Blow-up estimates for a semilinear coupled parabolic system

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This work deals with a semilinear parabolic system which is coupled both in the equations and in the boundary conditions. The blow-up phenomena of its positive solutions are studied using the scaling method, the Green function and Schauder estimates. The upper and lower bounds of blow-up rates are then obtained. Moreover we show the influences of the reaction terms and the boundary absorption terms on the blow-up estimates.

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1. Introduction and main results

In this work, we consider the following semilinear coupled parabolic system:

\begin{align*}
  u_t &= \Delta u + v^m, \quad x \in B(0; R), \quad t > 0 \\
  v_t &= \Delta v + u^n, \quad x \in B(0; R), \quad t > 0 \\
  \frac{\partial u}{\partial \eta} &= v^p, \quad x \in \partial B(0; R), \quad t > 0 \\
  \frac{\partial v}{\partial \eta} &= u^q, \quad x \in \partial B(0; R), \quad t > 0 \\
  u(x, 0) &= u_0(x), \quad x \in B(0; R) \\
  v(x, 0) &= v_0(x), \quad x \in B(0; R)
\end{align*}

(1)

where \( m, n, p \) and \( q \) are nonnegative constants, \( B(0; R) \) is the ball in IR\(^n\) centered at the origin with radius \( R \), \( \partial B(0; R) = \{ x : |x| = R \} \) is the boundary. \( \eta \) is the unit outward normal on \( \partial B(0; R) \). The initial data \( u_0(x) \) and \( v_0(x) \) are assumed to be nonnegative and radially symmetric non-increasing continuous functions, with \( \nu_r = \frac{\partial v}{\partial r} \geq 0 \) where \( r = |x| \), and satisfy the compatibility condition

\begin{align*}
  \frac{\partial u_0}{\partial \eta} &= u_0^p, \quad \frac{\partial v_0}{\partial \eta} = v_0^q.
\end{align*}

(2)

As is known to all, the principle of conservation is one of the basic theories in the formulation of governing equations for physical problems. When the problem under consideration involves a reaction process accompanied by diffusion, the principle leads to a set of partial differential equations for the unknown quantities of the system. These quantities may be
mass concentrations in chemical reaction processes, temperature in heat conduction, the neutron flux in nuclear reactors, population density in population dynamics, and many others. According to the principle of conservation, which states that “for any subdomain $S$ of $\Omega$ with boundary surface $\Gamma$, the rate of change of mass density is equal to the rate of flux across $\Gamma$ plus the rate of generation within $S$’, the problem can be described as the semilinear coupled parabolic system (1) when there are two different quantities. For when the flux across the boundary surface is prescribed as the nonlocal source terms in system (1), which represent the rate of flow of the density, the homogeneous questions are investigated widely. More detailed background of (1) can be found in [11].

A lot of works have been devoted to the study of the global existence and the blow-up in finite time for the semilinear parabolic system in the past few decades; cf. [1–10] and references therein. In [4], Escobedo and Herrero investigated a weakly coupled system:

$$\begin{align*}
    u_t &= \Delta u + v^p, & x \in \mathbb{R}^N, & t > 0 \\
    v_t &= \Delta v + u^q, & x \in \mathbb{R}^N, & t > 0 \\
    u(x, 0) &= u_0(x) \geq 0, & x \in \mathbb{R}^N \\
    v(x, 0) &= v_0(x) \geq 0, & x \in \mathbb{R}^N
\end{align*}$$

(3)

and showed that when $pq > 1$, if $\max(\alpha, \beta) \geq \frac{N}{q}$, where $\alpha = \frac{p+1}{pq-1}$ and $\beta = \frac{q+1}{pq-1}$, then all the nontrivial nonnegative solutions blow up with the rate $\alpha$ and $\beta$ in finite time.

Another system coupled in the boundary condition was considered by Deng in [2]:

$$\begin{align*}
    u_t &= \Delta u, \quad v_t = \Delta v, & x \in \mathbb{R}^N_+, & t > 0 \\
    -\frac{\partial u}{\partial \eta} &= v^p, -\frac{\partial v}{\partial \eta} = u^q, & x \in \mathbb{R}^N_+, & t > 0 \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{R}^N_+
\end{align*}$$

(4)

He proved that when $pq > 1$, all the nonnegative solutions blow up at the rate $\alpha = \frac{p+1}{2(pq-1)}$ and $\beta = \frac{q+1}{2(pq-1)}$, if $\max(\alpha, \beta) \geq \frac{N}{2}$ is satisfied.

In [5], Fila and Levine considered the system coupled in an equation and a boundary condition in the region $\mathbb{R}^N_+$; they proved that when $mq > 1$, either $\max(\alpha, \beta) > \frac{N}{q}$ or $\max(\alpha, \beta) = \frac{N}{q}$ and $m, q \geq 1$, the solutions blow up at the rate $\alpha = (m + 2)/(2(mq - 1))$, $\beta = (2q + 1)/(2(mq - 1))$. For while the region is a ball in $\mathbb{R}^N$ and $pq > 1$, Wang obtained in [9] blow-up estimates of the solutions of the system with the initial data $u_0(x)$ and $v_0(x)$ nonnegative non-increasing continuous functions and $\alpha, \beta \geq 0$. Also the authors in [6] studied a similar system and obtained its blow-up estimates and blow-up set.

The objective of this work is to study the more general system (1). By using the scaling method, the Green function and Schauder estimates, we establish the blow-up rate estimates of the positive solutions, and find out how the reaction terms and the boundary absorption terms affect the blow-up estimates of the solutions in finite time.

We will study the problem in the following cases:

(i) $m \geq \lambda_1, \quad n \geq \lambda_2$;
(ii) $p \leq \lambda_3, \quad n \leq \lambda_2$;
(iii) $m \geq \lambda_1, \quad q \leq \lambda_4$;
(iv) $p \geq \lambda_3, \quad q \geq \lambda_4$

(5)

where

$$\begin{align*}
    &\lambda_1 = \frac{2np + 2p - 1}{n + 2}, \quad \lambda_2 = \frac{2mq + 2q - 1}{m + 2}, \\
    &\lambda_3 = \frac{mq + m + 1}{2q + 1}, \quad \lambda_4 = \frac{np + n + 1}{2p + 1}.
\end{align*}$$

(6)

We define also

$$\begin{align*}
    &\alpha = \frac{m + 1}{mn - 1}, \quad \beta = \frac{n + 1}{mq - 1}, \quad \text{in the case of (5)(i)}; \\
    &\alpha = \frac{m + 2}{2(mq - 1)}, \quad \beta = \frac{n + 2}{2(mq - 1)}, \quad \text{in the case of (5)(ii)}; \\
    &\alpha = \frac{2(np - 1)}{2p + 1}, \quad \beta = \frac{q + 1}{2(np - 1)}, \quad \text{in the case of (5)(iii)}; \\
    &\alpha = \frac{p + 1}{2(pq - 1)}, \quad \beta = \frac{q + 1}{2(pq - 1)}, \quad \text{in the case of (5)(iv)}.
\end{align*}$$

(7)

**Lemma 1.** Let $\max\{mn - 1, mq - 1, np - 1, pq - 1\} > 0$; then we have $\alpha, \beta > 0$ when one of conditions in (5)(i)–(iv) is valid, where $\alpha$ and $\beta$ are the blow-up rates of the solution to problem (1).
Proof. We just need to prove $\alpha, \beta > 0$ under the condition (5)(i); the other cases can be proved similarly.

If $\max\{mn - 1, mq - 1, np - 1, pq - 1\} = mn - 1$, then $mn - 1 > 0$.
If $\max\{mn - 1, mq - 1, np - 1, pq - 1\} = mq - 1$, we obtain $mn - 1 \geq (mq - 1) + 2(q - n) > 0$ from $n \geq \lambda_2$, and then $mn - 1 > 0$.
If $\max\{mn - 1, mq - 1, np - 1, pq - 1\} = np - 1$, we obtain $mn - 1 \geq (np - 1) + 2(p - m) > 0$ from $m \geq \lambda_1$, and then $mn - 1 > 0$.
If $\max\{mn - 1, mq - 1, np - 1, pq - 1\} = pq - 1$, we obtain $(mn - 1) (p + 1) \geq (pq - 1)(m + 1) > 0$ from $m \geq \lambda_1$, $n \geq \lambda_2$, and then $mn - 1 > 0$.

As $m, n$ are nonnegative constants, $\alpha, \beta > 0$ can be simply proved. ■

By the results of [2–4,7,8], we know that all the solutions of the system (1) are nonnegative, and blow up in finite time if and only if $\max\{mn - 1, mq - 1, np - 1, pq - 1\} > 0$. Throughout this work we assume that the solution $(u, v)$ of (1) blows up in finite time $T$. It is obvious that $u$ and $v$ blow up simultaneously.

Our main results are as follows:

**Theorem 1.** Let $(u(x, t), v(x, t))$ be a smooth solution to problem (1); then for any case of (5), there exists a positive constant $c$ such that

$$
\max_{0 \leq t \leq T} \max_{x \in \Omega} u(x, t) \geq c(T - t)^{-\alpha}, \quad \max_{0 \leq t \leq T} \max_{x \in \Omega} v(x, t) \geq c(T - t)^{-\beta}.
$$

(8)

**Theorem 2.** Let $(u(x, t), v(x, t))$ be a smooth solution to problem (1); for any of the following cases:

(a) if (5)(i) or (iv) holds with $\max\{\alpha, \beta\} \geq N/2$;
(b) if (5)(ii) holds with $\max\{\alpha, \beta\} > N/2$;
(c) if (5)(ii) holds with $\max\{\alpha, \beta\} = N/2$ and $p \geq 1$;
(d) if (5)(iii) holds with $\max\{\alpha, \beta\} > N/2$;
(e) if (5)(iii) holds with $\max\{\alpha, \beta\} = N/2$ and $m, q \geq 1$,

there exists a positive constant $C$ such that

$$
\max_{0 \leq t \leq T} \max_{x \in \Omega} u(x, t) \leq C(T - t)^{-\alpha}, \quad \max_{0 \leq t \leq T} \max_{x \in \Omega} v(x, t) \leq C(T - t)^{-\beta}.
$$

(9)

2. Proof of the theorems

As $v_t \geq 0$, define $f(t) = \max_{0 \leq t \leq T} \max_{x \in \Omega} u(x, t), g(t) = \max_{0 \leq t \leq T} \max_{x \in \Omega} v(x, t) = \max_{0 \leq t \leq T} v(R, t)$. It is easy to see that $f(t_1) = \max_{0 \leq t \leq T} \max_{x \in \Omega} u(x, t) \leq \max_{0 \leq t \leq T} \max_{x \in \Omega} u(x, t) = f(t_2)$, for any $0 \leq t_1 < t_2$, i.e. $f(t)$ is a non-decreasing function. Similarly, $g(t)$ is a non-decreasing function too.

We need also the following result:

**Lemma 2.** Let $\alpha, \beta$ be positive constants and satisfy one of the following conditions:

(i) $2\alpha + 2 - 2\beta m = 0, 2\beta + 2 - 2\alpha n = 0, 2\alpha + 1 - 2\beta \geq 0$ and $2\beta + 1 - 2\alpha q \geq 0$;
(ii) $2\alpha + 2 - 2\beta m = 0, 2\beta + 2 - 2\alpha n \geq 0, 2\alpha + 1 - 2\beta \geq 0$ and $2\beta + 1 - 2\alpha q = 0$;
(iii) $2\alpha + 2 - 2\beta m \geq 0, 2\beta + 2 - 2\alpha n = 0, 2\alpha + 1 - 2\beta = 0$ and $2\beta + 1 - 2\alpha q \geq 0$;
(iv) $2\alpha + 2 - 2\beta m \geq 0, 2\beta + 2 - 2\alpha n \geq 0, 2\alpha + 1 - 2\beta = 0$ and $2\beta + 1 - 2\alpha q = 0$;

then there exists a positive constant $\epsilon$ such that for all $t \in \left(\frac{T}{2}, T\right)$,

$$
\varepsilon g^{\frac{1}{\beta}}(t) \leq f^{\frac{1}{\alpha}}(t), \quad \varepsilon f^{\frac{1}{\alpha}}(t) \leq g^{\frac{1}{\beta}}(t).
$$

(10)

**Proof.** Because $f$ and $g$ are non-decreasing functions, all the solutions of the system (1) are nonnegative, and blow up in finite time $T$, so either $f$ or $g$ never goes to zero. On the contrary, we assume that the first inequality of (10) is not true; then there exists a sequence $(t_n)$ with $t_n \to T$ as $n \to \infty$ such that $g^{\frac{1}{\beta}}(t_n) f^{\frac{1}{\alpha}}(t_n) \to 0$. As $v_t \geq 0$, and in view of the definition of $g(x)$, when $x_n$ belongs to $\partial\Omega_R$, for any $t_n$, there exists $(\tilde{x}_n, \tilde{t}_n) \in \partial\Omega_R \times [0, t_n]$ such that $v(\tilde{x}_n, \tilde{t}_n) = g(t_n)$, with $\tilde{t}_n \to T$ as $g(t_n) \to \infty$. Let $\lambda_n = g^{\frac{1}{\beta}}(t_n) \to 0$ as $n \to \infty$. Let $R_n$ be a normalized transformation in $x \in \mathbb{R}^n$ which maps $-e_1 = (-1, 0, \ldots, 0)$ to the unit outer normal vector to $\partial\Omega$ at $\tilde{x}_n$. Define

$$
\varphi_n(y, s) = \lambda_n^{\frac{2a}{\beta}} u(\lambda_n R_n y + \tilde{x}_n, \lambda_n^{\frac{2a}{\beta}} s + \tilde{t}_n),
$$

$$
\psi_n(y, s) = \lambda_n^{\frac{2b}{\beta}} v(\lambda_n R_n y + \tilde{x}_n, \lambda_n^{\frac{2b}{\beta}} s + \tilde{t}_n)
$$

(11)
where \( (y, s) \in J_n \times I_n(T), J_n = \{ y : \lambda_nR_ny + \hat{x}_n \in B_R \}, I_n(T) = (-\lambda_n^{-2}t_n, \lambda_n^{-2}(t - t_n)) \). Then, by some calculations we can get

\[
\begin{align*}
(\varphi_n)_s &= \Delta \varphi_n + \lambda_n^{2a+2-2\beta m} \psi_n^m, \quad (y, s) \in J_n \times I_n(t) \\
(\psi_n)_s &= \Delta \psi_n + \lambda_n^{2\beta + 2 - 2\alpha n} \varphi_n^n, \quad (y, s) \in J_n \times I_n(t) \\
\frac{\partial \varphi_n}{\partial \eta} &= \lambda_n^{2a+1 - 2\beta} \psi_n^p, \quad (y, s) \in J_n \times I_n(t) \\
\frac{\partial \psi_n}{\partial \eta} &= \lambda_n^{2\beta + 1 - 2\alpha} \varphi_n^q, \quad (y, s) \in J_n \times I_n(t)
\end{align*}
\]

and for \( y \in J_n, s \in (-\lambda_n^{-2}t_n, 0) \),

\[
\begin{align*}
0 &\leq \varphi_n(y, s) \leq \lambda_n^{2a}f(t_n) = g \frac{\lambda_n^{2a}}{\pi} (t_n)f(t_n), \\
0 &\leq \psi_n(y, s) \leq 1, \\
\psi_n(0, 0) &= 1.
\end{align*}
\]

It is clear that \( \forall k > 0, J_n \cap \{|y| \leq k\} \in C^{2+\delta}, \) and \( \partial J_n \) tends to \( \{y = 0\} \) as \( n \to \infty \). According to the Schauder estimates, there exists a \( \mu \in (0, 1) \), \( \forall k > 0 \)

\[
\begin{align*}
\|\varphi_n\|_{C^{2+\mu, 1+\mu/2}(J_n \cap \{|y| \leq k\} \times [-k, 0])} &\leq C_k, \\
\|\psi_n\|_{C^{2+\mu, 1+\mu/2}(J_n \cap \{|y| \leq k\} \times [-k, 0])} &\leq C_k
\end{align*}
\]

where \( C_k \) is a constant independent on \( n \). Thus \( (\varphi_n, \psi_n) \to (\varphi, \psi, (y, s) \in (J_n \cap \{|y| \leq k\}) \times [-k, 0] \) as \( n \to \infty \), and, if the condition (i) is satisfied, \( (\varphi, \psi) \) satisfies

\[
\begin{align*}
\varphi &= \Delta \varphi + \psi^m, \quad y \in R^d, s \in (-\infty, 0] \\
\psi &= \Delta \psi + \varphi^n, \quad y \in R^d, s \in (-\infty, 0] \\
-\frac{\partial \varphi}{\partial y_1} &= \delta_1 \psi^p, \quad y_1 = 0, s \in (-\infty, 0] \\
-\frac{\partial \psi}{\partial y_1} &= \delta_2 \varphi^q, \quad y_1 = 0, s \in (-\infty, 0]
\end{align*}
\]

and

\[
\psi(0, 0) = 1, \quad 0 \leq \psi \leq 1, \quad \varphi \equiv 0
\]

where \( \delta_1 = g(t_n) \frac{2a+1 - 2\beta d}{4a}, \delta_2 = g(t_n) \frac{2\beta + 1 - 2\alpha d}{4a} \); it is obvious that \( 0 \leq \delta_1, \delta_2 \leq 1 \) and \( \delta_1, \delta_2 \neq 0 \). This is a contradiction with \( \varphi_n = \Delta \varphi + \psi^m \), which means that the first inequality of (10) holds. When one of the other three conditions is satisfied, the first inequality of (10) can be obtained in the same way.

The second inequality of (10) can be proved similarly.

Let \( G(x, t; y, \tau) \) be the Green function of the heat equation in the ball \( B(0; R) \) with \( \partial G/\partial \eta = 0 \) on the boundary \( \partial B(0; R) \) [1]; then for all \( x \in B(0; R), 0 < z < t < T \),

\[
\begin{align*}
u(x, t) &= \int_{B_R} G(x, t; y, z)u(y, z)dy + \int_z^t \int_{B_R} G(x, t; y, \tau)v^m(y, \tau)dyd\tau + \int_z^t \int_{\partial B_R} G(x, t; y, \tau)v^p(R, \tau)dsd\tau \\
u(x, t) &= \int_{B_R} G(x, t; y, z)v(y, z)dy + \int_z^t \int_{B_R} G(x, t; y, \tau)v^p(y, \tau)dyd\tau + \int_z^t \int_{\partial B_R} G(x, t; y, \tau)v^q(R, \tau)dsd\tau
\end{align*}
\]

We are now going to prove Theorem 1.

According to Lemma 2 and (19), we have

\[
\begin{align*}
f(t) &\leq f(z) + g^m(t) \int_z^t d\tau + g^p(t) \int_z^t \frac{1}{\sqrt{\pi(\tau - t)}} d\tau, \\
&\leq f(z) + g^m(t)(T - z) + cg^p(t)\sqrt{T - z}, \\
&\leq f(z) + c(T - z)f^m\frac{m}{\sigma}(z) + c(T - z)^{\frac{1}{2}}f^p\frac{p}{\sigma}(z).
\end{align*}
\]

Since \( f(t) \to \infty \) as \( t \to T^- \), then for any \( z \in \left(\frac{1}{2}, T\right) \), there exists a \( t \in (z, T) \) satisfying \( f(t) = 2f(z) \).

Without loss of generality, we suppose that \( f(t), g(t) > 1 \); then for \( z \in (T/2, T) \), it follows that

\[
\begin{align*}
f(z) &\leq c(T - z)f^m\frac{m}{\sigma}(z) + c(T - z)^{\frac{1}{2}}f^p\frac{p}{\sigma}(z).
\end{align*}
\]
Similarly, we have
\[
g(z) \leq c(T-z)g^{n\bar{\alpha}}(z) + c(T-z)^{\frac{1}{2}}g^{\bar{\alpha}}(z).
\] (23)

In the case of (5)(i), we have
\[
2p \leq m + \frac{\alpha}{\beta}, \quad 2q \leq n + \frac{\beta}{\alpha}
\] (24)
or
\[
\frac{p\beta}{\alpha} \leq \frac{m\beta}{2\alpha} + \frac{1}{2}, \quad \frac{q\alpha}{\beta} \leq \frac{n\alpha}{2\beta} + \frac{1}{2}.
\] (25)

By (22),
\[
f(z) \leq c(T-z)f^{\frac{m\bar{\alpha}}{2}}(z) + c(T-z)^{\frac{1}{2}}f^{\frac{m\bar{\alpha}}{2}}(z)
\]
\[
\leq c(T-z)f^{\frac{m\bar{\alpha}}{2}}(z) + c(\varepsilon)(T-z)f^{\frac{m\bar{\alpha}}{2}}(z) + \varepsilon f(z).
\] (26)

Then
\[
f(z) \leq c(T-z)^{1/(1-\frac{m\bar{\alpha}}{2})} = c(T-z)^{-\alpha}.
\] (27)

Similarly, we can get
\[
g(z) \geq c(T-z)^{1/(1-\frac{m\bar{\alpha}}{2})} = c(T-z)^{-\beta}.
\] (28)

As the proof of Theorem 1 is completed.

As \(g(t)\) is a non-decreasing function, and satisfies \(\lim_{t \to T^-} g(t) = \infty\), according to the results of [2,5], in order to prove Theorem 2, we need the following lemma.

**Lemma 3.** For any \(t_0 \in \left(\frac{T}{2}, T\right)\), define
\[
t^{+}_0 = t^{+}(t_0) = \max\{t \in (t_0, T) \mid g(t) = 2g(t_0)\},
\] (29)

and let \(\lambda_0 = \lambda(t_0) = g^{-\frac{1}{\beta}}(t_0)\); we assert that \(\lambda^{-2}(t_0)(t^{+}_0 - t_0) \leq M\), where \(M\) is a positive constant independent of \(t_0\).

**Proof.** If the conclusion is false, then there exists a sequence \(\{t_n\}\) satisfying \(t_n \to T^-\) as \(n \to \infty\), such that \(\lambda^{-2}(t^{+}_n - t_n) \to \infty\), where \(t^{+}_n = t^{+}(t_n)\) and \(\lambda_n = \lambda(t_n)\). For \(t_n\), choose \((\hat{x}_n, \hat{v}_n) \in \partial B_R \times [0, t_n]\) such that \(\nu(\hat{x}_n, \hat{v}_n) = g(t_n)\).

For the case of (5)(i), let \(\varphi_n(y, s)\) and \(\psi_n(y, s)\) be defined in the same form as (11); thus in \(J_n \times I_n(t), \psi_n(0, 0) = 1\), from the definition of \(t^{+}_n\), for any \((y, s) \in J_n \times I_n(t^{+}_n)\),
\[
0 \leq \varphi_n(y, s) \leq \lambda_n^{2\alpha}g(t^{+}_n) = \lambda_n^{2\alpha} = 2,
\] (30)
\[
0 \leq \psi_n(y, s) \leq \lambda_n^{2\alpha}f(t^{+}_n) \leq \lambda_n^{2\alpha}e^{-\alpha}g^{\bar{\alpha}}(t^{+}_n) = \varepsilon^{-\alpha}2^{\bar{\alpha}}.
\] (31)

Thus when \(n \to \infty\), there exist \((\varphi, \psi)\) satisfying
\[
\begin{align*}
\varphi & = \Delta \varphi + \psi^m, \quad y \in R^N, \quad s \in (-\infty, \infty) \\
\psi & = \Delta \psi + \psi^n, \quad y \in R^N, \quad s \in (-\infty, \infty) \\
\frac{\partial \varphi}{\partial y_1} & = \delta_1 \psi^p, \quad y_1 = 0, \quad s \in (-\infty, \infty) \\
\frac{\partial \psi}{\partial y_1} & = \delta_2 \varphi^q, \quad y_1 = 0, \quad s \in (-\infty, \infty)
\end{align*}
\] (32)

and for \(y \in R^N, s \in (-\infty, \infty)\),
\[
\psi(0, 0) = 1, \quad 0 \leq \psi \leq 2, \quad 0 \leq \varphi(y, s) \leq \varepsilon^{-\alpha}2^{\bar{\alpha}},
\] (33)

where \(0 \leq \delta_1, \delta_2 \leq 1\) and \(\delta_1, \delta_2 \neq 0\). According to the condition of Theorem 2 and the results of [2], we know that \((\varphi, \psi)\) blows up in finite time. This is a contradiction. Hence the conclusion holds.

When \(p \leq \lambda_3, n \leq \lambda_2\) or \(m \leq \lambda_1, q \leq \lambda_4\) or \(p \geq \lambda_3, q \geq \lambda_4\); in a similar way, we can get
\[
\lambda^{-2}(t_0)(t^{+}_0 - t_0) \leq M.
\] (34)

Then Lemma 3 has been proved.
Now we begin to prove Theorem 2.

According to Lemma 3, let \( \lambda(t_0) = g^{-\frac{1}{\beta}}(t_0) \); thus \( t_0^+ - t_0 \leq Mg^{-\frac{1}{\beta}}(t_0), \forall t_0 \in \left( \frac{T}{2}, T \right) \).

Note that
\[
t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \ldots
\]
(35)
\[
t_j^+ - t_j = t_{j+1} - t_j \leq Mg^{-\frac{1}{\beta}}(t_j), \quad g(t_{j+1}) = 2g(t_j), \quad j = 0, 1, 2, \ldots
\]
(36)

Then
\[
T - t_0 = \sum_{j=0}^{\infty}(t_{j+1} - t_j) \leq M \sum_{j=0}^{\infty}g^{-\frac{1}{\beta}}(t_j)
\]
\[
= Mg^{-\frac{1}{\beta}}(t_0) \sum_{j=0}^{\infty}2^{-\frac{j}{\beta}} = g^{-\frac{1}{\beta}}(t_0) \frac{M}{1 - 2^{-\frac{1}{\beta}}},
\]
(37)

which implies that
\[
g(t_0) \leq \left( \frac{M}{1 - 2^{-\frac{1}{\beta}}} \right) \beta (T - t_0)^{-\beta} \leq C(T - t_0)^{-\beta},
\]
(38)

and instead \( t_0 \), with \( t \), we get
\[
g(t) \leq C(T - t)^{-\beta}.
\]
(39)

By Lemma 2, we have
\[
f(t) \leq C(T - t)^{\alpha}.
\]
(40)

Thus, we complete the proof of Theorem 2.

References