Some properties of fuzzy reasoning in propositional fuzzy logic systems

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In order to analyze the logical foundation of fuzzy reasoning, this paper first introduces the concept of generalized roots of theories in Łukasiewicz propositional fuzzy logic Łuk, Gödel propositional fuzzy logic Göd, Product propositional fuzzy logic P, and nilpotent minimum logic NM (the R₀-propositional fuzzy logic L/C₃). Next, it is proved that all consequences of a theory C, named D(C), are completely determined by its generalized root whenever C has a generalized root. Moreover, it is proved that every finite theory C has a generalized root, which can be expressed by a specific formula. Finally, we demonstrate the existence of a non-fuzzy version of Fuzzy Modus Ponens (FMP) in Łuk, Göd, P and NM (L/C₃), and we provide its numerical version as a new algorithm for solving FMP.

1. Introduction

Fuzzy reasoning is the theoretical foundation of fuzzy control. As fuzzy control became more widely applied and different kinds of fuzzy reasoning methods were introduced, researchers became more interested in the logical foundation of fuzzy reasoning. One of the main deduction rules in propositional logic is Modus Ponens (MP), which can be expressed as follows:

\[ \text{from } A \rightarrow B \text{ and } A \text{ infer } B. \]  

(1)

Where \( A, B \) are two propositions. \( A \rightarrow B \) and \( A \) are called premises and \( B \) is called a conclusion. In cases where \( A \rightarrow B \) and a proposition \( A^* \) that is close to \( A \) are given, a conclusion \( B^* \) close to \( B \) must be drawn. This deduction can be written as:

\[ \text{from } A \rightarrow B \text{ and } A^* \text{ infer } B^*. \]  

(2)

For example, if we know that young people (\( A \)) are usually physically fit (\( B \)), and that some person is almost young (\( A^* \)), then we can reasonably conclude that this person (\( A^* \)) is probably nearly physically fit (\( B^* \)). Examples like this can be found everywhere in common life. In what follows, we call the inference model (2) Generalized Modus Ponens (GMP).

GMP is much more general and useful than deduction form (1), because \( A^* \) is allowed to be different from \( A \), thus many different propositions \( A^* \) adapt to (2) (see [8]). More generally, consider the following inference model:

\[ \text{from } A_i \rightarrow B \text{ and } A^*_i \ (i = 1, 2, \ldots, m) \text{ infer } B^*, \]  

(3)

where the major premises \( A_i \rightarrow B(i = 1, \ldots, m) \) and the minor premise \( A^*_i (i = 1, \ldots, m) \) are given and a conclusion \( B^* \) is to be collectively determined by the pairs \( \{A_i \rightarrow B, A^*_i \} (i = 1, \ldots, m) \). We call this inference model (3) Collective Generalized Modus Ponens (CGMP).
GMP and CGMP have a lot of applications, therefore it is necessary to put forward a program to find a reasonable B* in (2) and (3). It is Zadeh who first investigated the GMP-like problem where the propositions A, A* and B, B* appeared in (2) were regarded as fuzzy subsets on universes X and Y, respectively. In this framework, (2) turns out to be

\[ A(x) \rightarrow B(y) \text{ and } A'(x), x \in X, y \in Y, \text{ calculate } B'(y). \]

(4)

We call (4) Fuzzy Modus Ponens (FMP) (see [19]).

Then, in 1973, Zadeh proposed an algorithm called Composition Rule of Inference (CRI) (see [19]) to solve FMP. By CRI the conclusion of FMP can be written as:

\[ B'(y) = \sup \{ A'(x) \wedge R(A(x), B(y)) | x \in X \}, \quad y \in Y. \]

(5)

where \( R : [0,1]^2 \rightarrow [0,1] \) is an implication operator, such as Zadeh’s operator \( R_L \), Łukasiewicz’s operator \( R_L \), Gödel’s operator \( R_G \), etc., \( a \wedge b = \min(a,b) \).

The well-known Triple I method is another algorithm to solve the problem of FMP proposed in [12,16], such that \( B^* \) is the smallest fuzzy subset of \( Y \) satisfying

\[ \forall x \in X, \quad \forall y \in Y, \quad (A(x) \rightarrow B(y)) \rightarrow (A'(x) \rightarrow B'(y)) = 1. \]

It has been proven in [15,18] that the Triple I conclusion \( B^* \) of FMP always exists and

\[ B'(y) = \sup \{ A'(x) \odot (A(x) \rightarrow B(y)) | x \in X \}, \quad y \in Y, \]

(6)

where \( \odot \) is a t-norm.

Formulas like (5) and (6) are called numerical solutions of GMP. Many numerical solutions were proposed under different assumptions by different authors (see [7,9,11]).

A more reasonable solution may be constructed from the syntactical point of view. One typical example is available in Wang’s work in classical logic (see [12]), where he proved that every finite theory has a root, and a numerical solution can be deduced based on this root.

In this paper, from the syntactic point of view, we are going to discuss the logical foundation of GMP/CGMP in the four logics mentioned above. A generalized root of a theory in these logics is defined, and its properties are carefully discussed, in order to draw some inference rules, which can help us find algorithms for determining their numerical solutions.

This paper is organized as follows: In the first section we discuss the aim of our work. Section 2 introduces preliminaries of this paper. The definition of a generalized root and its properties are given in Section 3, and in Section 4 we explain the concept of syntactical solutions of GMP and CGMP, and discuss GMP and CGMP’s generalized inference rule (syntactic-form solution) and some algorithms to compute numerical solutions.

2. Preliminaries

2.1. Logic systems: Łuk, Göd, II and Ł* (NM)

A t-norm is a binary operation \( \odot \) on \([0,1]\) satisfying the following conditions:

(i) \( \odot \) is commutative and associative, i.e., for all \( x, y, z \in [0,1], \) \( x \odot y = y \odot x, (x \odot y) \odot z = x \odot (y \odot z); \)
(ii) \( \odot \) is nondecreasing in both arguments, i.e., \( x_1 \leq x_2 \) implies \( x_1 \odot y \leq x_2 \odot y, \) \( y_1 \leq y_2 \) implies \( x \odot y_1 \leq x \odot y_2; \)
(iii) \( 1 \odot x = x \) and \( 0 \odot x = 0 \) for all \( x \in [0,1]. \) \( \odot \) is a continuous t-norm if it is a t-norm and is a continuous mapping of \([0,1]^2\) into \([0,1]. \)

An implication operator is a function \( R : [0,1]^2 \rightarrow [0,1] \) satisfying the following conditions:

(i) \( R(x,y) \) is non-increasing w.r.t. \( x \) and is non-decreasing w.r.t. \( y. \)
(ii) \( R(x,y) = 1 \) iff \( x \leq y. \)
(iii) \( R(x,y) \) is left continuous w.r.t. \( x \) and is right continuous w.r.t. \( y \) if the corresponding t-norm \( \odot \) is continuous.

It is well known that different implication operators and valuation lattices, \( L \) (i.e. the set of truth degrees for fuzzy logic), determine different logic systems (see [1,5,6,21]). In this paper, an implication operator \( R \) needs to fulfill the condition that there exists a t-norm \( \odot \) such that:

\[ x \odot y \leq z \text{ iff } x \leq R(y,z). \]

(7)

In that case \( (\odot, R) \) is called an adjoint pair.
Four popularly used implication operators and the corresponding t-norms are defined as follows:

\[ R_L(x, y) = \begin{cases} 
1, & x \leq y \\
(1 - x + y), & x > y
\end{cases}, \quad x \otimes_L y = \max(0, x + y - 1), x, y \in L; \]

\[ R_C(x, y) = \begin{cases} 
1, & x \leq y \\
y, & x > y
\end{cases}, \quad x \otimes_C y = \min(x, y), x, y \in L; \]

\[ R_H(x, y) = \begin{cases} 
1, & x \leq y \\
\frac{1}{y}, & x > y
\end{cases}, \quad x \otimes_H y = x \cdot y, x, y \in L; \]

\[ R_0(x, y) = \begin{cases} 
1, & x \leq y \\
(1 - x) \lor y, & x > y
\end{cases}, \quad x \otimes_n y = \begin{cases} 
\min(x, y), & x + y > 1 \\
0, & x + y \leq 1
\end{cases}, \quad x, y \in L; \]

where \( \otimes_n \) is called the standard nilpotent minimum t-norm (see [2]).

These four implication operators are called Łukasiewicz implication operator \( R_L \), Gödel implication operator \( R_C \), Product implication operator \( R_H \) and \( R_0 \)-implication operator \( R_0 \), respectively.

In 1995, Fodor [2,3] discovered a kind of new t-norm \( \otimes_n \) (nilpotent minimum) and corresponding fuzzy implications based on discussion of the so-called contrapositive symmetry. He wrote, “By these propitious characteristics, the nilpotent minimum can be admitted into investigations of many theoretical and practical problems soon.”

Since 1997, Wang has published some papers on fuzzy logic and fuzzy reasoning (see [12,14–17]). In [14], he proposed a fuzzy implication operator, called \( R_0 \) implication, and constructed a new formal system \( \mathcal{L}^* \) for fuzzy propositional calculus. We see that \( R_0 \) implication is indeed a particular implication based on the standard nilpotent minimum.

If we fix a t-norm \( \otimes \), then a propositional calculus (whose set of truth values is \( L \)) is fixed: \( \otimes \) is taken for the truth function of the strong conjunction \( \land \), and the residuum \( R \) of \( \otimes \) becomes the truth function of the implication operator and \( R(.,0) \) is the truth function of the negation. For more detail, we use the following definitions:

**Definition 1.** [5,6] The propositional calculus \( \text{PC}(\otimes) \) given by a t-norm \( \otimes \) has the set \( S \) of propositional variables \( p_1, p_2, \ldots \) and connectives \( \land, \lor, \neg \). The set \( F(S) \) of well-formed formulas in \( \text{PC}(\otimes) \) is defined inductively as follows: each propositional variable is a formula; if \( A, B \) are formulas, then \( \neg A \). \( A \land B \) and \( A \rightarrow B \) are all formulas.

**Definition 2.** [1,5,6,13] The formal deductive systems of \( \text{PC}(\otimes) \) given by \( \otimes \) corresponding to \( R_L, R_C, R_H \) and \( R_0 \), are called Łukasiewicz fuzzy logic \( \text{Ł} \), Gödel fuzzy logic \( \text{Gd} \), Product fuzzy logic \( \Pi \) and \( R_0 \)-fuzzy logic \( \mathcal{L}^* \) (the nilpotent minimum logic NM), respectively.

A logic system is called an \( n \)-valued logic system if the valuation lattice \( L = \{0, \frac{1}{n-1}, \ldots, \frac{n-1}{n}, 1\} \). A logic system is called a fuzzy logic system if the valuation lattice \( L = \{0, 1\} \).

The inference rule of each logic system above is Modus Ponens (MP): from \( A \) and \( A \rightarrow B \) infer \( B \).

The following formulas are axioms of the basic logic BL [6]:

1. (A1) \((\varphi \rightarrow \psi) \rightarrow ( (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\).
2. (A2) \((\varphi \land \psi) \rightarrow \varphi, \)
3. (A3) \((\varphi \land \psi) \rightarrow (\psi \land \varphi), \)
4. (A4) \((\varphi \land (\varphi \rightarrow \psi)) \rightarrow (\psi \land (\varphi \rightarrow \psi)), \)
5. (A5) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \land \psi) \rightarrow \chi), \)
6. (A6) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\psi \rightarrow (\varphi \rightarrow \chi)) \rightarrow \chi). \)
7. (A7) \(0 \rightarrow \varphi. \)

The following are Łukasiewicz’s axioms [6]:

1. (L1) \( \varphi \rightarrow (\psi \rightarrow \varphi), \)
2. (L2) \( \varphi \rightarrow (\psi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)), \)
3. (L3) \( (\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi), \)
4. ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\psi \rightarrow (\varphi \rightarrow \chi)) \rightarrow \chi). \)

The axioms of \( \Pi \) are those of BL plus [6]

1. (\Pi 1) \( (\varphi \rightarrow \chi) \rightarrow ((\varphi \land \chi) \rightarrow (\varphi \rightarrow \psi)), \) (\Pi 2) \( \varphi \land \neg \varphi \rightarrow 0. \)

The axiom system of Gödel logic is the extension of the axiom system of BL by the single axiom (G) \( \varphi \rightarrow (\varphi \land \varphi) \) stating (together with \( (\varphi \land \varphi) \rightarrow \varphi \) the idempotence of \( \land \).
Axioms of $R_0$ fuzzy logic $\mathcal{L}'$ are [14]:

\begin{align*}
&L'(1)\varphi \rightarrow (\psi \rightarrow \varphi), \quad L'(2)(\neg \varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi), \quad L'(3)(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi)), \\
&L'(4)(\psi \rightarrow \chi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)), \quad L'(5)(\varphi \rightarrow \neg \psi), \quad L'(6)\varphi \rightarrow \varphi \vee \psi, \quad L'(7)\varphi \vee \psi \rightarrow \psi \vee \varphi, \\
&L'(8)(\varphi \rightarrow \chi) \land (\psi \rightarrow \chi) \rightarrow (\psi \lor \psi \rightarrow \varphi), \quad L'(9)(\varphi \land \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi) \lor (\psi \rightarrow \chi), \quad L'(10) \\
(\varphi \rightarrow \psi) \lor ((\varphi \rightarrow \psi) \rightarrow \neg \varphi \rightarrow \psi), \quad \text{where } \varphi \land \psi = \neg (\neg \varphi \lor \neg \psi).
\end{align*}

In 2001, Esteva and Godo, based on Hájek’s work on the basic logic and Hohle’s work on the monoidal logic, proposed a new propositional calculus, called the monoidal t-norm based logic MTL and its two schematic extensions WNM and IMTL, called the weak nilpotent minimum logic and the involutive monoidal t-norm based logic, respectively. Furthermore, they obtained a natural common schematic extension NM of WNM and IMTL, called the nilpotent minimum logic NM (see [1]). Obviously, the system NM and the system $\mathcal{L}'$ are strongly related. In fact, it is easy to verify that two systems are equivalent, and related algebras (NM-algebras and $R_0$-algebras) are essentially the same kind of algebraic structures (see [10]).

**Remark 1.**

(i) One can further define connectives $\land$ and $\lor$ in these four logic systems, but the definition of $\land$ and $\lor$ in $\mathcal{L}'$ are different from those in the other three logic systems: in $\mathcal{L}'$, $\varphi \lor \psi$ is an abbreviation of $\neg((\neg \varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow \varphi) \rightarrow \varphi \rightarrow \psi)$, $\varphi \land \psi$ is that of $\neg(\neg \varphi \land \neg \psi)$. While in Luk, Gd and $\Pi$, $\land$ and $\lor$ can be defined uniformly as follows [6]: $\varphi \lor \psi$ is an abbreviation of $\varphi \lor \varphi \lor \psi$, and $\varphi \land \psi$ is that of $\neg((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi)$. 

(ii) Define on $L$ an unary operator and two binary operators as follows:

\[ x = R(x, 0), \quad xy = x \odot y, \quad x \Rightarrow y = R(x, y), \quad x, y \in L. \]  

where $(\odot, R)$ is an adjoint pair on $L$. From **Remark 1** (ii), the valuation lattice $L$ becomes an algebra of type $(-, \land, \Rightarrow)$. Defined in the above-mentioned logic systems

\[ A^m := \underbrace{A \land \cdots \land A}_{m}, \]  

and in the corresponding algebras $L$

\[ a^{(m)} := \underbrace{a \odot a \cdots \odot a}_{m}, \]  

where $\odot$ is the t-norm defined on $L$.

**Remark 2.** [6,13]. It is easy to verify that the following assertions are true for every $m \in \mathbb{N}$:

(i) in Gd, $a^{(m)} = a$.

(ii) in $\mathcal{L}'$, $a^{(m)} = a^m (m \geq 2)$.

(iii) in Luk, $a^{(m)} = (ma - (m - 1)) \lor 0$.

(iv) in $\Pi$, $a^{(m)} = a^m$.

**Definition 3.** [6,13].

(i) A homomorphism $\nu: F(S) \rightarrow L$ of type $(\neg, \land, \rightarrow)$ into the valuation lattice $L$, that is, $\nu(\neg A) = \neg \nu(A)$, $\nu(A \land B) = \nu(A) \land \nu(B)$, $\nu(A \rightarrow B) = \nu(A) \Rightarrow \nu(B)$, is called an R-valuation of $F(S)$. The set of all R-valuations will be denoted by $\Omega_R$.

(ii) A formula $A \in F(S)$ is called a tautology with respect to $R$ if $\forall \nu \in \Omega_R, \nu(A) = 1$ holds.

(iii) Let $A, B \in F(S)$. If $\nu(A) = \nu(B)$ for every $\nu \in \Omega_R$, then $A$ and $B$ are called logically equivalent, denoted $A \sim B$.

**Remark 3.** [6,13]. It is not difficult to verify in the above-mentioned four logic systems that $\nu(A \lor B) = \max\{\nu(A), \nu(B)\}$, and $\nu(A \land B) = \min\{\nu(A), \nu(B)\}$ for every valuation $\nu \in \Omega_R$. Moreover, one can check in Luk and $L^*$ that $A \land B$ and $\neg(A \rightarrow \neg B)$ are logically equivalent.

**Definition 4.** [6].

(i) A subset of $F(S)$ is called a theory.
3.1. Basic definitions and properties

3. Generalized roots of theories

Lemma 1. Let \( A \in F(S) \), and \( A \) is a finite subset of formulas \( A_1, \ldots, A_m \) such that for each \( 1 \leq i \leq m \), \( A_i \) is an axiom of \( L \), or \( A_i \in \Gamma \), or there are \( j, k \in \{1, \ldots, i - 1\} \) such that \( A_i \) follows from \( A_j \) and \( A_k \) by MP. Equivalently, we say that \( A \) is a conclusion of \( \Gamma \) (or \( \Gamma \)- conclusion). The set of all conclusions of \( \Gamma \) is denoted by \( D(\Gamma) \).

To simplify, we always replace symbol \( \vdash A \) with \( \vdash A \) in this paper.

Theorem 1. [1,6,13]. Completeness theorem holds in every above-mentioned logic system, i.e. \( \forall A \in F(S) \), \( A \) is a theorem in \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM) iff \( A \) is a tautology in \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM), respectively.

Theorem 1 points out that semantics and syntax in these four logic systems are in perfect harmony.

2.2. Generalized deduction theorems in \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM)

Theorem 2. [1,5,6]. Suppose that \( \Gamma \) is a theory, \( A, B \in F(S) \), then in \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM), \( \Gamma \cup \{A\} \vdash B \) iff \( \exists m \in \mathbb{N} \) s.t. \( \Gamma \vdash A^m \rightarrow B \).

Theorem 3. [13]. Suppose that \( \Gamma \) is a theory, \( A, B \in F(S) \), then

(i) in \( \L^+ \) (NM) the generalized deduction theorem holds, i.e. \( \Gamma \cup \{A\} \vdash B \) iff \( \Gamma \vdash A^2 \rightarrow B \).

(ii) in \( \Göd \), deduction theorem holds, i.e. \( \Gamma \cup \{A\} \vdash B \) iff \( \Gamma \vdash A \rightarrow B \).

It is easy for the reader to check the following Lemma:

Lemma 1. Let \( \Gamma \) be a theory and \( A \in F(S) \). If \( \Gamma \vdash A \), then there exists a finite subset of \( \Gamma \), say, \( \{A_1, \ldots, A_m\} \) such that \( \{A_1, \ldots, A_m\} \vdash A \).

Note that \( \&B \rightarrow C \) and \( A \rightarrow (B \rightarrow C) \) are provably equivalent by the definition of the deduction theorem and the generalized deduction theorem. It is easy for the reader to check the following Lemma:

Lemma 2. Suppose that \( \Gamma = (A_1, \ldots, A_m) \) is a finite theory, \( A \in F(S) \). Then

(i) in \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM), \( \Gamma \vdash A \) iff \( \exists n_1, \ldots, n_m \in \mathbb{N} \) such that \( \vdash (A_1^{n_1} \& \ldots \& A_m^{n_m}) \rightarrow A \).

(ii) in \( \Göd \), \( \Gamma \vdash A \) iff \( \vdash (A_1 \& \ldots \& A_m) \rightarrow A \).

(iii) in \( \L^+ \), \( \Gamma \vdash A \) iff \( \vdash (A_1^{n_1} \& \ldots \& A_m^{n_m}) \rightarrow A \).

The following Lemma can be obtained via Lemma 2.2.8 of [6,13]:

Lemma 3. Let \( \Gamma \) be a theory and let \( A, B, C, D \) be formulas. In \( \Luk \), \( \Göd \), \( II \) and \( \L^+ \) (NM), the following conclusions hold:

(i) If \( \Gamma \vdash A \), \( \Gamma \vdash B \), then \( \Gamma \vdash A \& B \).

(ii) If \( \Gamma \vdash A \rightarrow B \), \( \Gamma \vdash C \rightarrow D \), then \( \Gamma \vdash (A \& C) \rightarrow (B \& D) \).

(iii) If \( \Gamma \vdash A \rightarrow B \), then \( \Gamma \vdash (A \& C) \rightarrow (B \& C) \).

(iv) \( \vdash (A \& B) \rightarrow B \).

(v) If \( \Gamma \vdash A \), \( \Gamma \vdash B \), then \( \Gamma \vdash A \vee B \).

(vi) \( \Gamma \vdash (A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C) \).

(vii) \( \Gamma \vdash (A \& B) \rightarrow B \).

3. Generalized roots of theories

3.1. Basic definitions and properties

First let us briefly review the necessary concepts (see [12]).

\( \langle A \rangle \) in a logic system \( L \), define a binary relation \( \prec \) on \( F(S) \) as follows: \( A \prec B \) iff \( \vdash A \rightarrow B \), \( A, B \in F(S) \). Obviously \( \prec \) is reflexive and transitive and hence is a pre-order on \( F(S) \).
In a logic system \( L \), \( A \) is said to be a root of \( \Gamma \) iff \( A \in D(\Gamma) \) and \( \forall B \in D(\Gamma) \ A \not\prec B \), where \( D(\Gamma) \) is the set consisting of all conclusions of \( \Gamma \).

Reference [12] discusses some syntactical conclusions \( B^* \) on problems like GMP and CGMP in classical logic, and gives definitions for solutions of GMP and a root of a theory \( \Gamma \). Based on the root's definition and its properties, reference [12] obtains the result that the Triple I solution \( B^* \) of GMP is a root of \( \Gamma = \{ A \rightarrow B, A^* \} \). That is, the Triple I solution \( B^* \) of GMP is one of the smallest formulas in \( (F(S), \prec) \) satisfying the condition that \( \vdash (A \rightarrow B) \rightarrow (A^* \rightarrow B^*) \). This tells us that the definition of a root, as well as its properties, plays an important role in solving problems like GMP. Unfortunately, it has been proven that a theory \( \Gamma \) in Lukasiewicz propositional logic has no roots in general (see [20]).

This section defines another form of root called a generalized root in Lukasiewicz propositional logic, Gödel propositional logic, Product propositional logic, and \( R_0 \)-propositional logic. We will show that the generalized root and the root defined in [12] have some properties in common. In addition to generalized root properties, we will also discuss the structure of the set of all the conclusions of \( \Gamma \), called \( D(\Gamma) \), and the relations, specifically, inclusion \( \subset \) and equality =, between two conclusion sets, \( D(\Gamma_1) \) and \( D(\Gamma_2) \), when different theories \( \Gamma_1, \Gamma_2 \) are given.

**Definition 5.** Suppose that \( \Gamma \) is a theory and \( A \in F(S) \). If \( A \in D(\Gamma) \) and \( \forall B \in D(\Gamma), \exists m \in N \) such that \( \vdash A^m \rightarrow B \), then \( A \) is called a generalized root of \( \Gamma \).

**Definition 6.** Suppose that \( \Gamma_k \subseteq F(S), k = 1, 2, \ldots, l \). The members of \( \bigcap_{k=1}^{l} D(\Gamma_k) \) are called common conclusions of \( \Gamma_1, \ldots, \Gamma_l \) and \( A \) is said to be a generalized common root of \( \Gamma_1, \ldots, \Gamma_l \) if \( A \in \bigcap_{k=1}^{l} D(\Gamma_k) \), and \( \forall B \in \bigcap_{k=1}^{l} D(\Gamma_k) \exists m \in N \) such that \( \vdash A^m \rightarrow B \).

The following propositions hold in the four logic systems of concern:

**Proposition 1.** Suppose that \( \Gamma \) is a theory and \( A \in F(S) \). If \( A \) is a generalized root of \( \Gamma \), then \( D(\Gamma) = D(A) \), where \( D(A) \) is an abbreviation for \( D(\{A\}) \).

**Proof.** Suppose that \( A \) is a generalized root of \( \Gamma \). Then \( \forall B \in D(\Gamma) \), it follows from Definition 5 that \( \exists m \in N \) such that \( \vdash A^m \rightarrow B \) holds. Since \( \vdash A^m \) by Lemma 3, it follows from the inference rule MP that \( \vdash A \rightarrow B \). This means that \( B \in D(A) \), hence \( D(\Gamma) \subseteq D(A) \). For the converse, \( \forall B \in D(A) \), that is, \( \vdash A \rightarrow B \), it follows from the generalized deduction theorem that \( \exists m \in N \) such that \( \vdash A^m \rightarrow B \). \( \Gamma \vdash A^m \) follows from the assumption \( \Gamma \vdash A \). Therefore we have \( \Gamma \vdash A^m \) by MP, and \( D(A) \subseteq D(\Gamma) \).

**Proposition 2.** Suppose that \( A \in F(S) \). Then \( D(A) = \{ B \in F(S) | \exists m \in N, \vdash A^m \rightarrow B \} \).

**Proof.** Let \( V = \{ B \in F(S) | \exists m \in N, \vdash A^m \rightarrow B \} \). \( \forall C \in D(A) \), it follows from the generalized deduction theorem that \( \exists m \in N \) such that \( \vdash A^m \rightarrow C \). Thus \( C \in V \) and \( D(A) \subseteq V \). For the converse, \( \forall B \in V, \exists m \in N \) such that \( \vdash A^m \rightarrow B \). It is easy to prove that \( \vdash A \rightarrow A^m \) and it follows from the inference rule MP that \( \vdash A \rightarrow B \), that is, \( B \in D(A) \). Hence \( V \subseteq D(A) \) and so \( D(A) = V \).

**Theorem 4.** Every finite theory has a generalized root. More precisely, \( A_1 \& \cdots \& A_s \) is a generalized root of theory \( \Gamma = \{ A_1, \ldots, A_s \} \).

**Proof.** It is only necessary to prove that \( A_1 \& \cdots \& A_s \) is a generalized root of \( \Gamma \). By Lemma 3, \( A_1 \& \cdots \& A_s \in D(\Gamma) \), \( \forall B \in D(\Gamma) \); it follows from Lemma 2 that there exists \( n_1, \ldots, n_s \in N \) such that \( \vdash (A_1^{n_1} \& \cdots \& A_s^{n_s}) \rightarrow B \). Let \( m = \max(n_1, \ldots, n_s) \). We have \( \vdash (A_1 \& \cdots \& A_s)^m \rightarrow (A_1^{n_1} \& \cdots \& A_s^{n_s}) \) by Lemma 3. Hence by Hypothetical Syllogism, we get \( \vdash (A_1 \& \cdots \& A_s)^m \rightarrow B \). Thus \( A_1 \& \cdots \& A_s \) is a generalized root of \( \Gamma \).

**Proposition 3.** Suppose that \( \Gamma_k \subseteq F(S) \) and \( A_k \) is a generalized root of \( \Gamma_k, k = 1, \ldots, l \). Then \( A_1 \& \cdots \& A_s \) is a generalized root of \( \Gamma = \bigcup_{k=1}^{l} \Gamma_k \).

**Proof.** It is easy to prove that \( \Gamma \vdash A_i, i = 1, \ldots, l \) by assumption. Hence by Lemma 3, we get \( \vdash A_1 \& \cdots \& A_s \), that is, \( A_1 \& \cdots \& A_s \in D(\Gamma) \), \( \forall B \in D(\Gamma) \), there exists \( B_1, \ldots, B_s \in \Gamma \) and \( m_1, \ldots, m_s \in N \) such that \( \vdash (B_1^{m_1} \& \cdots \& B_s^{m_s}) \rightarrow B \) by Lemmas 1 and 2. \( \forall B \in \Gamma, i = 1, \ldots, k \), there exist \( j_i \in \{1, \ldots, l\} \) and \( s_i \in N \) such that \( \vdash A_i^{j_i} \rightarrow B_i \), where \( B_i \in \Gamma, i = 1, \ldots, k \). \( \vdash (A_1^{s_1} \& \cdots \& A_s^{s_s}) \rightarrow (B_1^{j_1} \& \cdots \& B_s^{j_s}) \) by Lemma 3. Clearly, there exists \( d_1, \ldots, d_s \in N \) such that \( \vdash (A_1^{d_1} \& \cdots \& A_s^{d_s}) \rightarrow (B_1^{j_1} \& \cdots \& B_s^{j_s}) \). Let \( d = \max(d_1, \ldots, d_s) \). By Lemma 3 and Hypothetical Syllogism, we have \( \vdash (A_1 \& \cdots \& A_s)^d \rightarrow (B_1^{j_1} \& \cdots \& B_s^{j_s}) \). Also by Hypothetical Syllogism \( \vdash (A_1 \& \cdots \& A_s)^k \rightarrow B \) is yielded. This shows that \( A_1 \& \cdots \& A_s \) is a generalized root of \( \Gamma \).

**Proposition 4.** Suppose that \( A \) and \( B \) are generalized roots of \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Then \( D(\Gamma_1) \subseteq D(\Gamma_2) \) iff \( \exists m \in N \) such that \( \vdash B^m \rightarrow A \).
Corollary 1. Suppose that \( \Gamma_1 = \{A_1, \ldots, A_k\} \) and \( \Gamma_2 = \{B_1, \ldots, B_l\} \). Then \( D(\Gamma_1) \subseteq D(\Gamma_2) \) iff \( \exists m \in \mathbb{N} \), such that \( \vdash (B_1 \& \cdots \& B_l)^m \rightarrow (A_1 \& \cdots \& A_k) \).

Proof. By Theorem 4 and Proposition 4, it is easy to check the corollary above.

3.2. Results in the \( n \)-valued Łukasiewicz logic

It has been proven that the semantics and syntax of the \( n \)-valued Łukasiewicz logic are in perfect harmony (see [4]). It is then easy to prove the following propositions:

Proposition 5. Suppose that \( m,n \in \mathbb{N} \). If \( m \geq n - 1 \), then \( \forall a \in L, a^{(m)} = a^{(n-1)} \) holds.

Proposition 6. Suppose that \( A \in F(S) \). If \( m \geq n - 1 \), then \( A^m \cong A^{n-1} \).

Theorem 5. Suppose that \( \Gamma \) is a theory and \( A \in F(S) \). If \( A \) is a generalized root of \( \Gamma \), then \( D(\Gamma) = \{B \in F(S) \mid -A^{n-1} \rightarrow B\} \).

Proof. The theorem above directly follows from Propositions 2 and 6.

Theorem 6. Suppose that \( A_i \) is a generalized root of \( \Gamma_i \) (\( i = 1, 2, \ldots, k \)), then \( \bigvee_{i=1}^{k} A_i^{n-1} \) is a generalized common root of \( \Gamma_1, \ldots, \Gamma_k \).

Proof. By Lemma 3, \( \vdash A_i^{n-1} \rightarrow \bigwedge_{i=1}^{k} A_i^{n-1} \) \( (t = 1, \ldots, k) \) and \( A_i^{n-1} \in D(\Gamma_i) \), it follows from the inference rule MP that \( \bigwedge_{i=1}^{k} A_i^{n-1} \in D(\Gamma_1) \). Since \( \forall B \in \bigcap_{i=1}^{k} D(\Gamma_i), \vdash A_i^{n-1} \rightarrow B \) \( (i = 1, \ldots, k) \) by Theorem 5, then we have \( \vdash (A_i^{n-1} \rightarrow B) \) \( \& \cdots \& (A_k^{n-1} \rightarrow B) \). Since \( (A_i^{n-1} \rightarrow B) \) \( \& \cdots \& (A_k^{n-1} \rightarrow B) \approx (A_1^{n-1} \lor \cdots \lor A_k^{n-1} \rightarrow B) \). Obviously \( \vdash A_i^{n-1} \lor \cdots \lor A_k^{n-1} \rightarrow B \) holds, therefore \( A_i^{n-1} \lor \cdots \lor A_k^{n-1} \) is a generalized common root of \( \Gamma_1, \ldots, \Gamma_k \).

3.3. Results in the Gödel fuzzy logic

Proposition 7. Suppose that \( A \) is a generalized root of \( \Gamma \). Then \( D(\Gamma) = \{B \in F(S) \mid -A \rightarrow B\} \).

Proof. The proposition above directly follows from the definition of the generalized root and the deduction theorem.

Proposition 8. Suppose that \( A \) and \( B \) are generalized roots of \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Then \( D(\Gamma_1) \subseteq D(\Gamma_2) \) iff \( \vdash B \rightarrow A \).

Proof. The proposition above directly follows from Propositions 4 and 7.

Theorem 7. Suppose that \( \Gamma_k \subseteq F(S) \) and \( A_k \) is a generalized root of \( \Gamma_k \) \( (k = 1, 2, \ldots, m) \). Then \( \bigvee_{i=1}^{m} A_i \) is a generalized common root of \( \Gamma_1, \ldots, \Gamma_m \).

Proof. The proof is analogous to that of Theorem 6.

3.4. Results in the \( R_0 \)-fuzzy logic \( L' \)

Lemma 4. [13] Suppose that \( A \in F(S) \). Then \( A^{m} \cong A^{2^m} \), \( m = 2, 3, \ldots \)

Proposition 9. Suppose that \( A \) is a generalized root of \( \Gamma \). Then \( D(\Gamma) = \{B \in F(S) \mid -A^{2} \rightarrow B\} \).

Proof. The proof is analogous to that of Theorem 5.
**Proposition 10.** Suppose that $A$ and $B$ are generalized roots of $\Gamma_1$ and $\Gamma_2$, respectively. Then $D(\Gamma_1) \subseteq D(\Gamma_2)$ iff $B^2 \rightarrow A^2$.

**Proof.** The proposition above directly follows from Lemmas 3, 4 and Proposition 9. □

**Theorem 8.** Suppose that $\Gamma_k \subseteq F(S)$ and $A_k$ is a generalized root of $\Gamma_k (k = 1, 2, \ldots, m)$. Then $\forall^m_i A_i^2$ is a generalized common root of $\Gamma_1, \ldots, \Gamma_m$.

**Proof.** The proof is analogous to that of Theorem 6. □

4. Definitions and expressions of the solutions of GMP and CGMP in Łuk, Göd, IJ and Ł’ (NM)

Now we turn to the problems of GMP and CGMP. Our aim is to find a suitable conclusion $B^*$ of the prerequisites $A \rightarrow B$ and $A^*$. $A \rightarrow B, A^*$ and $B^*$ should certainly satisfy the following condition:

$$\{A \rightarrow B, A^*\} \vdash B^*.$$ (11)

Obviously there are too many $B^*$ that satisfy (11). For example, (11) holds if $B^*$ is a tautology (theorem), but it is clearly not what we need since a tautology is a conclusion of any $A \rightarrow B$ and $A^*$ is given as prerequisites. Hence, a tautology is not a suitable conclusion for GMP. By (A1), it is reasonable to ask for $B^*$ in GMP to be as small as possible.

**Definition 7.** Let $A, B, A^*, B^* \in F(S)$. $B^*$ is called a Triple $I_2$ solution of GMP if it satisfies the following conditions:

$$(G_1) \{A \rightarrow B, A^*\} \vdash B^*;$$

$$(G_2)$$ if $C$ satisfies $(G_1)$, then $B^* \prec C \in F(S))$.

**Theorem 9.** Let $\Gamma = \{A \rightarrow B, A^*\}$. Then Triple $I_2$ solution of GMP $B^*$ is a generalized root of $\Gamma$.

**Proof.** By Definition 7 $(G_1)$, $B^* \in D(\Gamma)$.

Since $A^* \& (A \rightarrow B)$ is a generalized root of $\Gamma$ by Theorem 4, then, by Definition 5, $\forall C \in D(\Gamma)$, there is an $m$ such that $\vdash((A \rightarrow B) \& A^*)^m \rightarrow C$ and $(A \rightarrow B) \& A^* \in D(\Gamma)$. $B^* \rightarrow ((A \rightarrow B) \& A^*)$ by Definition 7 $(G_2)$, thus, $\vdash(B^*)^m \rightarrow ((A \rightarrow B) \& A^*)^m$ by Lemma 3. It follows from Hypothetical Syllogism that $\vdash(B^*)^m \rightarrow C$, that is, $B^*$ is a generalized root of $\Gamma$. □

**Theorem 10.** Suppose that $\Gamma = \{A \rightarrow B, A^*\}, A, B, A^* \in F(S)$.

(i) in the n-valued Łukasiewicz logic $L_n$, the generalized root $B^* = (A^*)^{n-1} \& (A \rightarrow B)^{n-1}$ of $\Gamma$ is a Triple $I_2$ solution of GMP.

(ii) in $\text{L}^*$, the generalized root $B^* = (A^*)^2 \& (A \rightarrow B)^2$ of $\Gamma$ is a Triple $I_2$ solution of GMP.

(iii) in Gödel, the generalized root $B^* = A^* \& (A \rightarrow B)$ of $\Gamma$ is a Triple $I_2$ solution of GMP.

**Proof.** Because the proofs of (i), (ii), and (iii) are all similar, here we only prove (i). In the $n$-valued Łukasiewicz logic $L_n$, since the deduction theorem means that

$$\Gamma \cup A \vdash B \text{ iff } \Gamma \vdash A^{n-1} \rightarrow B, \Gamma \subseteq F(S), A, B \in F(S).$$ (12)

Moreover, it is well known in $L_n$ that

$$A \rightarrow (B \rightarrow C) \Rightarrow (A \& B) \rightarrow C, A, B, C \in F(S),$$ (13)

where $\approx$ is the logical equivalence relation. Since $\vdash (A^*)^{n-1} \& (A \rightarrow B)^{n-1} \rightarrow B^*$ is clearly true, and it follows from (12) and (13) that the following assertions are also true:

$$\vdash (A^*)^{n-1} \rightarrow ((A \rightarrow B)^{n-1} \rightarrow B^*), \{A^*\} \vdash (A \rightarrow B)^{n-1} \rightarrow B^*, \{A \rightarrow B, A^*\} \vdash B^*.$$ (11)

Hence $B^*$ satisfies condition $(G_1)$. Assume that $C$ satisfies $(G_1)$, that is, $\{A \rightarrow B, A^*\} \vdash C \in F(S)$. Then it follows from the deduction theorem of $L_n$ that

$$\{A^*\} \vdash (A \rightarrow B)^{n-1} \rightarrow C, \{A^*\} \vdash (A^*)^{n-1} \rightarrow ((A \rightarrow B)^{n-1} \rightarrow C).$$

Therefore it follows from (13) that $\vdash (A^*)^{n-1} \& (A \rightarrow B)^{n-1} \rightarrow C$ and $B^* \prec C$, i.e. $B^*$ satisfies condition $(G_2)$. The proof of Theorem 10 is complete. □
Definition 8. Let $I_i = \{A_i \rightarrow B, A'_i\}$ and $A_i, B, A'_i, B' \in F(S)(i = 1, 2, \ldots, m)$. $B^*$ is called a Triple $I_2$ solution of CGMP if it satisfies the following conditions:

\begin{align*}
(\text{D}_1) & \quad I_i \vdash B^*(i = 1, 2, \ldots, m); \\
(\text{D}_2) & \quad \text{if } C \in \cap_{i=1}^m D(I_i), \text{ then } B^* \not\prec C, C \in F(S).
\end{align*}

Remark 4. It is easy to verify that if $B^*$ is a Triple $I_2$ solution of CGMP, then $B^*$ is a generalized common root of $\Gamma_1, \ldots, \Gamma_1$ by Definitions 5 and 8.

Theorem 11. Suppose that $\Gamma_1 = \{A_i \rightarrow B, A'_i\}, A_i, B, A'_i \in F(S), i = 1, 2, \ldots, m$.

1. in the $n$-valued Łukasiewicz logic $L_n$,
\[
((A_i)^{-1} \& (A_1 \rightarrow B)^{-1}) \vee \cdots \vee ((A'_m)^{-1} \& (A_m \rightarrow B)^{-1})
\]
2. is a Triple $I_2$ solution of CGMP.

Proof. Because the proofs of (i), (ii), and (iii) are all similar, here we only prove (i).

In the $n$-valued Łukasiewicz logic $L_n$, $A_i \& (A_i \rightarrow B)$ is a generalized root of $I_i$ by Theorem 4, and $((A_i)^{-1} \& (A_1 \rightarrow B)^{-1}) \vee \cdots \vee ((A'_m)^{-1} \& (A_m \rightarrow B)^{-1})$ is a generalized common root of $I_1, \ldots, I_m$ by Theorem 6. Then $((A_i)^{-1} \& (A_1 \rightarrow B)^{-1}) \vee \cdots \vee ((A'_m)^{-1} \& (A_m \rightarrow B)^{-1})$ satisfies condition (D1).

\[
\forall C \in \cap_{i=1}^m D(I_i), \quad \vdash ((A_i)^{-1} \& (A_1 \rightarrow B)^{-1}) \vee \cdots \vee ((A'_m)^{-1} \& (A_m \rightarrow B)^{-1}) \land C
\]

Remark 5. [12]. Suppose that $\Omega_R$ is the set consisting of all valuations from $F(S)$ to $L$ (see Definition 3), where $L$ is the set of truth degrees for fuzzy logic and $F(S)$ is the set of well-formed formulas (see Definition 1). Suppose that $v \in \Omega_R$ and $A \in F(S)$, then $v(A)$ is the value of $v$ at $A$. Let us also agree to use the symbol $A(v)$ to denote $v(A)$. A formula $A$ induces a mapping $A: \Omega_R \rightarrow L$ (denoted by $A$, itself) naturally defined by $\forall v \in \Omega_R, A(v) = v(A)$ where $v(A)$ is the value of $v$ at $A$.

By Theorems 9, 10 and Remark 5, the following algorithms are obvious.

Lemma 5. The FMP-type Triple $I_2$ solution $B^*$ of GMP can be computed as follows:

\begin{align*}
(\text{i}) & \quad \text{in the n-valued Łukasiewicz logic } L_n, B^*(v) = \sup \{((A(u))^n \& (A(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}; \\
(\text{ii}) & \quad \text{in } L^*, B^*(v) = \sup \{((A(u))^n \& (A(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}; \\
(\text{iii}) & \quad \text{in Gödel, } B^*(v) = \sup \{((A(u))^n \& (A(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}.
\end{align*}

Lemma 6. The FMP-type Triple $I_2$ solution $B^*$ of CGMP can be computed as follows:

\begin{align*}
(\text{i}) & \quad \text{in the n-valued Łukasiewicz logic } L_n, B^*(v) = \sup \{((A_i(u))^n \& (A_i(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}; \\
(\text{ii}) & \quad \text{in } L^*, B^*(v) = \sup \{((A_i(u))^n \& (A_i(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}; \\
(\text{iii}) & \quad \text{in Gödel, } B^*(v) = \sup \{((A_i(u))^n \& (A_i(u) \rightarrow B(v))^n)^{\uparrow} \mid u \in X, v \in Y\}.
\end{align*}

Example 1. Suppose that $X = Y = [0, 1], A, A^* \in F(X), B \in F(Y)$ are as follows:

$A(u) = \frac{u^2}{u^2 + 1}, B(v) = 1 - v, A^*(u) = 1 - u$ and $u, v \in [1, 0]$. In the 7-valued Łukasiewicz logic $L_7$, compute the FMP-type Triple $I_2$ solution $B^*$ of GMP by using Lemma 5.
**Solution.** Since $A'(u) \otimes (A(u) \rightarrow B(v)) = (1 - u) \otimes \left[\frac{1 + \frac{2}{u}}{u}\right] \rightarrow (1 - v)$, it follows from the definition of implication operator and the corresponding t-norm in this logic that

$$
A(u) \rightarrow B(v) = \left(1 - \frac{u + 2}{3} + 1 - v\right) \wedge 1 = \begin{cases} 
\frac{4 - u - 3v}{3}, & v > \frac{1 - u}{3} \\
1, & v \leq \frac{1 - u}{3}.
\end{cases}
$$

$$(A(u) \rightarrow B(v))^6 = \begin{cases} 
\frac{9 - 6u - 18v}{2}, & \frac{1 - u}{3} < v < \frac{3 - 2u}{6} \\
1, & v \leq \frac{1 - u}{3} \\
0, & v \geq \frac{3 - 2u}{6}.
\end{cases}$$

$$(A'(u))^6 = \begin{cases} 
1 - 6u, & u \leq \frac{1}{6} \\
0, & u > \frac{1}{6}.
\end{cases}$$

$$(A'(u))^6 \otimes (A(u) \rightarrow B(v))^6 = \begin{cases} 
3 - 8u - 6v, & \frac{1 - u}{3} < v < \frac{3 - 2u}{6}, u < \frac{1}{6} \\
1 - 6u, & v \leq \frac{1 - u}{3}, u < \frac{1}{6} \\
0, & \text{otherwise}.
\end{cases}$$

$$B'(v) = \sup\{(A'(u))^6 \otimes (A(u) \rightarrow B(v))^6|u \in [0, 1]\} = \begin{cases} 
1, & v \leq \frac{1}{2} \\
3 - 6v, & \frac{1}{2} < v < \frac{1}{2} \\
0, & \text{otherwise}.
\end{cases}$$

**Example 2.** Suppose that $X, Y, A, A'$, and $B$ are the same as those in Example 1. In $L^*$, compute the FMP-type Triple $l_2$ solution $B^*$ of GMP by using Lemma 5.

**Solution.** Since

$$(A'(u))^2 = \begin{cases} 
1 - u, & u < \frac{1}{2} \\
0, & u \geq \frac{1}{2}.
\end{cases}$$

$$(A(u) \rightarrow B(v)) = \begin{cases} 
1, & v \leq \frac{1 - u}{3} \\
\frac{1 - u}{3}, & v > \frac{2 - u}{3} \\
1 - v, & \frac{1 - u}{3} < v \leq \frac{2 - u}{3}.
\end{cases}$$

$$(A(u) \rightarrow B(v))^2 = \begin{cases} 
1, & v \leq \frac{1 - u}{3} \\
0, & v > \frac{2 - u}{3} \\
1 - v, & \frac{1 - u}{3} < v \leq \frac{2 - u}{3}, v < \frac{1}{2}. \\
0, & \frac{1 - u}{3} < v \leq \frac{2 - u}{3}, v \geq \frac{1}{2}.
\end{cases}$$

$$(A'(u))^2 \otimes (A(u) \rightarrow B(v))^2 = \begin{cases} 
1 - u, & \frac{1 - u}{3} < v \leq \frac{2 - u}{3}, u < \frac{1}{2}, v \leq u; \text{ or } u < \frac{1}{2}, v < \frac{1 - u}{3}, v \leq \frac{1 - u}{3} \\
1 - v, & v \leq \frac{1}{2}, u < v, \frac{1 - u}{3} < v \leq \frac{2 - u}{3} \\
0, & \text{otherwise}.
\end{cases}$$

$$B'(v) = \begin{cases} 
1, & v \leq \frac{1}{2} \\
1 - v, & \frac{1}{2} < v \leq \frac{1}{2} \\
0, & v \geq \frac{1}{2}.
\end{cases}$$

**Example 3.** Suppose that $X, Y, A, A'$, and $B$ are the same as those in Example 1. Compute the FMP-solutions $B^*$ by using Zadeh's CRI method, and the conclusion of FMP is as follows [13]:

$$B'(v) = \sup\{A'(u) \land (A(u) \rightarrow B(v))|u \in X\}, \quad v \in Y,$$

where the implication operator is a Łukasiewicz's implication operator.

**Solution.** Since $A'(u) \land (A(u) \rightarrow B(v)) = (1 - u) \land \left[\frac{1 + \frac{2}{u}}{u}\right] \rightarrow (1 - v)$, it follows from the definition of implication operator that

$$A(u) \rightarrow B(v) = (1 - \frac{u + 2}{3} + 1 - v) \land 1 = \begin{cases} 
\frac{4 - u - 3v}{3}, & v > \frac{1 - u}{3} \\
1, & v \leq \frac{1 - u}{3}.
\end{cases}$$

$$A'(u) \land (A(u) \rightarrow B(v)) = \begin{cases} 
\frac{4 - u - 3v}{3}, & v > \frac{1 - u}{3} \\
1 - u, & v \leq \frac{1 - u}{3}.
\end{cases}$$

Therefore we have $B'(v) = \begin{cases} 
\frac{1}{2} \land v, & v > \frac{1}{2} \\
\frac{1}{2}, & v \leq \frac{1}{2}.
\end{cases}$
6. Concluding remarks

This paper proposes a new concept called a generalized root of a theory, $\Gamma$, in Łukasiewicz propositional fuzzy logic Łuk, Gödel propositional fuzzy logic Göd, Product propositional fuzzy logic P, and the nilpotent minimum logic NM (the $R_0$-propositional fuzzy logic $L'C'$). This concept is based on the generalized deduction theorem and the completeness theorem. Then it is proved that $D(\Gamma)$ is completely determined by its generalized root whenever $\Gamma$ has a generalized root. In Łukasiewicz propositional fuzzy logic, the $n$-valued Łukasiewicz propositional logic, Gödel propositional fuzzy logic, Product propositional fuzzy logic and the nilpotent minimum logic (the $R_0$-propositional fuzzy logic), every finite theory has a generalized root. The conditions for the sets $D(\Gamma)$ of two different theories being equal and included are given. Finally, syntactic-form conclusions of problems like GMP, CGMP were drawn in Łuk, Göd, P and NM ($L'C'$), via the generalized root of a theory, $\Gamma$, and their numerical versions are provided as new algorithms for solving GMP and CGMP.

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