^+(t))$$

$$+ \frac{\partial^T H}{\partial \eta}(x(t)) B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t))$$

$$= -\kappa_i \frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t)) + (1 - \kappa_i) \frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t)) \quad (33)$$

Furthermore, when $\phi_i^+(t) < +\infty$, we have either

- $B^T_i \frac{\partial H_c}{\partial \eta}(\eta) \leq 0$ and $-B^T_i \frac{\partial H_c}{\partial \eta}(\eta) \leq 0$, which implies $\frac{\partial^T H}{\partial x}(x(t)) B_i \frac{\partial H_c}{\partial \eta}(\eta(t)) \leq -\frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t))$ or
- $B^T_i \frac{\partial H_c}{\partial \eta}(\eta) \leq 0$ and $-B^T_i \frac{\partial H_c}{\partial \eta}(\eta) \leq 0$, which implies $\frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t)) \leq 0$. So far, we can conclude that

$$\frac{\partial^T H}{\partial x}(x(t)) B_i \text{sat}(\mu_i(t), -\phi_i^+(t), \phi_i^+(t)) + \frac{\partial H_c}{\partial \eta}(\eta(t))$$

$$\leq -\frac{\partial^T H}{\partial x}(x(t)) B_i B^T_i \frac{\partial H_c}{\partial \eta}(\eta(t)) \quad (34)$$
i.e. \( \max \tilde{L}_T V g(x, \eta) \leq -\frac{\partial^T H}{\partial x}(x) B B^T \frac{\partial H}{\partial x}(x) \).

By LaSalle’s Invariance principle, the trajectories will converge to the largest invariant set, denoted as \( \mathcal{I} \), within the set where \( \{(x, \eta) \mid \dot{V} = 0\} \), i.e. \( \{(x, \eta) \mid B B^T \frac{\partial H}{\partial x}(x) = 0\} \). In \( \mathcal{I} \) we have

\[
B^T \frac{\partial^2 H}{\partial x^2} B \text{sat}(\frac{\partial H_e}{\partial \eta}(\eta(t)), -\phi^+(t), \phi^+(t)) = 0 \quad (35)
\]

which implies that \( x \) remains at a constant value, denoted by \( \nu \), in \( \mathcal{I} \) and \( \frac{\partial H}{\partial x}(\nu) = \alpha \mathbb{1} \). Furthermore, in view of \( \nu \geq \gamma \) and the convexity of \( H \) we can prove that \( \alpha > 0 \). By the optimal control protocol given in the previous section, we have that all the vertices will be white for large enough \( t \), which implies at steady state \( B \frac{\partial H}{\partial x}(\eta) = 0 \). This concludes the proof.

**Necessity.** If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value \( \alpha \) will be different for different components.

**Example 7.1 (Hydraulic network continued):** In this example we show the simulation results of the hydraulic network defined on the graph given in Figure 3, with flow constraints given as solution of the optimization problem (19). The values of the parameters are taken as components:

\[
S_i = 1 m^2, i = 1, \ldots, 5, \rho = 1 kg/m^3, \gamma = 0 \quad \text{and} \quad [x(0), x_e(0)] = [0, 0.5, 1, 2, 0.5, 9, 3, 0, -1, -2, -4].
\]

In Figure 4, it can be seen that the volume of each reservoir is kept nonnegative for all times. Furthermore the pressures of reservoirs converge to a common value (consensus).

![Network structure of Example 7.1](image)

**VIII. CONCLUSIONS**

We have considered a basic model of dynamical distribution networks with state inequality constraints. We have formulated a distributed PI controller structure with time-varying flow constraints which achieves consensus and maintains the state constraints. The flow constraints have been expressed in terms of solutions of an optimization problem. We have discussed the existence of solutions for the system in the sense of Filippov, and carried out the stability analysis of the network by taking the Hamiltonian of the system as the Lyapunov function.

The results of this paper can be extended in a straightforward way to the case where the flows on the edges obey a priori constraints; for instance a limitation on the capacity of the pipes in hydraulic networks.

**REFERENCES**


