On the harmonic index and the matching number of a tree*

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Abstract
The harmonic index of a graph $G$ is defined as the sum of weights $\frac{2}{d(u) + d(v)}$ of all edges $uv$ of $G$, where $d(u)$ and $d(v)$ are the degrees of the vertices $u$ and $v$ in $G$, respectively. In this paper, we give a sharp lower bound on the harmonic index of trees with a perfect matching in terms of the number of vertices. A sharp lower bound on the harmonic index of trees with a given size of matching is also obtained.

Key words: Harmonic index; matching number; bound.
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1 Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Its order is $|V(G)|$, denoted by $n$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of vertices which are adjacent to $v$ in $G$ and let $d_G(v)$ (or $d(v)$ for short) be the degree of $v$. Clearly, $d(v) = |N(v)|$. We will use $G - v$ to denote the graph that arises from $G$ by deleting the vertex $v \in V(G)$.

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies. The Randić

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index of a graph $G$ is defined in [10] as the sum of the weights $(d(u)d(v))^{-\frac{1}{2}}$ over all edges $uv$ of $G$. The mathematical properties of this graph invariant have been studied extensively (see recent book [6] and survey [8]). Motivated by the success of Randić index, various generalizations and modifications were introduced, such as the sum-connectivity index [11, 12] and the general sum-connectivity index [2,3].

Another variant of the Randić index, named the harmonic index $H(G)$, which is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)},$$

where the summation goes over all edges $uv$ of $G$. This index was first appeared in [4]. Estimating bounds for $H(G)$ is of great interest, and many results have been obtained. For example, Favaron et al. [5] considered the relationship between the harmonic index and the eigenvalues of graphs; Zhong [13, 14] found the minimum and maximum values of the harmonic index for simple connected graphs, trees and unicyclic graphs, and characterized the corresponding extremal graphs, respectively. Li and Shiu [9] studied how the harmonic index behaves when the graph is under perturbations and provided a simpler method for determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs, respectively. Moreover, the lower bound for harmonic index is also obtained in [9]. Deng et al. [1] studied the relationship between the harmonic index and the chromatic number of a graph $G$, and obtained the lower bound for $H(G)$ in terms of its chromatic number.

In this paper, we consider the relationship between the harmonic index and the matching number of a tree. Lower bounds on the harmonic index of trees with a perfect matching and trees with a given size of matching are obtained, respectively.

## 2 Preliminaries

Two distinct edges in a graph $G$ are called to be independent if they are not adjacent in $G$. A matching of $G$ is a set of mutually independent edges in $G$. The largest matching is called a maximum matching. The cardinality of a maximum matching of $G$ is commonly known as its matching number. Let $M$ be a matching of $G$. $M$ is called the $m$-matching of $G$ if $M$ contains
exactly $m$ edges of $G$. A vertex $v$ of $G$ is said to be $M$-saturated if it is incident with an edge of $M$, otherwise $v$ is called an $M$-unsaturated vertex. The matching $M$ of $G$ is called a perfect matching if all vertices of $G$ are $M$-saturated.

We begin with the following two important results due to Hou and Li [7] for trees with an $m$-matching.

**Lemma 2.1** ([7]) Let $T$ be a tree of order $n (n \geq 3)$ with a perfect matching. Then $T$ has at least two pendant vertices such that each of them is adjacent to a vertex of degree 2.

**Lemma 2.2** ([7]) Let $T$ be a tree of order $n$ with an $m$-matching, where $n > 2m$. Then there is an $m$-matching $M$ and a pendant vertex $v$ such that $v$ is an $M$-unsaturated vertex.

Let $e = uv$ be an edge of a graph $G$. Let $G'$ be the graph obtained from $G$ by contracting the edge $e$ into a new vertex $u_e$ and adding a new pendant edge $u_e v_e$, where $v_e$ is a new pendant vertex. We say that $G'$ is obtained from $G$ by separating an edge $uv$ (see Fig. 1).

![Figure 1: Separating an edge $uv$.](image)

**Lemma 2.3** ([9]) Let $e = uv$ be a cut edge of a connected graph $G$ and suppose that $G - uv = G_1 \cup G_2 ([|V(G_1)|, |V(G_2)| \geq 2])$, where $G_1$ and $G_2$ are two components of $G - uv$, $u \in V(G_1)$ and $v \in V(G_2)$. Let $G'$ be the graph obtained from $G$ by separating the edge $uv$. Then $H(G) > H(G')$.

**Lemma 2.4** (1) For $x \geq 3$, the function $f(x) = \frac{2x - 2}{x + 2} - \frac{2x - 6}{x + 1} - \frac{2}{3}$ is monotonically decreasing on $x$.

(2) For $x \geq 2$, the function $g(x) = \frac{2x - 2}{x + 2} - \frac{2x - 4}{x + 1}$ is monotonically decreasing on $x$. 

3
Proof. (1) We consider the derivative of $f(x)$. For $x \geq 3$, we have
\[
\frac{df(x)}{dx} = \frac{6}{(x+2)^2} - \frac{8}{(x+1)^2} + \frac{2}{x^2}
\]
\[
= \frac{6x^2(x+1)^2 + 2(x+2)^2(x+1)^2 - 8(x+2)^2x^2}{x^2(x+1)^2(x+2)^2}
\]
\[
= \frac{-8x^3 + 24x + 8}{x^2(x+1)^2(x+2)^2} < 0.
\]
Thus $f(x)$ is monotonically decreasing on $x$.

(2) Note that, for $x \geq 2$, we have
\[
\frac{dg(x)}{dx} = \frac{6}{(x+2)^2} - \frac{6}{(x+1)^2} < 0.
\]
Thus $g(x)$ is monotonically decreasing on $x$. □

Lemma 2.5  \( \text{Let } x, y \text{ be positive numbers with } 1 \leq y \leq x - 1. \text{ Let} \)
\[
h(x, y) = \frac{2x - 2y - 2}{x + 2} - \frac{2x - 4y - 4}{x + 1} - \frac{2y}{x}.
\]
Then $h(x, y)$ is monotonically decreasing on $x$ and $y$, respectively.

Proof. We consider some partial derivatives of $h(x, y)$. Note that
\[
\frac{\partial h(x, y)}{\partial y} = \frac{4}{x+1} - \frac{2}{x+2} - \frac{2}{x}
\]
\[
= \frac{-4}{(x+2)(x+1)x} < 0.
\]
Thus, $h(x, y)$ is monotonically decreasing on $y$. On the other hand,
\[
\frac{\partial h(x, y)}{\partial x} = \frac{2y + 6}{(x+2)^2} + \frac{2y}{x^2} - \frac{4y + 6}{(x+1)^2}
\]
\[
= \frac{12x^2y + 24xy + 8y - 12x^3 - 18x^2}{(x+2)^2(x+1)^2x^2}.
\]
Note that $y \leq x - 1$. Then we have
\[
\frac{\partial h(x, y)}{\partial x} \leq \frac{12x^2(x-1)^2 + 24x(x-1) + 8(x-1) - 12x^3 - 18x^2}{(x+2)^2(x+1)^2x^2}
\]
\[
= \frac{-6x^2 - 16x - 8}{(x+2)^2(x+1)^2x^2} < 0.
\]
Thus $h(x, y)$ is monotonically decreasing on $x$. □
3 Main results

Let \( n \) and \( m \) be positive integers and \( n \geq 2m \). Let \( T^0(n, m) \) be a tree of order \( n \), which is obtained from a star \( S_{n-m+1} \) by attaching a pendant edge to each of certain \( m - 1 \) non-central vertices of \( S_{n-m+1} \). Obviously, \( T^0(n, m) \) is a tree of order \( n \) with an \( m \)-matching. Let \( f(n, m) \) be the harmonic index of \( T^0(n, m) \). Then

\[
 f(n, m) = \frac{2(n - 2m + 1)}{n - m + 1} + \frac{2(m - 1)}{n - m + 2} + \frac{2(m - 1)}{3}.
\]

Now, we give the following initial result.

**Theorem 3.1** Let \( T \) be a tree of order \( n = 2m \) with a perfect matching. Then

\[
 H(T) \geq f(2m, m),
\]

the equality holds if and only if \( T \cong T^0(2m, m) \).

**Proof.** We prove the theorem by induction on \( m \). If \( m = 1, 2, 3 \), then the theorem holds clearly by the fact that there are at most two trees with \( n = 2m \) vertices and a perfect matching for \( m = 1, 2, 3 \).

Let \( T \) be a tree of order \( 2m \) with a perfect matching \((m \geq 4)\). Then \( T \) have a perfect matching \( M \). If \( M \) contains a non-pendent edge \( xy \) of \( T \), then let \( \bar{T} \) be the tree obtained from \( T \) by performing separating an edge \( xy \). Then \( \bar{T} \) contains a perfect matching \( \bar{M}_0 = M \cup \{e_0\} \setminus \{xy\} \), where \( e_0 \) is the new edge added into \( T \) after performing separating. Repeat this procedure until there is no non-pendent edge in the most updated perfect matching. Let \( T' \) be the resulting tree and the corresponding perfect matching be \( M' \). By Lemma 2.3, we have \( H(T) \geq H(T') \). Note that the equality does not hold if at least one separating is preformed. Clearly, each edge in \( M' \) is a pendent edge.

By Lemma 2.1, \( T' \) has a pendant vertex \( x_1 \) which is adjacent to a vertex \( x_2 \) of degree 2. Then \( x_1x_2 \in E(T') \) and there is a unique vertex \( x_3 \neq x_1 \) such that \( x_2x_3 \in E(T') \). Let \( T^* = T' - x_1 - x_2 \). Then \( T^* \) is a tree with \( 2(m - 1) \) vertices and with an \( (m - 1) \)-matching. Let \( d(x_3) = d \) and \( N(x_3) \setminus \{x_2\} = \{y_1, y_2, \ldots, y_d\} \), then \( d \geq 3 \). Without loss of generality, we can assume \( d(y_1) = 1 \), \( d(y_i) \geq 2 \) for \( i = 2, \ldots, d - 1 \). By the induction assumption, we have
Moreover, note that $d \leq m$. Then Lemma 2.4(1) implies that

$$\frac{2d - 2}{2 + d} - \frac{2d - 6}{1 + d} = \frac{2m - 2}{2 + m} - \frac{2m - 6}{1 + m} = \frac{2}{m}.$$ \hspace{1cm} \diamondsuit$$

The equality $H(T) = f(2m, m)$ holds if and only if separating is not preformed and equality holds throughout the above inequalities. That is, if and only if $T^* \cong T^0(2(m - 1), m - 1)$, $d(y_1) = 1, d(y_i) = 2$ for $i = 2, \ldots, d - 1$ and $d = m$. Thus $T \cong T^0(2m, m)$.

Another result of the present paper is to give a sharp lower bound on the harmonic index of trees with a given size of matching as follows.
Theorem 3.2 Let $T$ be a tree of order $n$ with an $m$-matching, where $n > 2m$. Then

$$H(T) \geq f(n, m),$$

the equality holds if and only if $T \cong T^0(n, m)$.

Proof. We prove the theorem by induction on $n$. Suppose that $n = 2m$. Then the result follows from Theorem 3.1. Now we suppose that $n > 2m$. Let $T$ be a tree of order $n$ with an $m$-matching. By Lemma 2.2, $T$ has an $m$-matching $M$ and a pendant vertex $v$ which is $M$-unsaturated. Let $uv \in E(T)$ with $d(u) = d$ and $N(u) \setminus \{v\} = \{v_1, v_2, \ldots, v_{d-1}\}$. Obviously, $d \geq 2$. Let $T' = T - v$. Then $T'$ is a tree with $n - 1$ vertices and with an $m$-matching. By the induction assumption, we have

$$H(T) = H(T') + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left( \frac{2}{d + d(v_i)} - \frac{2}{d + d(v_i) - 1} \right)$$

$$\geq f(n-1, m) + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left( \frac{2}{d + d(v_i)} - \frac{2}{d + d(v_i) - 1} \right)$$

$$= f(n, m) + \frac{2(n-2m)}{n-m} - \frac{2(n-2m+1)}{n-m+1} + \frac{2(m-1)}{n-m+1} - \frac{2(m-1)}{n-m+2}$$

$$+ \frac{2}{d+1} + \sum_{i=1}^{d-1} \left( \frac{2}{d + d(v_i)} - \frac{2}{d + d(v_i) - 1} \right).$$

Now, we consider the following two cases.

Case 1. $d(v_i) \geq 2$ for $i = 1, 2, \ldots, d-1$.

In the case, we have

$$H(T) \geq f(n, m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2}$$

$$+ \frac{2}{d+1} + (d-1) \left( \frac{2}{d + 2} - \frac{2}{d + d-1} \right)$$

$$= f(n, m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2}$$

$$+ \frac{2d-2}{d+2} - \frac{2d-4}{d+1}.$$
Moreover, note that $T$ has an $m$-matching and $d \leq n - m$. Then Lemma 2.4(2) implies that

$$\frac{2d - 2}{d + 2} - \frac{2d - 4}{d + 1} \geq \frac{2n - 2m - 2}{n - m + 2} - \frac{2n - 2m - 4}{n - m + 1}.$$ 

That is,

$$H(T) \geq f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} - \frac{2m - 2}{n - m + 2} + \frac{2n - 2m - 2}{n - m + 2} - \frac{2n - 2m - 4}{n - m + 1} = f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} + \frac{2n - 4m}{n - m + 2} = f(n, m) + \frac{2n - 4m}{n - m}(n - m + 1)(n - m + 2) > f(n, m).$$

Case 2. There exists some $i$ ($1 \leq i \leq d - 1$) such that $d(v_i) = 1$. Without loss of generality, we assume that $d(v_1) = d(v_2) = \cdots = d(v_k) = 1$ and $d(v_i) \geq 2$ for $k + 1 \leq i \leq d - 1$, where $k \geq 1$. Then we have

$$H(T) \geq f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} - \frac{2m - 2}{n - m + 2} + \frac{2}{d + 1} + k\left(\frac{2}{d + 1} - \frac{2}{d}\right) + \sum_{i=k+1}^{d-1} \left(\frac{2}{d + d(v_i)} - \frac{2}{d + d(v_i) - 1}\right) \\
\geq f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} - \frac{2m - 2}{n - m + 2} + \frac{2}{d + 1} + \frac{2k}{d + 1} - \frac{2k}{d} + (d - k - 1)\left(\frac{2}{d + 2} - \frac{2}{d + 1}\right) \\
= f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} - \frac{2m - 2}{n - m + 2} + \frac{2d - 2k - 2}{d + 2} - \frac{2d - 4k - 4}{d + 1} - \frac{2k}{d}.
$$

Note that $T$ has an $m$-matching, $k \leq n - 2m$ and $d \leq n - m$. Then Lemma 2.5 implies that

$$\frac{2d - 2k - 2}{d + 2} - \frac{2d - 4k - 4}{d + 1} - \frac{2k}{d} \geq \frac{2n - 2m - 2}{n - m + 2} + \frac{2n - 6m + 4}{n - m + 1} - \frac{2n - 4m}{n - m}.$$
That is, 

\[ H(T) \geq f(n, m) + \frac{2n - 4m}{n - m} - \frac{2n - 6m + 4}{n - m + 1} - \frac{2m - 2}{n - m + 2} \]

\[ + \frac{2n - 2}{n - m + 2} + \frac{2n - 6m + 4}{n - m + 1} - \frac{2n - 4m}{n - m} \]

\[ = f(n, m). \]

The equality \( H(T) = f(n, m) \) holds if and only if equality holds throughout the above inequalities. That is, if and only if \( T' \cong T^0(n - 1, m) \), \( d(v_i) = 1 \) for \( 1 \leq i \leq n - 2m \), \( d(y_i) = 2 \) for \( n - 2m + 1 \leq i \leq d - 1 \) and \( d = n - m \). Thus \( T \cong T^0(n, m) \). \( \square \)

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References


