DEFORMATION OF TRAVELING WAVES
IN DELAYED CELLULAR NEURAL NETWORKS

PEIXUAN WENG*
Department of Mathematics, South China Normal University,
Guangzhou 510631, P.R. China

JIANHONG WU†
Department of Mathematics and Statistics, York University,
Toronto, Ontario, Canada M3J 1P3

Received December 11, 2001 Revised January 18, 2002

In this paper, we establish the existence and describe the global structure of traveling waves for a class of lattice delay differential equations describing cellular neural networks with distributed delayed signal transmission. We describe the transition of wave profiles from monotonicity, damped oscillation, periodicity, unboundedness and back to monotonicity as the wave speed is varied. We also describe an interval of the wave speed where the structure of the wave solution is unknown since the corresponding profile equation involves distributed argument of both advanced and retarded types, and we present some preliminary numerical simulation to illustrate the complexity.

Keywords: CNN; delay; traveling wave; monotone iteration.

1. Introduction

We consider a CY–CNN (Chua–Yang Cellar Neural Network, see [Chua, 1998; Chua & Yang, 1988]) distributed in a one-dimensional integer lattice, where the output of a given cell is fed to its next neighbor with a distributed delay due to, for example, finite switching speed and finite velocity of signal transmission (see [Chua, 1998; Chua & Yang, 1988; Hsu et al., 1999; Hsu & Lin, 2000]). Let \( x_i(t) \), \( i \in \mathbb{Z} \), be the internal state of the cell located at the node \( i \). Then we have

\[
\frac{dx_i(t)}{dt} = -x_i(t) + \alpha f(x_i(t)) + \beta \int_0^\tau k(u)f(x_{i+1}(t-u))du, \quad i \in \mathbb{Z},
\]

where \( \alpha > 0 \) and \( \beta > 0 \) measure the synaptic weights of self-feedback and neighborhood interaction, \( f \) is the usual signal function given by

\[
f(x) = \begin{cases}
1, & x \geq 1, \\
x, & x \in (-1, 1), \\
-1, & x \leq -1,
\end{cases}
\]

and \( k : [0, \tau] \rightarrow [0, \infty) \) is piecewise continuous and satisfies

\[
\int_0^\tau k(u)du = 1.
\]

It is assumed, in the above model, that the self-feedback is instantaneous but neighborhood interaction is delayed. The general situation when the self-feedback is also delayed will be discussed in the final section.

*Research partially supported by Natural Science Foundation of Guangdong Province, P.R. China.
†Research partially supported by Natural Sciences and Engineering Research Council of Canada, by Canada Research Chairs Program, and by MITACS (Network of Centers of Excellence).
Our focus here is a traveling wave solution given by
\[ x_i(t) = \phi(i - ct), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R} \quad (3) \]
for a wave profile \( \phi(s) = \phi(s; c) \), \( s = i - ct \in \mathbb{R} \), with a given wave speed \( c \in \mathbb{R} \). Understanding the structure of such traveling waves is important in a number of applications including image processing, see (Chow et al., 1998; Chua, 1998; Chua & Yang, 1988; Hsu et al., 1999; Hsu, & Lin, 2000; Mallet-Paret 1999; Roska et al., 1990, 1993, 1992; Wu & Zou, 1997, 2001; Zou & Wu, 1998]. Clearly, such a profile must satisfy the functional differential equation
\[ -c\phi'(s) = -\phi(s) + \alpha f(\phi(s)) + \beta \int_0^\tau k(u)f(\phi(s + 1 + cu))du. \quad (4) \]

Assuming the synaptic connection is sufficiently large so that
\[ \alpha + \beta > 1, \quad (5) \]
we find easily that there exist three constant solutions of (4) given by
\[ x^- = - (\alpha + \beta), \quad x^0 = 0, \quad x^+ = \alpha + \beta, \quad (6) \]
respectively. We are interested in those profiles with
\[ \lim_{s \to +\infty} \phi(s) = x^+. \quad (7) \]
In applications, we also need
\[ \phi(s) < 1 \quad \text{for some} \quad s \in \mathbb{R}, \quad (8) \]
for otherwise, the output \( f(\phi(s)) = 1 \) for all \( s \in \mathbb{R} \). Under the conditions (7) and (8), there exists \( s^* \in \mathbb{R} \) so that
\[ \phi(s^*) = 1 \leq \phi(s) \quad \text{for all} \quad s \geq s^*, \quad \text{and} \]
\[ \lim_{s \to +\infty} \phi(s) = \alpha + \beta. \quad (9) \]
Consequently, after a simple translation, we need only consider a profile \( \phi \) satisfying
\[ \phi(0) = 1 \leq \phi(s) \quad \text{as} \quad s \geq 0, \quad \text{and} \]
\[ \lim_{s \to +\infty} \phi(s) = \alpha + \beta. \quad (10) \]
As \( f \) has the particular form given by (2), if \( 1 + \sigma \geq 0 \), then Eq. (4) is a functional differential equation with advanced arguments, and this equation can be explicitly solved to obtain
\[ \phi(s) = \phi(s; c) = (1 - \alpha - \beta) e^{s^*} + \alpha + \beta, \quad s \geq 0. \quad (11) \]
In this paper, we describe completely all possible backward extensions of the above \( \phi \) to \((-\infty, 0]\). In particular, we show that

(I) There exists a \( c_* < -\beta/(1 + \beta \int_0^\tau u k(u)du) < 0 \) such that for every \( c < c_* \), (11) has an extension to \( \mathbb{R} \) which satisfies (4) and
\[ \lim_{s \to -\infty} \phi(s) = x^0, \]
and moreover, \( \phi : \mathbb{R} \to \mathbb{R} \) is monotonically nondecreasing.

(II) If \( 1 \leq \alpha < 1 + \beta \), then there are real numbers \( c_*, c^*, c_p, c_0 \) with \( c_* < c^* \leq c_p \leq c_0 < 0 \) such that

(IIa) if \( c_* \leq c \leq c^* \), then \( |\phi(s)| \leq 1 \) for \( s < s_0 \) with some \( s_0 < 0 \). Furthermore, if \( \bar{c} := \max \{c_*, -1/\tau\} < c^* \) and \( \bar{c} < c^* \leq c_0 \), then \( \phi \) oscillates about zero in the sense that there exists a sequence \( t_n \to -\infty \) as \( n \to \infty \), so that \( \phi(t_n) = 0 \) for \( n = 1, 2, \ldots \);

(IIb) if \( c^* \leq \bar{c} \) and \( \bar{c} < c \leq c_p \), or \( \bar{c} < c^* \) and \( c^* < c < c_p \), then \( \phi \) has an extension to \( \mathbb{R} \) which satisfies (4); \( \phi(s) \) is not eventually monotone in the sense that for any \( s_0 < 0 \), \( \phi([s_0, \infty]) \) is not monotone;

(IIc) if \( \bar{c} < c_p \) and \( c_p \leq c < c^* \), then \( \phi \) has an extension to \( \mathbb{R} \) which satisfies (4) and \( \phi \) is eventually periodic in the sense that there exist \( T > 0 \) and \( s_0 \leq 0 \) such that \( \phi(s + T) = \phi(s) \) for all \( s \leq s_0 - T \); and
\[ \sup_{s < 0} |\phi(s)| > 1; \]

(IId) if \( c = c^* > \bar{c} \), then \( \phi \) has an extension to \( \mathbb{R} \) which satisfies (4), and \( \phi \) is non-decreasing on \( \mathbb{R} \), and there exists some \( s^* < 0 \) that satisfies
\[ \phi(s) \equiv x^- \quad \text{for} \quad s \leq s_*, \]
\[ \phi(s) \in (x^-, x^+) \quad \text{for} \quad s > s_*; \]

(IIe) if \( c^* < c < 0 \), then \( \phi \) has an extension to \( \mathbb{R} \) which satisfies (4), and \( \phi \) is non-decreasing and unbounded on \((-\infty, 0]\).

(III) If \( \alpha \geq 1 + \beta \), then for each \( \bar{c} < c < 0 \), \( \phi \) has an extension to \( \mathbb{R} \) which satisfies (4), and \( \phi(s) \) oscillates about zero and \( |\phi(s)| < 1 \) for \( s < 0 \).

(IV) For every \( c > 0 \), Eq. (4) has a solution \( \phi \) which is monotonically nonincreasing and satisfies
\[ \lim_{s \to -\infty} x(s) = x^+ \quad \text{and} \quad \lim_{s \to -\infty} x(s) = x^0. \]
Regarding (4) as a nonlinear bifurcation prob-
lem with the parameter $c$, the above description gives a relatively complete picture of the deformation of the profile $\phi$. In particular, we notice the transition from monotonicity, damped oscillations, periodic oscillations, unboundedness and back to monotonicity.

The proof of (I) and (IV) will be given in Sec. 2, and the extension to the case where the self-feedback also involves a distributed delay will be given in Sec. 4. The transition properties (II) and (III) will be described in Sec. 3, where we provide numerical evidence where $c^{*} < -1/\tau$ may occur if $\beta$ or $\tau$ is large. This case does not occur for a CNN with discrete delay, for which a complete discussion of the qualitative global structure of traveling waves was obtained in the work [Hsu et al., 1999]. If $c^{*} < -1/\tau$, then there is a gap of $c$ for which our results cannot be applied as the equation for the wave profile involves distributed delayed and advanced arguments simultaneously.

2. Monotone Traveling Waves

Let

$$
\Delta(\sigma, c, x^{0}) = -c\sigma + 1 + \sigma - \beta \int_{0}^{T} k(u) e^{(1+cu)^{\sigma}} du. 
$$

We first consider the equation $\Delta(\sigma, c, x^{0}) = 0$. That is

$$
-c\sigma + 1 - \beta \int_{0}^{T} k(u) e^{(1+cu)^{\sigma}} du = 0. 
$$

(13)

Simple calculations show that

$$
\frac{\partial \Delta}{\partial \sigma}(\sigma, c, x^{0}) = -c - \beta \int_{0}^{T} k(u)(1 + cu)^{\sigma} du, 
$$

$$
\frac{\partial^{2} \Delta}{\partial \sigma^{2}}(\sigma, c, x^{0}) = -\beta \int_{0}^{T} k(u)(1 + cu)^{2} e^{(1+cu)^{\sigma}} du < 0. 
$$

(14)

Lemma 2.1. Assume that $\alpha \geq 1$. There exists a unique pair of $(c_{*}, \sigma_{*})$ with $c_{*} < 0$ and $\sigma_{*} = \sigma(c_{*}) > 0$ such that

(i) $(c_{*}, \sigma_{*})$ are given by the equations

$$
\Delta(\sigma_{*}, c_{*}, x^{0}) = 0, \quad \frac{\partial \Delta}{\partial \sigma}(\sigma_{*}, c_{*}, x^{0}) = 0. 
$$

(15)

(ii) For any $c < c_{*}$, there exist $\sigma = \sigma(c) > 0$ and $\varepsilon_{0} = \varepsilon_{0}(c) > 0$ such that

$$
\Delta(\sigma, c, x^{0}) = 0 \quad \text{and} \quad \Delta(\sigma + \varepsilon_{*}, c, x^{0}) > 0 
$$

for $0 < \varepsilon < \varepsilon_{0}. 
$$

(16)

(iii) For any $c_{*} < c < 0$, Eq. (13) has no real roots $\sigma$.

Proof. Note that $\Delta(0, c, x^{0}) = 1 - \alpha - \beta < 0$, and that $\lim_{\sigma \to -\infty} \Delta(\sigma, c, x^{0}) = +\infty$ for any $\sigma > 0$. By (14), $\Delta = \Delta(\sigma, c, x^{0})$ is a concave function of $\sigma$. Therefore, there is a unique pair of $(c_{*}, \sigma_{*})$ with $c_{*} < 0$ and $\sigma_{*} = \sigma(c_{*}) > 0$ such that the conclusions in (i) and (ii) hold. Moreover, for any $c > c_{*}$, there is no positive root $\sigma$ of (13).

Since $\alpha \geq 1$, we have

$$
\lim_{\sigma \to -\infty} \Delta(\sigma, 0, x^{0}) = 1 - \alpha \leq 0. 
$$

Furthermore, for any fixed $\sigma < 0$, we have

$$
\frac{\partial \Delta}{\partial c} = -\sigma \int_{0}^{T} k(u) e^{(1+cu)^{\sigma}} du > 0. 
$$

Therefore, there is no negative root $\sigma$ of (13) for any $c < 0$. Note also that $\Delta(0, c, x^{0}) < 0$ for any $c < 0$. Therefore, (13) has no real roots for any $c \in (c_{*}, 0)$. This completes the proof. $\blacksquare$

Remark 2.1. Note that

$$
\frac{\partial \Delta}{\partial \sigma}(\sigma, c, x^{0})|_{\sigma=0} = -c - \beta \int_{0}^{T} k(u) du < 0. 
$$

Therefore, the unique $c$ such that $(\partial \Delta/\partial \sigma)(\sigma, c, x^{0})|_{\sigma=0} = 0$ is given by

$$
c_{1} := -\frac{\beta}{1 + \beta \int_{0}^{T} k(u) du}. 
$$

As $(\partial^{2} \Delta/\partial \sigma^{2})(\sigma, c, x^{0}) < 0$, we conclude that $(\partial \Delta/\partial \sigma)(\sigma, c, x^{0}) < 0$ for $c > c_{1}$, consequently $c_{*} < c_{1}$.

We now consider the existence of monotone traveling waves of (1) for $c < c_{*}$. Our approach is based on monotone iteration, coupled with the concept of upper and lower solutions introduced below.

Definition 2.1. A function $U : \mathbb{R} \to \mathbb{R}$ is called an upper solution of (4) if it is differentiable almost everywhere and satisfies

$$
-\alpha U'(s) \geq -U(s) + \alpha f(U(s)) 
$$

$$
+ \beta \int_{0}^{T} k(u) f(U(s + 1 + cu)) du \quad \text{a.e.} 
$$

(17)
Similarly, one can define a lower solution by considering

\[-cL'(s) \leq -L(s) + \alpha f(L(s)) + \beta \int_0^r k(u)f(L(s + 1 + cu))du \text{ a.e.}
\]

We consider the following pair

\[U(s) = \begin{cases} x^+, & s \geq 0, \\ x^+e^{\sigma s}, & s \leq 0, \end{cases} \]

where \(L(s)\) is specified below.

**Lemma 2.2.** For any given \(c < c_*\), define \(\sigma = \sigma(c)\) and \(\varepsilon_0 = \varepsilon_0(c)\) as in (ii) of Lemma 2.1. Fix \(0 < \varepsilon < \varepsilon_0\). Then there exists \(\zeta_0 > 0\) such that

(i) if \(0 < \zeta < \zeta_0\), then \(U(s) \geq L(s) \geq 0\) for \(s \in \mathbb{R}\);

(ii) \(U(s)\) is a upper solution of (4) and \(L(s)\) is a lower solution of (4).

**Proof.** The conclusion (i) is obvious.

If \(s \geq 0\), we have

\[cL'(s) - L(s) + \alpha f(L(s)) + \beta \int_0^r k(u)f(L(s + 1 + cu))du
\]

\[= 0 - 0 + \begin{cases} \beta \int_0^r k(u)(1 - e^{(s+1+cu)})e^{\sigma(s+1+cu)}du & \text{if } 0 \leq \frac{-(s+1)}{c} < \tau, \\ 0 & \text{if } \frac{-(s+1)}{c} \geq \tau, \end{cases}
\]

\[\geq 0. \]

If \(s \leq 0\), we have

\[cL'(s) - L(s) + \alpha f(L(s)) + \beta \int_0^r k(u)f(L(s + 1 + cu))du
\]

\[= c\sigma \zeta e^{\sigma s} - c(\sigma + \varepsilon)\zeta e^{(\varepsilon+\sigma)s} - \zeta(1 - e^{\varepsilon s})e^{\sigma s} + \alpha \zeta(1 - e^{\varepsilon s})e^{\sigma s}
\]

\[+ \begin{cases} \beta \zeta \int_0^r k(u)(1 - e^{(s+1+cu)})e^{\sigma(s+1+cu)}du & \text{if } \frac{-(s+1)}{c} \leq 0, \\ \beta \zeta \int_{\frac{-(s+1)}{c}}^r k(u)(1 - e^{(s+1+cu)})e^{\sigma(s+1+cu)}du & \text{if } \tau > \frac{-(s+1)}{c} \geq 0, \\ 0 & \text{if } \frac{-(s+1)}{c} \geq \tau, \end{cases}
\]

\[= \begin{cases} -\zeta e^{\sigma s} \Delta(\sigma, c, x^0) + \zeta \Delta(\sigma + \varepsilon, c, x^0)e^{(\varepsilon+\sigma)s} > 0 & \text{if } \frac{-(s+1)}{c} \leq 0, \\ -\zeta e^{\sigma s} \Delta(\sigma, c, x^0) + \zeta \Delta(\sigma + \varepsilon, c, x^0)e^{(\varepsilon+\sigma)s} & \text{if } \tau > \frac{-(s+1)}{c} \geq 0, \\ -\zeta e^{\sigma s} \Delta(\sigma, c, x^0) + \zeta \Delta(\sigma + \varepsilon, c, x^0)e^{(\varepsilon+\sigma)s} & \text{if } \frac{-(s+1)}{c} \geq \tau, \end{cases}
\]
If \(-(s + 1)/c \geq 0\), then \(u \in [0, -(s + 1)/c]\) implies that \(s + 1 + cu \geq 0\). Thus \(1 - e^{(s+1+cu)} \leq 0\). For such \(s\), we have

\[
\zeta \Delta (\sigma + \varepsilon, c, x^0)e^{(\varepsilon + \sigma)s} - \beta \zeta \int_t^s k(u)(1 - e^{(s+1+cu)})e^{(s+1+cu)}du > 0,
\]

and

\[
\zeta \Delta (\sigma + \varepsilon, c, x^0)e^{(\varepsilon + \sigma)s} - \beta \zeta \int_0^s k(u)(1 - e^{(s+1+cu)})e^{(s+1+cu)}du > 0.
\]

The discussion shows that \(L(s)\) is a lower solution of (4). Similarly, we can show that \(U(s)\) is an upper solution of (4). This completes the proof. □

Define, for every \(\phi \in C(R, [x^0, x^+])\), the mapping \(H\) by

\[
H(\phi)(s) = \left(\frac{1}{c} + \mu\right)\phi(s) - \frac{\alpha}{c} f(\phi(s)) - \frac{\beta}{c} \int_0^s k(u)f(\phi(s + 1 + cu))du,
\]

where \(\mu > -1/c\) is given so that \(H(\phi_1)(s) \leq H(\phi_2)(s)\) for \(s \in R\) if \(\phi_1(s) \leq \phi_2(s)\) for \(s \in R\).

It is easy to show that (4) is equivalent to

\[
\phi(s) = e^{-\mu s} \int_{-\infty}^{-s} e^{\mu t} H(\phi(t))dt.
\]

We are going to consider (19) subject to the boundary conditions

\[
\lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+.
\]

Let \(C = C(R, [x^0, x^+])\), and

\(S_1 = \{\phi \in C | \phi\) is nondecreasing and satisfies (20)\}\).

We define an operator \(T\) on \(C\) by

\[
(T\phi)(s) = e^{-\mu s} \int_{-\infty}^{-s} e^{\mu t} H(\phi(t))dt, \quad \phi \in C.
\]

Then, by (19) a fixed point of \(T\) is a solution of (4) and vice versa.

We need the following technical lemma, the proof is straightforward.

**Lemma 2.3.** \(T\) has the following properties:

(i) if \(\phi \in S_1\), then \(T\phi \in S_1\);

(ii) \(\phi\) is an upper (a lower) solution of (4) if and only if \(\phi(s) \geq (\leq)(T\phi)(s)\) for \(s \in R\);

(iii) if \(\phi, \tilde{\phi} \in C\) and \(\phi(s) \leq \tilde{\phi}(s)\) for \(s \in R\), then 

\((T\phi)(s) \leq (T\tilde{\phi})(s)\) for \(s \in R\);

(iv) if \(\phi\) is an upper (a lower) solution of (4), then 

\(T\phi\) is also an upper (a lower) solution of (4).

**Theorem 2.1.** There exists a \(c_\ast < -\beta/(1 + \beta \int_0^\tau uk(u)du) < 0\) such that for every \(c < c_\ast\), (1) has a monotone traveling wave solution \(\phi\) satisfying boundary conditions (20), and (1) also has a monotone traveling wave solution \(\phi\) satisfying boundary conditions

\[
\lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^+.
\]

**Proof.** For any positive integer \(n\), define \(U_n(s)\) by

\[
U_n(s) = (T^n U)(s), \quad s \in R,
\]

with \(U_0 = U\). Then using the properties of \(T\) described in Lemma 2.3, we have

\[
x^0 \leq L(s) \leq \cdots \leq U_n(s) \leq \cdots \leq U_1(s) \leq U(s) \leq x^+.
\]

By Lebesgue’s dominated convergence theorem, the limiting function \(U_\ast(s)\) defined by \(U_n(s) = \lim_{n \to \infty} U_n(s)\) exists and is a fixed point of \(T\). Furthermore, \(U_\ast(s)\) is nondecreasing and satisfies (4). Since every \(U_n\) satisfies (20), and \(L(s)\) is a non-trivial lower solution of (4) and \(L \leq U_\ast\). \(U_\ast\) lies in \(S_1\) and thus satisfies (20). The above argument implies the existence of a monotone solution of (4) satisfying (20).

Now noting that \(f\) is an odd function, if we let \(\psi(s) = -\phi(s)\), then (4) is changed to

\[
-c\psi'(s) = -\psi(s) + \alpha f(\psi(s)) + \beta \int_0^s k(u)f(\psi(s + 1 + cu))du,
\]

which is exactly of the same form as (4), thus we know that (4) has a monotone solution satisfying (21). This completes the proof. □

In the rest of this section, we discuss the existence of monotone traveling waves of (1) for \(c > 0\). By the facts

\[
\Delta(0, c, x^0) = 1 - \alpha - \beta < 0,
\]

\[
\frac{\partial \Delta}{\partial \sigma} (\sigma, c, x^0) < 0, \quad \frac{\partial^2 \Delta}{\partial \sigma^2} (\sigma, c, x^0) < 0,
\]

and \(\lim_{\sigma \to -\infty} \Delta(\sigma, c, x^0) = +\infty\) for any fixed \(c > 0\), we know that \(\Delta(\sigma, c, x^0) = 0\) has a unique real root
\( \sigma = \sigma(c) < 0 \) for any \( c > 0 \). Furthermore, there is 
\( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \), one has
\[ \Delta(\sigma - \varepsilon, c, z^0) > 0. \quad (23) \]

We note that if \( \phi \geq 1 \) for large \( |s| \), then (4) becomes
\[ \frac{d\phi}{ds} = \frac{1}{c}(\phi - \alpha - \beta), \quad (24) \]
and
\[ \phi(s) = (\phi(0) - \alpha - \beta)e^{\frac{s}{c}} + \alpha + \beta \quad (25) \]
is the solution of (24) if \( |s| \) is large. We can therefore see from (25) that (24) has no monotone solutions satisfying (20), so we consider monotone solutions with boundary conditions
\[ \lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} \phi(s) = x^0. \]

Define
\[ S_2 = \left\{ \phi \in C | \phi \text{ is nonincreasing,} \right\}, \]
\[ \lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} \phi(s) = x^0 \}
\[ S_2^+ = \left\{ \phi \in C | \phi \text{ is nonincreasing,} \lim_{s \to \infty} \phi(s) = x^0 \right\}. \]

A function \( U \in S_2^+ \) is called an upper solution of (4) with \( c > 0 \) if it is differentiable almost everywhere and satisfies
\[ -cU'(s) \geq -U(s) + \alpha f(U(s)) + \beta \int_0^t k(u)f(U(s + 1 + cu))du \quad \text{a.e.} \quad (26) \]
A lower solution is defined by considering
\[ -cL'(s) \leq -L(s) + \alpha f(L(s)) + \beta \int_0^t k(u)f(L(s + 1 + cu))du \quad \text{a.e.} \quad (27) \]
Let
\[ U(s) = \begin{cases} x^+, & s \leq 0, \\ x^+e^{\sigma s}, & s \geq 0. \end{cases} \]
If \( s \geq 0 \), we have
\[ -U(s) + \alpha f(U(s)) + \beta \int_0^t k(u)f(U(s + 1 + cu))du \leq -x^+e^{\sigma s} + \alpha x^+e^{\sigma s} + \beta x^+ \int_0^t k(u)e^{\sigma(s+1+cu)}du \\
= x^+(-1 + \alpha + \beta \int_0^t k(u)e^{\sigma(1+cu)}du)e^{\sigma s} \\
= -cU'(s). \]
If \( s \leq 0 \), we have
\[ -U(s) + \alpha f(U(s)) + \beta \int_0^t k(u)f(U(s + 1 + cu))du \\
= -x^+ + \alpha + \beta \int_0^t k(u)f(U(s + 1 + cu))du \\
\leq -x^+ + \alpha + \beta = 0 = -cU'(s). \]
Therefore, \( U \) is an upper solution of (4). Let \( 0 < \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 \) defined by (23). Choose \( \zeta > 0 \) so that \( L(s) \leq U(s) \), where
\[ L(s) = \begin{cases} 0, & s \leq 0, \\ \zeta(1 - e^{-\varepsilon_0}e^{\sigma s}), & s \geq 0, \end{cases} \]
where \( \sigma < 0 \) is the solution of (13). We can show that \( L(s) \) is a lower solution of (4), with the argument similar to that for the situation where \( c < 0 \).
Let
\[ F(\phi)(s) = \left( \frac{1}{c} - \mu \right) \phi(s) - \frac{\alpha}{c}f(\phi(s)) + \frac{\beta}{c} \int_0^t k(u)f(\phi(s + 1 + cu))du, \]
where \((1/c) - \mu < 0\) is given so that \( F(\phi_1)(s) \leq F(\phi_2)(s) \) for \( s \in \mathbb{R} \) if \( \phi_1, \phi_2 \in C \) and \( \phi_1(s) \geq \phi_2(s) \) for \( s \in \mathbb{R} \). (4) is equivalent to
\[ \phi(s) = -e^{\mu s} \int_s^\infty e^{-\mu t} F(\phi)(t)dt. \quad (28) \]
If we define the operator \( Q \) on \( S_2^+ \) by
\[ (Q\phi)(s) = -e^{\mu s} \int_s^\infty e^{-\mu t} F(\phi)(t)dt, \quad (29) \]
then a fixed point of \( Q \) is a solution of (4) and vice versa.

We can easily verify the following properties of the operator \( Q \):
(ii) if \( \phi, \tilde{\phi} \in C \) and \( \phi(s) \leq \tilde{\phi}(s) \) for \( s \in \mathbb{R} \), then
\[ (Q\phi)(s) \leq (Q\tilde{\phi})(s) \]
(iii) \( \phi \in S_2^+ \) is an upper (resp. a lower) solution of (4) if and only if \( \phi(s) \geq (\text{resp. } \leq) (Q\phi)(s) \) for \( s \in \mathbb{R} \);
(iv) If \( \phi \in S_2^+ \) is an upper (a lower) solution of (4), then \( Q\phi \in S_2^+ \) is also an upper (a lower) solution of (4).

Thus, we can obtain the existence of a monotone solution in \( S_2 \) of (4) by monotone iteration with \( U_0(s) = U(s) \). In particular, we have the following.

**Theorem 2.2.** For any \( c > 0 \), (1) has a monotone traveling wave solution satisfying
\[
\lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} \phi(s) = x^0,
\]
and a monotone traveling wave solution satisfying
\[
\lim_{s \to -\infty} \phi(s) = x^-; \quad \lim_{s \to \infty} \phi(s) = x^0.
\]

**3. Deformation of Profiles:**

**Oscillation, Periodicity and Monotonicity**

Let \( \bar{c} = \max\{c_*, -1/\tau\} \). For any \( c \in (\bar{c}, 0) \), we have
\[
1 + cu \geq 1 + c\tau =: \gamma(c) > 0 \quad \text{for } u \in [0, \tau].
\]
Thus equation (4) is of advanced type. Assume that \( \phi \) is a solution of (4) with
\[
\lim_{t \to +\infty} \phi(s) = x^+.
\]
For reasons explained in the introduction, we consider the case where \( \phi(0) = 1 \leq \phi(s) \) for \( s \geq 0 \). Therefore
\[
\phi(s) = (1 - \alpha - \beta)e^{\int_0^s c} + \alpha + \beta \quad \text{for } s \geq 0.
\]
We shall consider those solutions of (4) whose restrictions on \([0, \infty)\) satisfy (32).

We start with a lemma from [Gyori & Ladas, 1991] about the following initial value problem of a linear integrodifferential equation:
\[
\begin{cases}
\frac{dx(t)}{dt} + bx(t) + \int_0^\infty K(s)x(t-s)ds = 0 \quad \text{for } t \geq 0, \\
x(t) = \psi(t) \quad \text{for } t \in (-\infty, 0], \\
\psi \in B^+,
\end{cases}
\]
where \( b \in \mathbb{R} \), \( K: \mathbb{R}^+ \to \mathbb{R}^+ \) is piecewise continuous and
\[
0 < \int_0^\infty K(s)e^{-\gamma s}ds < \infty \quad \text{for some } \gamma \in \mathbb{R},
\]

and
\[
B^+ = \{ \psi \in C((-\infty, 0], \mathbb{R}^+) : [0, \infty)
\]
\[
\exists \int_{-\infty}^0 K(t-s)\psi(s)ds \in \mathbb{R} \quad \text{is continuous}\}.
\]

**Lemma 3.1.** [Gyori et al., 1991] The following two statements are equivalent to each other:

(i) There is no \( \psi \in B^+ \) such that the initial value problem (33) has a solution which is positive for \( t \geq 0 \).

(ii) The characteristic equation
\[
\lambda + b + \int_0^\infty K(s)e^{-\lambda s}ds = 0
\]

has no real roots.

**Remark 3.1.** It should be pointed out that the lemma on p. 235 of [Gyori & Ladas, 1991] requires that \( K \) is continuous, but the argument there applies, without any change, to the case where \( K \) is piecewise continuous.

**Lemma 3.2.** Assume that \( \alpha \geq 1 \). If there exists \( s_0 < 0 \) so that \( |\phi(s)| \leq 1 \) for \( s \leq s_0 \), then for any \( s_1 < 0, \phi(s) \) cannot be positive on \((-\infty, s_1]\) nor negative on \((-\infty, s_1]\) (in this sense, we say that \( \phi \) oscillates about 0 on \((-\infty, s_1]\).

**Proof.** Note that if \( |\phi(s)| \leq 1 \) for all \( s \leq s_0 \), we have
\[
\frac{d\phi(s)}{ds} + \frac{\alpha-1}{c} \phi(s) + \frac{\beta}{c} \int_0^\tau k(u)\phi(s+1+cu)du = 0
\]
for \( s \leq s_0 - 1 \).

This, with a change of variable \( \psi(s) = \phi(-s) \), can then be written as
\[
\frac{d\psi(s)}{ds} + \frac{1-\alpha}{c} \psi(s) - \frac{\beta}{c} \int_0^\tau k(u)\psi(s-1-cu)du = 0
\]
for \( s \geq -s_0 + 1 \).

That is,
\[
\frac{d\psi(s)}{ds} + \frac{1-\alpha}{c} \psi(s) + \frac{\beta}{c^2} \int_1^1 k\left(\frac{v-1}{c}\right)\psi(s-v)dv = 0
\]
for \( s \geq -s_0 + 1 \).
By way of contradiction, assume that there exists $s_1 < 0$ such that $\phi(s) > 0$ for $s \leq s_1$. Then $\psi(s) < 0$ for $s \geq -s_1$. Let

$$T^* = \max\{-s_1, -s_0 + 1\} + 1,$$

and $\chi(s) = \psi(s + T^*)$ for $s \geq -1$. Extend $\chi$ to $(-\infty, -1)$ so that $\chi([-\infty, 0] \in B^+$. Then we have

$$\frac{d\chi(s)}{ds} = \frac{1 - \alpha}{c} \chi(s) + \int_0^\infty K(v)\chi(s - v)dv = 0 \text{ for } s \geq 0,$$

where $K : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$K(v) = \begin{cases}
0, & v \in [0, \infty)/[1 + ct, 1], \\
\frac{\beta}{c^2} k \left(\frac{v - 1}{c}\right), & v \in [1 + ct, 1].
\end{cases}$$

(39)

Clearly, $K$ is piecewise continuous and satisfies condition (34). The associated characteristic equation (35) of (39) is $\Delta(\sigma, c, x^0) = 0$. By Lemma 2.1, this characteristic equation has no real roots if $\alpha \geq 1$ and if $c \in (c_\alpha, 0)$. Thus we obtain from Lemma 3.1 the conclusion. 

**Lemma 3.3.** Assume that $\alpha \geq 1$. If there exists $s_0 < -1$ such that $|\phi(s)| < x^+$ for $s < s_0$, then for any $s_1 < 0$, $\phi$ is not monotone on $(-\infty, s_1)$.

**Proof.** Assume, by way of contradiction, that there exists $s_1 < 0$ such that $\phi$ is monotone on $(-\infty, s_1)$. Then we have

$$\lim_{s \to -\infty} \phi(s) = \phi_{-\infty} \in [x^-, x^+], \quad \lim_{s \to -\infty} \phi'(s) = 0.$$ 

(41)

If $\phi_{-\infty} \in [1, x^+]$ and $\phi$ is monotonically nondecreasing, then if $s << -1$, $\phi(s)$ satisfies

$$-c\phi'(s) = -\phi(s) + \alpha + \beta.$$ 

(42)

This generates a contradiction to (41). Similarly, it is impossible that $\phi$ is monotonically nonincreasing and $\phi_{-\infty} \in [1, x^+]$. By a similar argument, (4) does not have a monotone solution with (41) and $\phi_{-\infty} \in (x^-, -1]$. Also, by Lemma 3.2, (4) does not have a monotone solution with $\phi_{-\infty} \in (-1, 1)$. This completes the proof. 

We also need a few technical lemmas (Lemmas 3.4–3.7). These, and our arguments, are similar to the corresponding results in [Hsu et al., 1999].

**Lemma 3.4.** If there is $s_0 < 0$ such that $1 \leq \phi(s) < x^+$ or $x^- < \phi(s) \leq -1$ for $s \in [s_0 - 1, s_0]$, then $\phi$ is not monotone for $s \in [s_0 - 1, 0]$ and $\phi$ is periodic for $s < 0$ (that is, $\phi(s + T) = \phi(s)$ for some $T > 0$ and for all $s < -T$).

**Proof.** We only consider the case where there exists $s_0 < 0$ such that $1 \leq \phi(s) < x^+$ for $s \in [s_0 - 1, s_0]$, the other case is similar. Since $\phi(0) = 1$ and $\phi'(0) = \frac{1}{c}(1 - \alpha - \beta) > 0$, we have $\phi(-s) < 1$ for $0 < s << 1$. Hence, $\phi(s)$ is not monotone for $s \in [s_0 - 1, 0]$.

Note that

$$\phi'(s) = \frac{1}{c}[\phi(s) - \alpha - \beta] > 0.$$ 

Hence, for any $s_1 < s_0 - 1$ such that $\phi(s) \geq 1$ for $s \in [s_1, s_0 - 1]$, we have

$$\phi'(s) \geq \frac{1}{c}[\phi(s) - \alpha - \beta] > 0.$$ 

This and Lemma 3.1 imply that there is a $s_2 \leq s_0 - 1$ such that $\phi(s_2) = 1$ and $1 \leq \phi(s) < x^+$ for $s \in [s_2, s_0]$.

We now show that $\phi(s) = \phi(s - s_2)$ for $s < s_2$. Let $\psi(s) = \phi(s - s_2)$. Then

$$\psi(s_2) = \phi(s_2),$$ 

$$f(\psi(s + 1 + cu)) = f(\phi(s + 1 + cu)) = 1,$$ 

(43)

for $s \in [s_2 - \gamma, s_2], u \in [0, \tau],$

where $\gamma = 1 + ct$. Note that $\phi$ and $\psi$ satisfy

$$\phi'(s) = \frac{1}{c} \left[\phi(s) - \alpha f(\phi(s)) - \beta \int_0^s k(u)f(\phi(s + 1 + cu))du\right],$$ 

(44)

and

$$\psi'(s) = \frac{1}{c} \left[\psi(s) - \alpha f(\psi(s)) - \beta \int_0^s k(u)f(\psi(s + 1 + cu))du\right],$$ 

(45)

respectively. Therefore, the uniqueness of the Cauchy initial value problem for the ODE

$$x'(s) = \frac{1}{c} \left[x(s) - \alpha f(x(s)) - \beta \int_0^s k(u)f(x(s + 1 + cu))du\right]$$
on $[s_2 - \gamma, s_2]$ ensures that
\[ \phi(s) = \psi(s) \quad \text{for } s \in [s_2 - \gamma, s_2]. \] (46)

Now by (46) and (43), we obtain
\[ \psi(s_2 - \gamma) = \phi(s_2 - \gamma), \]
\[ f(\psi(s + 1 + cu)) = f(\phi(s + 1 + cu)), \] (47)
for $s \in [s_2 - 2\gamma, s_2 - \gamma], u \in [0, \tau].$

Therefore, using the same argument as above, we have
\[ \phi(s) = \psi(s) \quad \text{for } s \in [s_2 - 2\gamma, s_2 - \gamma]. \]

Continuing the above process, we have
\[ \phi(s) = \psi(s) = \phi(s - s_2) \quad \text{for } s \leq s_2. \]

Therefore, $\phi(s)$ is periodic with a period $\omega = -s_2$ for $s < 0$. This completes the proof.  

**Lemma 3.5.** Suppose that there is $s_* < 0$ such that $\phi(s_*) = x^-$ and $\phi(s) > x^-$ and is monotone for $s > s_*$. Then $s_* \geq s^*-1$, where $s^*>s_*$ is such that $\phi(s^*) = -1$. Moreover, in case where $s_* = s^*-1$, we have $\phi(s) = x^-$ for $s < s_*$, and in case where $s_* > s^*-1$, we have that $\phi(s)$ is monotone and unbounded on $(-\infty, 0]$.

**Proof.** Note that $x^- < \phi(s) \leq -1$ for $s_* < s < s^*$. Clearly, $s_* < s^* - 1$ is impossible, otherwise, Lemma 3.4 implies that $\phi$ is not monotone on $(s_*, 0)$. Thus $s_* \geq s^* - 1$.

If $s_* > s^* - 1$, then
\[ \phi'(s_*) = \frac{1}{c} \left[ -x^- - \alpha + \beta \int_0^\tau k(u)f(\phi(s_*)+1+cu))du \right] \]
\[ > \frac{1}{c}(-x^--\alpha-\beta) = 0. \]

Therefore, $\phi(s) < x^-$ for $s < s_*$ and close to $s_*$. Let $\bar{s} = \inf\{Q < s_* : \phi(s) < x^- \text{ for } s \in (Q, s_*)\}.$

Then for each $s \in (\bar{s}, s_*), we have
\[ \phi'(s) = \frac{1}{c} \left[ -\phi(s) - \alpha \right. \]
\[ + \beta \int_0^\tau k(u)f(\phi(s+1+cu))du \]
\[ > \frac{1}{c}[\phi(s) + \alpha + \beta] \geq \frac{1}{c}(x^- + \alpha + \beta) = 0. \]

From this, it follows that $\bar{s} = -\infty$ and $\phi$ is monotonically nondecreasing and unbounded on $(-\infty, s_*)$.

If $s_* = s^* - 1$, then for $s \in [s_* - \gamma(c), s_*]$, we have
\[ \phi'(s) = -\frac{1}{c} \left[ -\phi(s) - \alpha f(\phi(s)) \right. \]
\[ + \beta \int_0^\tau k(u)f(\phi(s+1+cu))du \]
\[ = \frac{1}{c}[\phi(s) + \alpha f(\phi(s)) + \beta]. \]

Moreover, $\phi(s_*) = x^-$. Therefore, the uniqueness implies that $\phi(s) = x^-$ for $s \in [s_* - \gamma(c), s_*]$. Repeating this argument on $[s_* - n\gamma(c), s_* - (n - 1)\gamma(c)]$ for $n = 1, 2, \ldots$, we conclude that $\phi(s) = x^-$ for $s \leq s_*$. This completes the proof. ■

**Lemma 3.6.** If there is $s_0 < 0$ such that $\phi(s)$ is not monotone for $s_0 < s < 0$. Then $\phi(s) \in (x^-, x^+)$ for all $s \leq 0$.

**Proof.** We have shown that $\phi'(0) > 0$. Therefore, $\phi(s)$ is increasing on $[-\delta, 0]$ with some small $\delta > 0$. Let $s_1 \in (s_0, -\delta)$ so that $\phi(s)$ is increasing on $[s_1, 0]$, but is not monotone on $[s_1 - \sigma, s_1 + \sigma]$ with any $\sigma > 0$.

We claim that $x^- < \phi(s_1) < 0$. In fact, if $\phi(s_1) \geq 0$, clearly, $\phi'(s_1) = 0$. Then
\[ 0 = \phi'(s_1) \]
\[ = \frac{1}{c} \left[ (1-\alpha)\phi(s_1) - \beta \int_0^\tau k(u)f(\phi(s_1+1+cu))du \right] \]
\[ > \frac{1}{c}[1-\alpha)\phi(s_1) - \beta \phi(s_1)] \geq 0, \]
a contradiction. If $\phi(s_1) \leq x^-$, then there exist $s_s$ and $s^*$ such that $s_* \leq s_s < s^*$ such that $\phi(s_*) = x^-$, $\phi(s^*) = -1$ and $\phi(s) > x^-$ for $s > s_*$. By Lemma 3.5, $\phi(s)$ is monotone for $s < 0$, a contradiction. Therefore, we must have $x^- < \phi(s_1) < 0$.

It then follows that
\[ x^- < \phi(s_1) \leq \phi(s) < 1 = \phi(0) \quad \text{for } s_1 \leq s < 0. \] (48)

Next we shall show that
\[ \phi(s_1) \leq \phi(s) \leq -\phi(s_1) \quad \text{for } s < s_1. \] (49)

Let $\hat{s}_1 > \hat{s}_2 > s_1 > s_2$ be such that
\[ \phi(\hat{s}_1) = -\phi(s_1), \quad \phi(\hat{s}_2) = \phi(s_2) > \phi(s_1) \]
and \( \phi(s) \) is decreasing for \( s_2 < s < s_1 \). Hence
\[
\phi(s_1) \leq \phi(s) \leq -\phi(s_1) \text{ for } s_2 \leq s < s_1.
\]  
(50)  

Let \( \bar{s} = \min\{1 + cr, \bar{s}_1 - s_1, \bar{s}_2 - s_2\} \), and define
\[
\psi(s) = -\phi(s) \text{ for } s \in \mathbb{R}.
\]

Then we have
\[
\phi(s) \geq \psi(s + \bar{s}_1 - s_1) \text{ for } s \geq s_1
\]
(51)  
and
\[
\phi(s) \leq \phi(s + \bar{s}_2 - s_2) \text{ for } s \geq s_2.
\]
(52)  

Therefore, using
\[
\eta_1(s_1) = \eta_2(s_1), \quad \eta_1(s_2) = \eta_3(s_2)
\]

and by the well-known comparison theorem of ODEs, we obtain
\[
\begin{cases}
\eta_1(s) = \phi(s) \leq \eta_2(s) = \psi(s + \bar{s}_1 - s_1) \leq -\phi(s_1) \quad \text{for } s_1 - \bar{s} \leq s \leq s_1, \\
\eta_3(s) = \phi(s) \geq \phi(s + \bar{s}_2 - s_2) \geq \phi(s_1) \quad \text{for } s_2 - \bar{s} \leq s \leq s_2.
\end{cases}
\]
(54)  

By (54), there are \( \bar{s}_3 \in [\bar{s}_1 - \bar{s}, \infty) \) and \( \bar{s}_4 \in [\bar{s}_2 - \bar{s}, \infty) \) such that
\[
\phi(s_1 - \bar{s}) = \psi(\bar{s}_3), \quad \phi(s_2 - \bar{s}) = \phi(\bar{s}_4).
\]

We claim
\[
\begin{cases}
\phi(s) \geq \psi(s + \bar{s}_3 - s_1 + \bar{s}) \quad \text{for } s \geq s_1 - \bar{s}, \\
\phi(s) \leq \phi(s + \bar{s}_4 - s_1 + \bar{s}) \quad \text{for } s \geq s_2 - \bar{s}
\end{cases}
\]
(55)  

and
\[
\begin{cases}
\phi(s) \leq \psi(s + \bar{s}_3 - s_1 + \bar{s}) \leq -\phi(s_1) \quad \text{for } s_1 - 2\bar{s} \leq s \leq s_1 - \bar{s}, \\
\phi(s) \geq \phi(s + \bar{s}_4 - s_2 + \bar{s}) \geq \phi(s_1) \quad \text{for } s_2 - 2\bar{s} \leq s \leq s_2 - \bar{s}.
\end{cases}
\]
(56)  

In fact, letting
\[
\eta_4(s) = \psi(s + \bar{s}_3 - s_1 + \bar{s}), \quad \eta_5(s) = \phi(s + \bar{s}_4 - s_2 + \bar{s}),
\]

then we have
\[
\eta_i(s) = \frac{1}{c} \left[ \eta_i(s) - \alpha f(\eta_i(s)) - \beta \int_0^\tau k(u) f(\eta_i(s + 1 + cu)) du \right], \quad i = 4, 5.
\]
(57)  

Note that
\[
\eta_4(s_1 - \bar{s}) = \psi(\bar{s}_3) = \phi(s_1 - \bar{s}), \quad \eta_5(s_2 - \bar{s}) = \phi(\bar{s}_4) = \phi(s_2 - \bar{s}).
\]

Since \( \bar{s}_3 > \bar{s}_1 - \bar{s}, \bar{s}_4 > \bar{s}_2 - \bar{s} \), by (51) and (52), we have
\[
\begin{cases}
-\frac{\beta}{c} f(\eta_1(s + 1 + cu)) \geq -\frac{\beta}{c} f(\eta_4(s + 1 + cu)) \quad \text{for } s \geq s_1 - \bar{s}, u \in [0, \tau], \\
-\frac{\beta}{c} f(\eta_1(s + 1 + cu)) \leq -\frac{\beta}{c} f(\eta_5(s + 1 + cu)) \quad \text{for } s \geq s_2 - \bar{s}, u \in [0, \tau].
\end{cases}
\]
(58)
Therefore, (55) follows again from the well-known comparison theorem of ODEs. Similarly, we can obtain (56).

Continuing the above procedure, we have
\[
\begin{cases}
\phi(s) \leq -\phi(s_1) & \text{for } s \leq s_1, \\
\phi(s) \geq \phi(s_1) & \text{for } s \leq s_2.
\end{cases}
\]

(59)

This, together with (48), proves (49). The conclusion of the lemma then follows from (48) and (49).

Remark 3.2. Assume that \( \alpha \geq 1 \). We conclude from Lemmas 3.3 and 3.6 that the following statements are equivalent:

(i) there exists some \( s_0 < 0 \) such that \( \phi(s) \) is not monotone on \((s_0, 0)\);

(ii) \( |\phi(s)| < x^+ \) for \( s < 0 \);

(iii) for any \( s_0 < 0 \), \( \phi(s) \) is not monotone on \((-\infty, s_0)\);

(iv) \( \phi(s) \) is not monotone on \((-\infty, 0)\).

Corollary 3.1. We have the following conclusions.

(i) If \( \phi(s_*) = x^- \) for some \( s_* < 0 \), then \( \phi \) is monotone on \((-\infty, 0)\).

(ii) \( \lim_{s \to -\infty} \phi(s) = x^- \) if and only if there is some \( s_* < 0 \) such that \( \phi(s) = x^- \) for \( s \leq s_* \) and \( \phi \) is nondecreasing on \((-\infty, 0)\).

Proof. (i) is an immediate consequence of Remark 3.2. If \( \lim_{s \to -\infty} \phi(s) = x^- \), then by Lemma 3.4, there exists \( s_* < 0 \) such that \( \phi(s_*) = x^- \). By (i), \( \phi \) is monotone and hence \( \phi(s) = x^- \) for \( s \leq s_* \). This proves (ii). \( \blacksquare \)

Lemma 3.7. Suppose that \( 1 \leq \alpha < 1 + \beta \). If there is \( c^* \in (c, 0) \) such that \( \lim_{s \to -\infty} \phi(s; c^*) = x^- \), then for any \( c \in (c, c^*) \) we have that \( x^- < \phi(s; c) < x^+ \) for \( s < 0 \), and for \( c \in (c^*, 0) \) we have \( \phi(s; c) \) is nondecreasing and unbounded on \((-\infty, 0)\).

Proof. By Corollary 3.1, \( \phi(s; c^*) \) is nondecreasing and there is \( s_* < 0 \) such that \( \phi(s; c^*) = x^- \) for \( s \leq s_* \).

Assume now \( c \in (c^*, 0) \) is given. We want to show that \( \phi(s) = \phi(s; c) \) is unbounded. By Lemma 3.5 and Corollary 3.1, it is sufficient to prove that \( \phi(s) < x^- \) for some \( s < 0 \). First of all, we note that
\[
\phi(s; c) > \phi(s; c^*) \text{ for } s > 0 =: s_0.
\]

We claim
\[
\phi(s; c) < \phi(s; c^*) \text{ for } s \in [-\gamma, 0),
\]
where \( \gamma = 1 + c^* \tau \).

To verify the claim, we first note
\[
\phi(0; c) = \phi(0; c^*) = 1, \quad \phi'(0; c) > \phi(0; c^*).
\]

If (61) fails, then there exists \( \tilde{s} \in (-\gamma, s_0) \) so that
\[
\phi(s; c) < \phi(s; c^*) \text{ for } s \in (\tilde{s}, s_0)
\]
and \( \phi(\tilde{s}; c) = \phi(\tilde{s}; c^*) \).

Therefore,
\[
\phi'(\tilde{s}; c) \leq \phi'(\tilde{s}; c^*). \tag{62}
\]

Since \( f(\phi(\tilde{s} + 1 + cu; c)) = f(\phi(\tilde{s} + 1 + cu; c^*)) = 1 \) and \( 1 < \alpha < 1 + \beta \), we have
\[
\phi(\tilde{s}; c) - \alpha f(\phi(\tilde{s}; c)) - \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + cu; c))du = \phi(\tilde{s}; c^*) - \alpha f(\phi(\tilde{s}; c^*))
\]
\[
- \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + c^* u; c^*))du.
\]

Note that
\[
\phi(\tilde{s}; c^*) - \alpha f(\phi(\tilde{s}; c^*))
\]
\[
- \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + c^* u; c^*))du
\]
\[
\leq 1 + \alpha - \beta < 0 \text{ if } \phi(\tilde{s}; c^*) \leq -1, \tag{63}
\]
and
\[
\phi(\tilde{s}; c^*) - \alpha f(\phi(\tilde{s}; c^*))
\]
\[
- \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + c^* u; c^*))du
\]
\[
= (1 - \alpha)\phi(\tilde{s}; c^*) - \beta
\]
\[
\leq (1 - \alpha)(-1) - \beta < 0 \text{ if } \phi(\tilde{s}; c^*) \in (-1, 1). \tag{64}
\]

Thus, we have
\[
\phi'(\tilde{s}; c) = \frac{1}{c} \left[ \phi(\tilde{s}; c) - \alpha f(\phi(\tilde{s}; c)) \right.
\]
\[
- \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + cu; c))du \left. \right] > \frac{1}{c^*} \left[ \phi(\tilde{s}; c^*) - \alpha f(\phi(\tilde{s}; c^*)) \right.
\]
\[
- \beta \int_0^\tau k(u)f(\phi(\tilde{s} + 1 + c^* u; c^*))du \right] = \phi'(\tilde{s}; c^*),
\]
a contradiction to (62). Therefore, (61) holds.

If \( \phi(s_0 - \gamma; c^*) \leq x^- \), then \( \phi(s_0 - \gamma; c) < x^- \) and the conclusion of the Lemma follows. If \( \phi(s_0 - \gamma; c^*) > x^- \), let \( s_1 = s_0 - \gamma \) and we can find \( \bar{s}_1 \in (s_1, s_0) \) such that \( \phi(\bar{s}_1; c) = \phi(s_1; c^*) \). We shall show

\[
\phi'(\bar{s}_1; c) > \phi'(s_1; c^*), \tag{65}
\]

\[
\phi(s - s_1 + \bar{s}_1; c) > \phi(s; c^*) \quad \text{for } s > s_1, \tag{66}
\]

\[
\phi(s - s_1 + \bar{s}_1; c) < \phi(s; c^*) \quad \text{for } s_1 - \gamma \leq s < s_1. \tag{67}
\]

Since \( s_0 = 0 \), we have \( f(\phi(s_1 + 1 + cu; c^*) = 1 \) for \( u \in [0, \tau] \). Then by an argument similar to that in (63) and (64) and using (60), we have

\[
\phi(\bar{s}_1; c) - \alpha \int_0^r k(u)f(\phi(s_1 + 1 + cu; c))du \\
\leq \phi(s_1; c^*) - \alpha \int_0^r k(u)f(\phi(s_1; c^*)) \\
- \beta \int_0^r k(u)f(\phi(s_1 + 1 + cu; c^*))du \\
\leq \phi(s_1; c^*) - \alpha \int_0^r k(u)f(\phi(s_1; c^*)) \\
- \beta \int_0^r k(u)f(\phi(s_1 + 1 + cu; c^*))du \\
< 0, \tag{68}
\]

which implies (65).

We now verify (66). If (66) does not hold, then by (65) there is \( \tilde{s} > s_1 \) such that

\[
\phi(s - s_1 + \tilde{s}_1; c) > \phi(s; c^*) \quad \text{for } s_1 < s < \tilde{s},
\]

\[
\phi(\tilde{s}_1 - s_1 + \bar{s}_1; c) = \phi(\tilde{s}_1; c^*). \]

Therefore,

\[
\phi'(\tilde{s} - s_1 + \bar{s}_1; c) > \phi'(\tilde{s}; c^*). \tag{69}
\]

But by an argument similar to that for (65), we have

\[
\phi(\tilde{s} - s_1 + \bar{s}_1; c) - \alpha \int_0^r k(u)f(\phi(s_1 + 1 + cu; c)) \\
- \beta \int_0^r k(u)f(\phi(s_1 + 1 + cu; c)) < 0,
\]

and thus

\[
\phi'(\tilde{s} - s_1 + \bar{s}_1; c) > \phi'(\tilde{s}; c^*). \]

This is a contradiction to (69). Thus (66) holds. Similar argument and (66) leads to (67).

Again, if \( \phi(s_1 - \gamma; c^*) \leq x^- \), then \( \phi(s_1 - \gamma; c) < x^- \) and the conclusion of the Lemma follows. If

\[
\phi(s_1 - \gamma; c^*) > x^- \), we can find \( \tilde{s}_2 \in (s_2, s_1) \) with \( s_2 = s_1 - \gamma \) such that \( \phi(\tilde{s}_2 - s_1 + \tilde{s}_1; c) = \phi(\tilde{s}_2; c^*) \). Using an argument similar to that above, we obtain

\[
\phi'(\tilde{s}_2 - s_1 + \tilde{s}_1; c) > \phi'(s_2; c^*) \tag{70},
\]

\[
\phi(s - s_2 + \tilde{s}_2 - s_1 + \tilde{s}_1; c) > \phi(s; c^*) \tag{71}
\]

for \( s > s_2 \),

\[
\phi(s - s_2 + \tilde{s}_2 - s_1 + \tilde{s}_1; c) < \phi(s; c^*) \tag{72}
\]

for \( s_2 - \gamma \leq s < s_2 \).

Continuing the procedure, we obtain that there is \( \tilde{s} \) such that \( \phi(\tilde{s}; c) < x^- = \phi(s_*, c^*) \). Therefore, the conclusion of the lemma regarding \( c > c^* \) follows.

Now we consider the case where \( \tilde{c} < c < c^* \). If there is \( \tilde{s} \) such that \( \phi(\tilde{s}; c) = x^- \), then Corollary 3.1 implies that \( \phi(s; c) \) is monotone on \((-\infty, 0)\). By an argument similar to that above for \( c > c^* \), we can find \( \tilde{s}_* \) such that \( \phi(\tilde{s}_*; c^*) < x^- \), a contradiction. Therefore, \( \phi(s; c) > x^- \) for \( s < 0 \). We can then apply the argument of Lemma 3.6 to show \( x^- < \phi(s; c) < x^+ \) for \( s < 0 \). This completes the proof. \( \blacksquare \)

**Corollary 3.2.** Suppose that \( 1 \leq \alpha < 1 + \beta \), then there is at most one \( c^* \in (\tilde{c}, 0) \) such that \( \lim_{s \to -\infty} \phi(s) = x^- \).

**Proof.** This is an immediate consequence of Corollary 3.1 and Lemma 3.7. \( \blacksquare \)

Now we start to construct monotone and unbounded solutions of (13) for certain \( c \).

**Lemma 3.8.** Assume that \( \alpha < 1 + \beta \). Then there is \( c_0 < 0 \) such that \( \phi(s) = \phi(s; c) \) satisfies

\[
\phi(s^*; c) = -1, \tag{73}
\]

\[
\phi(s; c) > -1 \quad \text{is monotone for } s > s^* \]

with some \( s^* \geq -(1 + \alpha \tau) = -\gamma \) if and only if \( c \geq c_0 \), where

\[
c_0' = \frac{c_0}{1 - c_0 \tau}, \tag{74}
\]

\[
c_0 = \begin{cases} 
\frac{(\alpha - 1) \left\{ \ln \frac{1 - \alpha + \beta}{\alpha + \beta - 1} \right\}^{-1}}{-\frac{\beta}{2}} & \text{as } \alpha \neq 1, \\
\frac{(\alpha - 1) \left\{ \ln \frac{1}{\alpha + \beta - 1} \right\}^{-1}}{-\frac{\beta}{2}} & \text{as } \alpha = 1.
\end{cases}
\]
Lemma 3.9. Assume that $\alpha < 1 + \beta$. Let
\[
\tilde{c} = \left\{ \left[ \frac{1}{c_0} - \ln \frac{2\beta}{1 + \beta - \alpha} \right] - \tau \right\}^{-1}.
\]
Then $\phi(-\gamma; \tilde{c}) = x^-$, where $\gamma = 1 + \tilde{c}\tau$.

Proof. For any $c_0' \leq c < 0$, by Lemma 3.8, there is $s^* \geq -\gamma = -\gamma(c)$ satisfying (73). Therefore, for such a $c$, if $\phi$ is nondecreasing for $s \in [-\gamma, s^*]$, we have
\[
\phi'(s) = \frac{1}{c}(\phi(s) + \alpha - \beta), \quad s \in [-\gamma, s^*].
\]
It then follows that
\[
\phi(s) = (\alpha - 1 - \beta)e^{\frac{1}{c}(s - s^*)} - \alpha + \beta \quad \text{for} \quad s \in [-\gamma, s^*].
\]
Let $\phi(-\gamma) = x^-$. Then we have
\[
\frac{2\beta}{1 + \beta - \alpha} = e^{\frac{1}{c}(-\gamma - s^*)}.
\]

Thus
\[
\ln \frac{2\beta}{1 + \beta - \alpha} = - \frac{1}{c}(\gamma + s^*) = - \frac{1 + ct + s^*}{c}.
\]

Noting that $s^* = -c/c_0$ as $\alpha \neq 1$, we obtain from (76) that
\[
\ln \frac{2\beta}{1 + \beta - \alpha} = - \frac{1 + ct + s^*}{c} + \frac{1}{c_0}.
\]

Let $B := 1/c_0 - \ln 2\beta/(1 + \beta - \alpha)$. Then $Bc = 1 + ct$ and thus, $\tilde{c} = 1/(B - \tau)$. It is obvious that $\phi(-\gamma(\tilde{c}); \tilde{c}) = x^-$.

If $\alpha = 1$, then $s^* = 2c/\beta$. Thus
\[
\ln \frac{2\beta}{1 + \beta - \alpha} = - \frac{1 + ct}{c} - \frac{2}{\beta}.
\]

Let
\[
B := - \frac{2}{\beta} - \ln \frac{2\beta}{1 + \beta - \alpha} = \frac{1}{c_0} - \ln \frac{2\beta}{1 + \beta - \alpha}.
\]

We can then reach the same conclusion as in the case $\alpha \neq 1$. This completes the proof. ■

It is obvious that $\tilde{c} > -1/\tau$. On the other hand, according to the definition of $\tilde{c}$, we must have $\tilde{c} > c_\ast$. Thus $\tilde{c} > \tilde{c}$. Assume that $1 \leq \alpha < 1 + \beta$. Then Lemma 3.9 implies that there is $c > \tilde{c}$ such that (4) has an unbounded traveling solution. Now let
\[
C^\circ = \{c \mid \phi(s; c) < 1 \quad \text{for} \quad s < -1\}. \tag{78}
\]

Because $c \leq c_\ast$ implies $c \in C^\circ$, we have $C^\circ \neq \emptyset$. Let
\[
c^\circ = \sup\{c \mid c \in C^\circ\}.
\]

Then $c_\ast \leq c^\circ$, $|\phi(s; c) | \leq 1$ for $s < -1$.

Define
\[
C = \{c_1 \mid \text{for any } c_1 < c < 0, \phi(s; c) \text{ is monotone and unbounded}\}.
\]

Then by Lemmas 3.7 and 3.9, we have $C \neq \emptyset$. Define
\[
c^\ast = \inf\{c \mid c \in C\}. \tag{79}
\]

Then we have $c^\circ \leq c^\ast < 0$ and $\phi(s; c^\ast)$ is bounded and monotone on $(-\infty, 0)$. There exist three cases:

1. $c^\circ = c^\ast$. For this case, we define $c_p = c^\ast = c^\circ$.
2. $c_\ast < c^\ast \leq \tilde{c}$. For this case, we define $c_p = c^\circ$.
3. $c_\ast \leq \tilde{c} < c^\ast$. For this case, we define
where $s^* = s^*(c) < 0$ is such that $\phi(s^*) = -1$ and
$\phi(s; c) > -1$ for $s > s^*$. By Lemma 3.7, $C_p \not= \emptyset$. Define
\[ c_p = \inf \{ c | c \in C_p \} . \]

Then $c_p \leq c^*$ and $x^- \leq \phi(s; c_p) \leq -1$ for $s^*(c_p) - 1 \leq s \leq s^*(c_p)$. We again have three subcases:

(i) $c_p < c_*$
(ii) $c_1 < c_p < c_*$
(iii) $c_* \leq c < c_p$.

By Theorem 2.1 and from the definition of $c_*$, we know the property of traveling wave solution $\phi(s; c)$ for $c < c_*$ and $c > c^*$. If $c^* > c$, then the qualitative behaviors of $\phi(s; c)$ for $c_p \leq c \leq c^*$ are described by the definition of $c_p$, $c^*$ and Lemma 3.4, Corollary 3.1. Furthermore, if $c_p > c$, then the nonmonotonicity or oscillation of $\phi(s; c)$ for $s < -1$ is obtained for $c < c_*$ by Lemmas 3.7, 3.3 and 3.2 and the definitions of $c_p$ and $c^*$. Thus, we have the following.

**Theorem 3.1.** Assume that $1 \leq \alpha < 1 + \beta$. There is $c_*, c^*, c_p, c_e$ with $c_0 < c^* < c_p < c_e < 0$ such that

(i) if $c < c_*$, then $\phi(s; c)$ is nondecreasing and satisfies (20);
(ii) if $c_0 \leq c < c^*$, then $|\phi(s; c)| \leq 1$ for $s < -1$; furthermore, if $c^* > c$ and $c < c^* < c^*$, then $\phi(s; c)$ oscillates about zero on $(-\infty, s_0]$ with some $s_0 < 0$;
(iii) if $c^* > c$ and $c < c_p$, or $c < c^*$ and $c_p < c < c^*$, then $\phi(s; c)$ is not monotone on $(-\infty, s_0)$ with every $s_0 < 0$;
(iv) if $c < c_p$ and $c_p \leq c < c^*$, then $\phi(s; c)$ is eventually periodic on $(-\infty, 0)$, and $\sup_{s < 0} |\phi(s; c)| > 1$;
(v) if $c = c^* > c$, then $\phi(s; c)$ is nondecreasing and there is $s_* < 0$ so that
\[ \phi(s) = x^- \text{ for } s \leq s_* , \]
\[ \phi(s) \in (x^-, x^-) \text{ for } s > s_* . \]
(vi) if $c^* < c < 0$, then $\phi(s; c)$ is nondecreasing and unbounded on $(-\infty, 0)$.

**Remark 3.3.** If $c_* \geq -1/\tau$, then $c^* = c_*$ and our result here coincides with those for discrete delays stated in Theorem 3.1 of [Hsu et al., 1999]. However, as Table 1 shows, the situation $c_* < -1/\tau$ can happen. In this case, there is a gap: either $c^* < c < c_p$ if $c_p \leq c$, or $c^* < c \leq c$ if $c^* \leq c < c_p$, where our results cannot be applied.

To obtain the table, we derive first from Eq. (15) that
\[ -c_* - \beta \int_0^\tau k(u)(1 + \epsilon u)(1+c_*)^\sigma_* du = 0 . \tag{80} \]
Integrating by parts leads to
\[ \int_0^\tau k(u)(1 + \epsilon u)(1+c_*)^\sigma_* du = (1 + c_* \tau) \int_0^\tau k(u)(1+c_*)^\sigma_* du . \tag{81} \]
Substituting (81) into (80), we obtain
\[ \beta(1 + c_* \tau) \int_0^\tau k(u)(1+c_*)^\sigma_* du = -c_* \left[ 1 - \beta \int_0^\tau \int_0^\tau k(u)(1+c_*)^\sigma_* du \right] . \]
Thus $1 + c_* \tau \geq 0$ if and only if
\[ \int_0^\tau \int_0^\tau k(u)(1+c_*)^\sigma_* du \leq \frac{1}{\beta} . \]
In particular, we note that $\beta > 0$ and $\tau > 0$ should be sufficiently small in order for $c_* \geq -1/\tau$ to hold. The datum in Table 1, obtained from solving (15) by using Maple, support the above observation.

The following lemma implies that the conditions $c_* \geq -1/\tau$ and $1 \leq \alpha < 1 + \beta$ guarantee that $c^* > c_*$, and hence nonmonotone traveling solutions of (4) do exist.

**Lemma 3.10.** If $c_* \geq -1/\tau$ and $1 \leq \alpha < 1 + \beta$, then $c^* > c_*$.

**Proof.** We have the condition that $c^* = c_*$. Now it is sufficient to show that $\phi(s; c)$ is not monotone for some $c' \in (c_*, 0)$.

For any $c \in (c_*, 0)$, we claim that the set
\[ S = \{ s < 0 | \phi(s; c) = 0 \} \]
Table 1. Values of $c_*$ and $\sigma_*$ with different parameters and different choices of the delay distribution kernel $k(u)$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\tau$</th>
<th>$k(u)$</th>
<th>$-\frac{1}{\tau}$</th>
<th>$c_*$</th>
<th>$\sigma_*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>0.3</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>$-2$</td>
<td>$-0.8591013001$</td>
</tr>
<tr>
<td>1.2</td>
<td>0.3</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>$-0.6873153510$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>$-1.589949648$</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>0.5</td>
<td>1</td>
<td>$\frac{e}{e-1}e^{-u}$</td>
<td>$-1$</td>
<td>$-1.039077935$</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>0.4</td>
<td>1</td>
<td>$\frac{e}{e-1}e^{-u}$</td>
<td>$-1$</td>
<td>$-0.9127034926$</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$-0.5$</td>
<td>$-1.0816959095$</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>0.2</td>
<td>2</td>
<td>$\frac{e}{2(e-1)}e^{-\frac{s}{2}}$</td>
<td>$-0.5$</td>
<td>$-0.3602431414$</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.3</td>
<td>2</td>
<td>$\frac{e}{2(e-1)}e^{-\frac{s}{2}}$</td>
<td>$-0.5$</td>
<td>$-0.5962264581$</td>
<td></td>
</tr>
</tbody>
</table>

Thus we have

\[
\phi'(s_0 - 1; \ c') = \frac{1}{c'} \left\{ (1 - \alpha)\phi(s_0 - 1; \ c') - \beta \int_0^\tau k(u)f(\phi(s_0 + c'u; \ c'))du \right\} = \frac{1}{c'} \left\{ (1 - \alpha)\phi(s_0 - 1; \ c') - \beta \int_0^\tau k(u)\phi(s_0 + c'u; \ c')du \right\} < 0.
\]

Thus $\phi(s; \ c')$ is not monotone. This completes the proof. ■

Now we consider the situation where $\alpha \geq 1 + \beta$.

\textbf{Lemma 3.11.} Suppose that $\alpha \geq 1 + \beta$. Then $-1 < \phi(s; \ c') < 1$ for $c < c' < 0$ and $s < 0$.

\textbf{Proof.} Fix $c \in (\bar{c}, 0)$. Let $\gamma = 1 + c\tau > 0$. Consider the following initial value problem:

\[
\begin{align*}
\psi(s) &= \frac{1}{c}[\psi(s) - \alpha\psi(s) - \beta], \quad s < 0, \\
\psi(0) &= 1 .
\end{align*}
\]

We have

\[
\psi(s) = \frac{1 - \alpha - \beta}{1 - \alpha} e^{\frac{1 - \alpha}{\alpha}s} + \frac{\beta}{1 - \alpha}, \quad s < 0,
\]

\[
\psi'(s) > 0, \quad s < 0,
\]

\[
-1 \leq \frac{\beta}{1 - \alpha} < \psi(s) < 1, \quad s < 0.
\]

This implies that

\[-1 < \psi(s) = \phi(s) < 1 \quad \text{for} \quad -\gamma < s < 0.\]
If there is $s_1 \in [-2\gamma, -\gamma)$ such that $\phi(s_1) = -1$ and $\phi(s) > -1$ for $s \in (s_1, -\gamma]$, then
\[
0 \leq \phi'(s_1) = \frac{1}{c}(\alpha - 1) - \frac{\beta}{c} \int_0^\tau k(u)f(\phi(s_1 + 1 + cu))du < \frac{1}{c}(\alpha - 1) - \frac{\beta}{c} \leq 0,
\]
which is a contradiction. If there is $s_2 \in [-2\gamma, -\gamma)$ such that $\phi(s_2) = 1$ and $\phi(s) < 1$ for $s \in (s_2, -\gamma]$, then
\[
0 \geq \phi'(s_2) = \frac{1}{c}(1 - \alpha) - \frac{\beta}{c} \int_0^\tau k(u)f(\phi(s_2 + 1 + cu))du > \frac{1}{c}(1 - \alpha) + \frac{\beta}{c} \geq 0,
\]
a contradiction again. Therefore, $\phi(s) \in (-1, 1)$ for $s \in [-2\gamma, -\gamma]$.

Continuing the procedure, we have $\phi(s) \in (-1, 1)$ for $s < 0$. This completes the proof. $\blacksquare$

Combining this lemma with Theorem 2.1 and Lemma 3.2, we have the following.

**Theorem 3.2.** Assume that $\alpha \geq 1 + \beta$. Then there is $c_\ast < 0$ such that

(i) if $c < c_\ast$, then $\phi(s; c)$ is nondecreasing and satisfies (20);

(ii) if $c = c_\ast < 0$, then $\phi(s; c)$ oscillates about zero and $|\phi(s; c)| < 1$ for $s < 0$.

4. Remarks on Extensions to CNN with Delay Self-Feedback

In this section, we shall briefly discuss the existence of monotone waves of the CNN model with delays present in both self-feedback and neighborhood interaction:

\[
\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \int_0^\tau k_i(u)f(x_i(t - u))du + \beta \int_0^\tau k_i(u)f(x_{i+1}(t - u))du, \quad i \in \mathbb{Z};
\]

(82)

where $k_i(u) \geq 0 (i = 1, 2)$ are piecewise continuous on $[0, \tau]$ and satisfy
\[
\int_0^\tau k_i(u)du = 1, \quad i = 1, 2.
\]

Let $\phi(s) = \phi(s; c) = x(i - ct)$, then $\phi(s)$ satisfies

\[
-c\phi'(s) = -\phi(s) + \alpha \int_0^\tau k_1(u)f(\phi(s + cu))du + \beta \int_0^\tau k_2(u)f(\phi(s + 1 + cu))du.
\]

(83)

The associated characteristic equation is

\[
\Delta(\sigma, c, x^0) = 0,
\]

where

\[
\Delta(\sigma, c, x^0) = -c\sigma + 1 - \alpha \int_0^\tau k_1(ue\sigma)du - \beta \int_0^\tau k_2(u)e^{(1+cu)\sigma}du.
\]

We derive that

\[
\frac{\partial \Delta}{\partial \sigma}(\sigma, c, x^0) = -c - \alpha \int_0^\tau k_1(ue\sigma)du - \beta \int_0^\tau k_2(u)(1+cu)e^{(1+cu)\sigma}du,
\]

\[
\frac{\partial^2 \Delta}{\partial \sigma^2}(\sigma, c, x^0) = -\alpha \int_0^\tau k_1(ue\sigma)du - \beta \int_0^\tau k_2(u)(1+cu)^2e^{(1+cu)\sigma}du < 0.
\]

Corresponding to Lemma 2.1, we have the following.

**Lemma 4.1.** Assume that $c < 0$ and $\alpha \geq 1$. We have the following conclusions.

(i) There is a pair of $(c_\ast, \sigma_\ast)$ with

\[
c_\ast < -\frac{\beta}{1 + \alpha \int_0^\tau k_1(du) + \beta \int_0^\tau k_2(du)} < 0,
\]

\[
\sigma_\ast = \sigma(c_\ast) > 0
\]

and satisfying

\[
\Delta(\sigma_\ast, c_\ast, x^0) = 0, \quad \frac{\partial \Delta}{\partial \sigma}(\sigma_\ast, c_\ast, x^0) = 0.
\]

(85)

(ii) For any $c < c_\ast$, there exist $\sigma = \sigma(c) > 0$, $\varepsilon_0 = \varepsilon_0(c) > 0$ satisfying

\[
\Delta(\sigma, c, x^0) = 0 \quad \text{and} \quad \Delta(\sigma + \varepsilon, c, x^0) > 0 \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0.
\]
(iii) For any $c_0 < c < 0$, \textit{Eq. (8)} has no real roots in $\sigma$.

Using the Definition 2.1 and the following upper and lower solutions

\begin{align*}
U(s) &= \begin{cases}
x(s), & s \geq 0, \\
x(s)e^{\sigma}, & s \leq 0,
\end{cases} \\
L(s) &= \begin{cases}
0, & s \geq 0, \\
\zeta(1-c)e^{\sigma}, & s \leq 0,
\end{cases}
\end{align*}

we obtain the following.

\textbf{Theorem 4.1.} \textit{There exists a}

\[c_0 < -\frac{\beta}{1 + \alpha \int_{0}^{\pi} u_k(u)du + \beta \int_{0}^{\pi} u_k(u)du} < 0\]

\textit{such that for every $c < c_0$, (82) has a monotone traveling wave solution $\phi$ satisfying boundary conditions (20), and a monotone traveling wave solution $\phi$ satisfying boundary conditions}

\[\lim_{s \to -\infty} \phi(s) = x^0, \quad \lim_{s \to \infty} \phi(s) = x^-.
\]

\textit{For any fixed $c > 0$, the characteristic equation $\Delta(\sigma, c, x^0) = 0$ of (82) has a unique real root $\sigma = \sigma(c) < 0$. Using the following pair of upper and lower solutions}

\begin{align*}
U(s) &= \begin{cases}
x(s), & s \leq 0, \\
x(s)e^{\sigma}, & s \geq 0,
\end{cases} \\
L(s) &= \begin{cases}
0, & s \leq 0, \\
\zeta(1-c)e^{\sigma}, & s \geq 0,
\end{cases}
\end{align*}

\textit{we can show the conclusion of following.}

\textbf{Theorem 4.2.} \textit{For any $c > 0$, (82) has a monotone traveling wave solution satisfying}

\[\lim_{s \to -\infty} \phi(s) = x^+, \quad \lim_{s \to \infty} \phi(s) = x^0,
\]

\textit{and a monotone traveling wave solution satisfying}

\[\lim_{s \to -\infty} \phi(s) = x^-, \quad \lim_{s \to \infty} \phi(s) = x^0.
\]