An ILOWG operator based group decision making method and its application to evaluate the supplier criteria

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ABSTRACT

The aim of this work is to present some cases of the induced linguistic ordered weighted geometry (ILOWG) operators and study their desired properties, which are very suitable to deal with group decision making (GDM) problems involving multiplicative linguistic preference relations. First, the concepts of compatibility index (CI) for two multiplicative linguistic preference relations are defined. Then, we provide some ILOWG operators to aggregate multiplicative linguistic preference relations in GDM problems. In particular, we present the compatibility index ILOWG (CI-ILOWG) operator, which induces the order of argument values by utilizing the compatibility index of experts; and the importance ILOWG (I-ILOWG) operator, which induces the order of argument values based on the importance index of the experts. Next, the reciprocity, consistency and compatibility properties of the collective multiplicative linguistic preference relations obtained by these cases of ILOWG operators are verified. Finally, the aggregation of individual judgements (AIJ) and the aggregation of individual priorities (AIP) provide the same priorities of alternatives by utilizing the row geometric mean method (RGMM) as a prioritization procedure and the ILOWG operators as an aggregation procedure. Our results show that if all the individual decision makers have an acceptable consensus degree, then the collective preference relation is also of an acceptable consensus degree. Moreover, the compatibility index induced linguistic ordered weighted geometric mean complex judgement matrix (CI-ILOWGCJM) guarantees that the group compatibility degree is at least as good as the arithmetic mean of all the individual compatibility degrees. Accordingly, a theoretic basis has been developed for the application of these cases of ILOWG operators in linguistic group decision making. Finally, a numerical example for evaluating criteria of supply selection is given to illustrate the application of the results.

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1. Introduction

Group decision making (GDM) consists of multiple individuals interacting to reach a final decision, which follows a common resolution scheme composed by the following two phases [1]: (1) Aggregation phase and (2) Exploitation phase. The aggregation phase combines individual preferences to obtain a collective value for each alternative, which reflects all of the properties contained in all individual preferences. However, according to Arrow’s impossible theory, it is impossible to aggregate individual preferences into group preference in a completely rational way [2]. There arise situations of conflict and disagreement among preferences of experts. Therefore, finding a group consensus to represent a common opinion of the group is a fundamental issue under a group decision making environment [3–8].

As a result, there is one problem must be solved, that is to say, how to choose an aggregation operator to achieve a higher degree of the group experts’ consensus or agreement. Yager [9] provided a family of averaging operator called the
ordered weighted averaging (OWA) operators, which allow the implementation of the concept of fuzzy majority and has been implemented extensively in the resolution process of different problems [10–19]. Afterwards, Yager and Filev [20] presented an induced OWA (IOWA) operator in which the ordering of the \( a_i \) (\( i \in n \)) is induced by other variables \( u_i \) (\( i \in n \)) called the order inducing variables, where \( a_i \) and \( u_i \) are the components of the OWA pairs \( (u_i, a_i) \) (\( i \in n \)). The IOWA operator is a more general type of the OWA operator and the arithmetic mean (AM) operator. But, it behaves in a similar way as the arithmetic mean, and is somewhat unsuitable for aggregating decision information taking the form of multiplicative preference relation. Therefore, Xu [21] introduced the induced ordered weighted geometric (IOWG) operator to deal with multiplicative preference relations, which behaves in a similar way as the geometric mean operator. Furthermore, Chiclana et al. [22] presented some extended IOWG operators: the importance IOWG (I-IOWG) operator, the consistency IOWG (C-IOWG) operator, and the preference IOWG (P-IOWG) operator. However, these IOWG operators are somewhat unsuitable for dealing with multiplicative decision-making problems under uncertainty. For example, it is unsuitable for aggregating decision information taking the form of interval multiplicative preference relation [23,24]: \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \) where \( \tilde{a}_{ij} = [a^L_{ij}, a^U_{ij}] \), \( \tilde{a}_{ij} = a^L_{ij}, a^L_{ij} \times a^L_{ji} = a^U_{ij} + a^L_{ji} = 1, a^+_j \geq a^-_j > 0, a^-_n = a^+_n = 1, \) for all \( i,j = 1,2,\ldots n \). To deal with interval multiplicative preference relations in GDM, we [25] developed the induced continuous ordered weighted geometric (ICOWG) operator, which is able to contain the reciprocity and consistency properties of the collective preference relations. In particular, we presented the reliability induced COWG (R-ICOWG) operator and the relative consensus degree induced COWG (RCD-ICOWG) operator. 

The aim of this work is to introduce some extended ILOWG operators for GDM problems with multiplicative linguistic preference relations. To that end, we define the concept of compatibility index for two multiplicative linguistic preference relations. The smaller the compatibility index, the higher the agreement degree of two preference relations. Then, we present the two particular cases of ILOWG operators:

(I) In a homogeneous GDM problem, the experts have equal importance. However, in this situation, each expert always has a compatibility index (CI) value associated with them. So, we present the compatibility index ILOWG (CI-ILOWG) operator, which induces the order of argument values based on the compatibility index of the experts and aggregates of experts’ preferences in such a way that more importance is placed on the most consistent or agreeable ones. Recently, Xu [26] introduced the induced linguistic ordered weighted geometric (ILOWG) operator, which is suitable to aggregate linguistic multiplicative preference relations. However, knowledge on the use of the ILOWG operator as an aggregation method is still quite limited. For example, few studies have been focused on the question of whether the ILOWG operator can improve the group consensus degree.

The aim of this work is to introduce some extended ILOWG operators for GDM problems with multiplicative linguistic preference relations. To do that, we define the concept of compatibility index for two multiplicative linguistic preference relations. The smaller the compatibility index, the higher the agreement degree of two preference relations. Then, we present the two particular cases of ILOWG operators:

(I) In a homogeneous GDM problem, the experts have equal importance. However, in this situation, each expert always has a compatibility index (CI) value associated with them. So, we present the compatibility index ILOWG (CI-ILOWG) operator, which induces the order of argument values based on the compatibility index of the experts and aggregates of experts’ preferences in such a way that more importance is placed on the most compatible one. The compatibility index induced linguistic ordered weighted geometric mean complex judgement matrix (CI-ILOWGCJM) guarantees that the group compatibility degree is more than the arithmetic mean of all the individual compatibility degrees.

(II) The second one is the importance ILOWG (I-ILOWG) operator, which induces the order of argument values based on the importance index of the experts. Obviously, the I-ILOWG operator may be applied just when the GDM problem is classified as heterogeneous. We prove that if all the experts have an acceptable compatibility degree, then importance induced linguistic ordered weighted geometric mean complex judgement matrix (I-ILOWGCJM) is also of an acceptable compatibility degree.

Furthermore, we shall show that the collective preference relations obtained by these cases of ILOWG operators verified the reciprocity and consistency properties. Utilizing the row geometric mean method (RGMM) as a prioritization procedure and the ILOWG operators as an aggregation procedure, it is proved that the aggregation of individual judgements (AIJ) and the aggregation of individual priorities (AIP) provide the same priorities of alternatives.

In order to do that, this work is set out as follows. Section 2 briefly reviews some basic concepts such as the IOWA, IOWG and ILOWG operators. Section 3 defines the concept of compatibility index for two multiplicative linguistic preference relations. In Section 4, we present two particular cases of ILOWG operator: the CI-ILOWG operator and the I-ILOWG operator, and analyze their desirable properties. Section 5 gives a practical example to illustrate the developed procedures. Section 6 analyses the limitations of the ILOWG operator. Finally, in Section 7 we draw our conclusions.

2. Preliminaries: IOWA, IOWG and ILOWG operators

We start this section by providing a summary of the concepts of IOWA, IOWG and ILOWG operators.

2.1. The IOWA and IOWG operator

Recently, Yager and Filev [20] presented an induced OWA (IOWA) operator in which the ordering of the \( a_i \) (\( i \in n \)) is induced by other variables \( u_i \) (\( i \in n \)) called the order inducing variables, where \( a_i \) and \( u_i \) are the components of the OWA pairs \( (u_i, a_i) \) (\( i \in n \)).

**Definition 1.** An IOWA operator of dimension \( n \) is a mapping, \( \Phi_W : (R \times R) \rightarrow R \), to which a set of weights or a weighting vector is associated, \( W = (w_1, w_2, \ldots, w_n)^\top, w_j \in [0, 1] \) and \( \sum_{j=1}^n w_j = 1 \), and it is defined to aggregate the set of second
arguments of list of two tuples $\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle$ given on the basis of a positive ratio scale, according to the following expression:

$$\Phi_W (\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j a_{\sigma(j)}$$  \hspace{1cm} (1)

where $a_{\sigma(j)}$ is the jth largest element of the collection of the aggregated objects $a_1, a_2, \ldots, a_n$. $u_i$ in $\langle u_i, a_i\rangle$ is referred to as the order inducing variable and $a_i$ as the argument variable.

The IOWA operator is a more general type of the OWA operator and the arithmetic mean (AM) operator, which is somewhat unsuitable for aggregating decision information taking the form of multiplicative preference relation. Based on the IOWA operator and OWG operator, Xu [21] introduced the induced ordered weighted geometric (IOWG) operator, which is able to deal with multiplicative preference relations. The IOWG operator is defined in the following:

**Definition 2.** An IOWG operator of dimension n is a mapping, $\Phi_W : (R \times R) \rightarrow R$, to which a set of weights or a weighting vector is associated, $W = (w_1, w_2, \ldots, w_n)^T$, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and it is defined to aggregate the set of second arguments of a list of two tuples $\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle$, given on the basis of a positive ratio scale, according to the following expression:

$$\Phi^G_W (\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \prod_{j=1}^{n} (a_{\sigma(j)})^{w_j}$$  \hspace{1cm} (2)

where $a_{\sigma(j)}$ is the jth largest element of the collection of the aggregated objects $a_1, a_2, \ldots, a_n$. $u_i$ in $\langle u_i, a_i\rangle$ is referred to as the order inducing variable and $a_i$ as the argument variable.

2.2. The ILOWG operator

Xu [27] introduced some basic notations and operational laws of linguistic variables. Let $S = \{s_\alpha | \alpha = 1/t, 1/2, 1, 2, \ldots, t\}$ be a multiplicative linguistic label set with odd cardinality. Any label, $s_\alpha$, represents a possible value for a linguistic variable, and it is required that the multiplicative linguistic label set should satisfy the following characteristics:

1. The set is ordered: $s_\alpha > s_\beta$ if $\alpha > \beta$;
2. There is the reciprocal operator: $\text{rec}(s_\alpha) = s_\beta$ such that $\alpha \beta = 1$.

This linguistic label set $S$ is called the multiplicative linguistic scale. For example, $S$ can be defined as:

$$S = \{s_{1/5} = \text{extremely poor}, s_{1/4} = \text{very poor}, s_{1/3} = \text{poor}, s_{1/2} = \text{slightly poor}, s_1 = \text{fair}, s_2 = \text{slightly good}, s_3 = \text{good}, s_4 = \text{very good}, s_5 = \text{extremely good}\}$$

To preserve all the given information, Xu [27] extended the discrete linguistic label set $S$ to a continuous linguistic term set $\hat{S} = \{s_\alpha | \alpha \in [1/q, q]\}$, where $q$ ($q > 1$) is a sufficiently large positive integer. If $s_\alpha \in S$, then call $s_\alpha$ the original linguistic term, otherwise, we call $s_\alpha$ the virtual linguistic term. Considering any two linguistic terms $s_\alpha, s_\beta \in \hat{S}$, and $u_1, u_2 \in [0, 1]$, several operational laws are introduced as follows:

1. $(s_\alpha)^0 = s_\alpha$;
2. $(s_\alpha)\alpha_1 \otimes (s_\beta)^\omega_2 = (s_\alpha)^{\omega_1 + \omega_2}$;
3. $(s_\alpha \otimes s_\beta)^n = (s_\alpha)\omega_1 \otimes (s_\beta)\omega_2$;
4. $s_\alpha \otimes s_\beta = s_{\alpha + \beta}$;
5. $s_\alpha \otimes s_\beta = s_{\alpha + \beta}$.

In [28], Xu developed an induced linguistic ordered weighted geometric (ILOWG) operator to aggregate multiplicative linguistic information.

**Definition 3.** An ILOWG operator can be defined as follows:

$$\text{ILOWG}_\gamma (\langle u_1, s_{a_1}\rangle, \langle u_2, s_{a_2}\rangle, \ldots, \langle u_n, s_{a_n}\rangle) = (s_{\gamma_1})^{\omega_1} \otimes (s_{\gamma_2})^{\omega_2} \otimes \cdots \otimes (s_{\gamma_n})^{\omega_n} = s_\gamma$$  \hspace{1cm} (3)

where $\gamma = \prod_{j=1}^{n} \gamma_j$ $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the exponential weighting vector of the $s_\gamma_j$, and $\omega_j \in [0, 1]$. $s_\gamma_j$ is the value of the ILOWG pair $\langle u_j, s_j \rangle$ having the jth largest $u_j$, and $u_i$ in $\langle u_j, s_j \rangle$ is referred to as the order-inducing variable and $s_\gamma_j$ as the multiplicative linguistic argument variable.

**Definition 4.** If an expert provides his/her preference relation $R = (r_{ij})_{n \times n}$ on $X$ according to the multiplicative linguistic scale $S = \{s_\alpha | \alpha = 1/t, 1/2, 1, 2, \ldots, t\}$, whose element $r_{ij}$ estimates the preference degree of alternative $x_i$ over $x_j$, and satisfies

$$r_{ij} \geq s_{1/t}, \quad r_{ij} \otimes r_{ji} = s_1, \quad s_\beta = s_1, \quad \text{for all } i, j \in N$$

then we call $R$ the multiplicative linguistic preference relation.
**Definition 5.** Let \( R = (r_{ij})_{n \times n} \) be multiplicative linguistic preference relation, we call \( R \) a consistent multiplicative linguistic preference matrix, if
\[
    r_{ij} = r_{ik} \otimes r_{kj}, \quad \text{for all } i, j \in N.
\]

**Definition 6.** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times n \) multiplicative linguistic matrices, \( a_{ij}, b_{ij} \in S \), the Hardmard product of \( A, B \) can be denoted by
\[
    C = A \circ B = (c_{ij}),
\]
where \( c_{ij} = a_{ij} \otimes b_{ij} \), for all \( i, j \in N \).

### 3. The definition of compatibility index

In GDM problems, there usually arise situations of conflict and agreement in the preferences of experts. Thus, finding a group consensus to represent a common opinion of the group is an important issue under a group decision environment. The concept of compatibility was defined by Saaty [29] and used to reflect the consensus degree or agreement degree in GDM [30,31]. Motivated by the ideas of these works, we express the definition of compatibility index under a group decision environment with multiplicative linguistic preference relation as follows.

**Definition 7.** Let \( A = (a_{ij})_{n \times n} \in R \) and \( b = (b_{ij})_{n \times n} \in R \), then
\[
    \text{Cl}(A, B) = \frac{1}{n^2} e^T A \circ B^T e = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}
\]
is called the compatibility index of \( A \) and \( B \), where \( e^T = (1, 1, \ldots, 1) \).

If we let the lower indices of linguistic values correspond to \( I(a_{ij}) \) and \( I(b_{ij}) \) respectively, then Eq. (5) can be rewritten as
\[
    \text{Cl}(A, B) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} I(a_{ij}) I(b_{ij}).
\]

Obviously, the smaller the value of compatibility index \( \text{Cl}(A, B) \), the greater the agreement degree of the multiplicative linguistic preferences \( A \) and \( B \).

**Theorem 1.** Let \( A = (a_{ij})_{n \times n} \in R \) and \( b = (b_{ij})_{n \times n} \in R \), then
(1) \( \text{Cl}(A, B) \geq 1 \);
(2) \( \text{Cl}(A, B) = 1 \) if and only if \( A \) and \( B \) are perfectly compatible.

**Proof.** (1)
\[
    \text{Cl}(A, B) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} I(a_{ij}) I(b_{ji})
    = \frac{1}{n^2} \left[ \sum_{i,j}^{n} \left( I(a_{ij}) I(b_{ji}) + I(a_{ji}) I(b_{ij}) \right) + \sum_{i=1}^{n} I(a_{ii}) I(b_{ii}) \right]
    \geq \frac{1}{n^2} \left( \frac{n(n-1)}{2} \times 2 + n \right) = 1.
\]

(2) Necessity. If \( \text{Cl}(A, B) = 1 \), then \( I(a_{ij}) I(b_{ji}) = I(a_{ji}) I(b_{ij}) \) for all \( i, j \in N \). Thus, we can obtain \( a_{ij} = b_{ij} \) for all \( i, j \in N \). Therefore, \( A \) and \( B \) are perfectly compatible.

(3) Sufficiency. If \( A \) and \( B \) are perfectly compatible, then \( a_{ij} = b_{ij} \) for all \( i, j \in N \). Thus, we have \( I(a_{ij}) \times I(b_{ij}) = 1 \) for all \( i, j \in N \). Therefore, \( \text{Cl}(A, B) = 1 \). \( \Box \)

**Theorem 2.** Let \( A, B, G \in R \), then we have
(1) Reflexivity: \( \text{Cl}(A, A) = 1 \),
(2) Symmetry: \( \text{Cl}(A, B) = \text{Cl}(B, A) \),
(3) Transitivity: If \( \text{Cl}(A, B) = 1 \) and \( \text{Cl}(B, G) = 1 \), then \( \text{Cl}(A, G) = 1 \).

**Proof.** (1)
\[
    \text{Cl}(A, A) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} I(a_{ij}) I(a_{ji}) = 1.
\]
Since  
If  
If a set of experts
Let $A$ be the set of compatibility index values obtained by using the CI-ILOWG operator.

**Definition 8.** $CI-ILOWG$, which is obtained by using CI-ILOWG operator.

4. Some ILOWG operators to aggregate multiplicative linguistic preference relations

In this section, we implement a CI-ILOWG operator to aggregate individual preferences in such a way that the greater weighting value is placed on that information.

Theorem 3. Let $A = (a_{ij})_{n \times n} \in R$, $B = (b_{ij})_{n \times n} \in R$ and $G = (g_{ij})_{n \times n} \in R$, then

1. $\min_{ij}(I(a_{ij})I(b_{ij}))CI(B, G) \leq CI(A, G) \leq \max_{ij}(I(a_{ij})I(b_{ij}))CI(B, G)$
2. $\min_{ij}(I(d_{ij})I(g_{ij}))CI(A, B) \leq CI(A, G) \leq \max_{ij}(I(d_{ij})I(g_{ij}))CI(A, B)$.

Proof. (1) Since

$$CI(A, G) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} l(a_{ij})l(g_{ij}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} l(a_{ij})l(b_{ij})l(g_{ij}).$$

Thus, we can obtain

$$\min_{ij}(I(a_{ij})I(b_{ij}))CI(B, G) \leq CI(A, G) \leq \max_{ij}(I(a_{ij})I(b_{ij}))CI(B, G).$$

(2) The proof of Eq. (2) is similar to Eq. (1). □

4. Some ILOWG operators to aggregate multiplicative linguistic preference relations

In this section, we present two special cases of ILOWG operators for GDM with multiplicative linguistic preference relations: the compatibility index ILOWG (CI-ILOWG) operator and the importance degree ILOWG (I-ILOWG) operator.

4.1. The compatibility index ILOWG (CI-ILOWG) operator

In a homogeneous GDM problem, the experts have equal importance. However, in a homogeneous situation, each expert always can have a compatibility index (CI) value associated with them, which measures the level of consensus between individual preferences and group preference. Thus, the more compatible the information provided by the expert, the higher the weighting value that should be placed on that information.

In this section, we implement a CI-ILOWG operator to aggregate individual preferences in such a way the greater weight is given the most compatible index one. We analyze the reciprocity, consistency and compatibility properties of the CI-ILOWG operator.

**Definition 8.** If a set of experts $E = \{e_1, e_2, \ldots, e_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ by means of multiplicative linguistic preference relations $\{R^{(1)}, \ldots, R^{(i)}, \ldots, R^{(m)}\}$, $R^{(i)} \in R$, then a CI-ILOWG operator is an ILOWG operator in which its order- inducing values is the set of compatibility index values $\{CI^{(1)}, \ldots, CI^{(i)}, \ldots, CI^{(m)}\}$.

**Definition 9.** If $R^{(1)}, \ldots, R^{(i)}, \ldots, R^{(m)}$ are the multiplicative linguistic preference relations provided by $m$ experts, then the CI-ILOWG operator $R = (\vec{r}_{\vec{g}})_{n \times n}$ is expressed as follows:

$$\vec{R} = CI-ILOWG \left( \{CI^{(1)}, R^{(1)}\}, \ldots, \{CI^{(m)}, R^{(m)}\} \right)$$

$\begin{align*}
\vec{R} &= CI-ILOWG \left( \{CI^{(\sigma(1))}, R^{(\sigma(1))}\}, \ldots, \{CI^{(\sigma(m))}, R^{(\sigma(m))}\} \right) \\
&= \left( R^{(\sigma(1))} \gamma_{e(1)} \circ \left( R^{(\sigma(2))} \gamma_{e(2)} \circ \cdots \circ \left( R^{(\sigma(m))} \gamma_{e(m)} \right) \right) \right) \end{align*}$

$$\vec{R} = (r_{\vec{g}}^{(\sigma(1))})^{\gamma_{e(1)}} \otimes \left( r_{\vec{g}}^{(\sigma(2))} \gamma_{e(2)} \otimes \cdots \otimes \left( r_{\vec{g}}^{(\sigma(m))} \gamma_{e(m)} \right) \right) \otimes \gamma_{e(1)} = \prod_{i=1}^{m} (\gamma_{e(i)}^{(\sigma(i))})^{\gamma_{e(i)}},$$
Theorem 4. If \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is a permutation of \((1, 2, \ldots, n)\) such that \( \text{CI}(^{\sigma(l-1)}) \leq \text{CI}(^{\sigma(l)}) \) and \( \gamma_{\sigma(l-1)} \geq \gamma_{\sigma(l)} \) for all \( l = 2, \ldots, m; \{\text{CI}(^{\sigma(1)}), \text{CI}(^{\sigma(2)}), \ldots, \gamma_{\sigma(m)}\}^{T} \) is the two tuple with \( \text{CI}(^{\sigma(i)}) \) the \( i \)-th smallest value in the set \( \{\text{CI}(^{1}), \ldots, \text{CI}(^{n})\} \); \( y = (\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(m)}) \) is a weighting vector, such that \( \sum_{i=1}^{m} \gamma_{\sigma(i)} = 1, \gamma_{\sigma(i)} \in [0, 1] \).

Yager [32] proposed a procedure to determine the weighting vector associated to an IOWA operator. In this case, each comment in the aggregation consists of a triple \((p^{(l)}_{y}, u, v) : p^{(l)}_{y} \) is the argument value to aggregate, \( u \) is the importance weight value associated to \( p^{(l)}_{y} \), and \( v \) is the order inducing value. Thus, the aggregation is

\[
\text{IOWA}_{Q}(p^{(1)}_{y}, \ldots, p^{(n)}_{y}) = \sum_{l=1}^{n} u_{l} p^{\sigma(l)}_{y},
\]

with

\[
u_{l} = Q \left( \frac{S(l)}{S(n)} \right) - Q \left( \frac{S(l - 1)}{S(n)} \right)
\]

where \( S(l) = \sum_{k=1}^{l} u_{\sigma(k)} \), and \( \sigma \) is the permutation such that \( u_{\sigma(l)} \) in \( (p^{\sigma(l)}_{y}, u_{\sigma(l)}, v_{\sigma(l)}) \) is the \( l \)-th largest value in the set \( \{v_{1}, \ldots, v_{n}\} \). \( Q \) is a function: \( [0, 1] \rightarrow [0, 1] \) such that \( Q(0) = 0, Q(1) = 1 \) and if \( x > y \) then \( Q(x) \geq Q(y) \) [33].

In our case, we propose to use the compatibility index values associated to each one of the experts both as a weight associated to the argument and as the order inducing values \( u_{l} = v_{l} = 1/\text{CI}(^{l}) \). Thus, the ordering of the preference values is first induced by the ordering of the experts from the most to least compatible one, and the weights of the CI-ILOWG operator are obtained by applying the above Eq. (10), which reduces to

\[
\gamma_{\sigma(l)} = Q \left( \frac{S(\sigma(l))}{S(\sigma(n))} \right) - Q \left( \frac{S(\sigma(l - 1))}{S(\sigma(n))} \right)
\]

where \( S(\sigma(l)) = \sum_{k=1}^{l} 1/\text{CI}(^{\sigma(k)}) \), and \( \sigma \) is the permutation such that \( \text{CI}(^{\sigma(l)}) \) in \( (r^{\sigma(l)}_{ij}, \text{CI}(^{\sigma(l)}), \text{CI}(^{\sigma(l)})) \) is the \( l \)-th smallest value in the set \( \{\text{CI}(^{1}), \ldots, \text{CI}(^{m})\} \).

In an aggregation process, we consider that the weighting value of experts should be implemented in such a way that the effect from those experts who are less compatible is reduced, and therefore the above is obtained if the linguistic quantifier \( Q \) verifies that the more the compatibility of an expert, the higher the weighting value of that expert in the aggregation, i.e.:

\[
\text{CI}(^{\sigma(1)}) \leq \text{CI}(^{\sigma(2)}) \leq \cdots \leq \text{CI}(^{\sigma(n)}) \Rightarrow \gamma_{\sigma(1)} \geq \gamma_{\sigma(2)} \cdots \gamma_{\sigma(n)}.
\]

**Theorem 4.** Assuming the parameterized family of RIM quantifiers \( Q(r) = r^{\alpha} \), \( \alpha \geq 0 \), if \( \alpha \in [0, 1] \) and \( S(\sigma(l)) = \sum_{k=1}^{l} 1/\text{CI}(^{\sigma(k)}) \), then \( \gamma_{\sigma(l)} \geq \gamma_{\sigma(l-1)} \), for all \( l = 1, 2, \ldots, m \).

**Proof.** If \( \alpha \in [0, 1] \), then the function \( Q(r) = r^{\alpha} \) is concave and we can have

\[
Q(\bar{T}_{l}) - Q(\bar{T}_{l-1}) \geq Q(\bar{T}_{l+1}) - Q(\bar{T}_{l}).
\]

Suppose \( \bar{T}_{l} = \frac{S(\sigma(l))}{S(\sigma(n))} \), and \( S(\sigma(l)) = \sum_{k=1}^{l} 1/\text{CI}(^{\sigma(k)}) \), then

\[
\gamma_{\sigma(l)} = Q \left( \frac{S(\sigma(l))}{S(\sigma(n))} \right) - Q \left( \frac{S(\sigma(l-1))}{S(\sigma(n))} \right) = Q(\bar{T}_{l}) - Q(\bar{T}_{l-1})
\]

and

\[
\gamma_{\sigma(l+1)} = Q \left( \frac{S(\sigma(l+1))}{S(\sigma(n))} \right) - Q \left( \frac{S(\sigma(l))}{S(\sigma(n))} \right) = Q(\bar{T}_{l+1}) - Q(\bar{T}_{l}).
\]

Thus, we can obtain

\[
\gamma_{\sigma(l)} \geq \gamma_{\sigma(l+1)}
\]

which completes the **Theorem 4.**

In GDM models with multiplicative linguistic scale preference assessments, it is usually assumed that the multiplicative linguistic preference relations to express the judgments are reciprocal. The CI-ILOWG operator is able to maintain both the reciprocity and compatibility properties in the collective multiplicative linguistic preference relation. To study these desired properties, we derive the following theorems.
Theorem 5. Let \( \{R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)}\} \) be multiplicative linguistic preference relations provided by \( m \) experts, where \( R^{(l)} = (r^{(l)}_{ij})_{n \times n}, \) \( r^{(l)}_{ij} \in S \) \((i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\), then their CI-ILOWGCJM \( \bar{R} = (\bar{r}_{ij})_{n \times n} \) is also a multiplicative linguistic preference relation, where

\[
\bar{r}_{ij} = \text{CI-ILOWG} \left( \left\{ (CI^{(1)}), r^{(1)}_{ij} \right\}, \left\{ (CI^{(2)}), r^{(2)}_{ij} \right\}, \ldots, \left\{ (CI^{(m)}), r^{(m)}_{ij} \right\} \right)
\]

\[
= \text{CI-ILOWG} \left( \left\{ (CI^{(1)}), r^{(1)}_{ij} \right\}, \left\{ (CI^{(2)}), r^{(2)}_{ij} \right\}, \ldots, \left\{ (CI^{(m)}), r^{(m)}_{ij} \right\} \right)
\]

\[
= \left( r^{(1)}_{ij} \right)^{\gamma_{(1)}} \otimes \left( r^{(2)}_{ij} \right)^{\gamma_{(2)}} \otimes \cdots \otimes \left( r^{(m)}_{ij} \right)^{\gamma_{(m)}}
\]

and

\[
r_{ij} \geq s_{1/1}, \ r_{ij} \otimes r_{ij} = s_{1}, \ r_{ii} = s_{1}, \ \text{for all } i, j \in N.
\]

Furthermore, if all the \( R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)} \) are consistent, then \( \bar{R} \) is also consistent.

Proof. (i) Since \( R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)} \) are multiplicative linguistic preference relations, we have \( r^{(l)}_{ij} \geq s_{1/1}, \ r^{(l)}_{ij} \otimes r^{(l)}_{ij} = s_{1}, \ r^{(l)}_{ii} = s_{1}, \) for all \( i, j \in N, \) and then

\[
\bar{r}_{ij} = \left( r^{(1)}_{ij} \right)^{\gamma_{(1)}} \otimes \left( r^{(2)}_{ij} \right)^{\gamma_{(2)}} \otimes \cdots \otimes \left( r^{(m)}_{ij} \right)^{\gamma_{(m)}}
\]

\[
\geq (s_{1/1})^{\gamma_{(1)}} \otimes (s_{1/1})^{\gamma_{(2)}} \otimes \cdots \otimes (s_{1/1})^{\gamma_{(m)}}
\]

\[
= s_{1/1}
\]

\[
r_{ij} \otimes r_{ij} = (s_{1})^{\gamma_{(1)}} \otimes (s_{1})^{\gamma_{(2)}} \otimes \cdots \otimes (s_{1})^{\gamma_{(m)}}
\]

\[
= s_{1}
\]

Thus, \( \bar{R} = (\bar{r}_{ij})_{n \times n} \) is also a multiplicative linguistic preference relation.

(ii) Since all the \( R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)} \) are consistent, i.e., \( r^{(l)}_{ij} = r^{(l)}_{ik} \otimes r^{(l)}_{kj} \), for all \( l = 1, 2, \ldots, m; i, j \in N \).

Then

\[
r_{ij} \otimes r_{jk} = (r^{(l)}_{ij} \otimes r^{(l)}_{kj})^{\gamma_{(l)}} \otimes (r^{(l)}_{ij} \otimes r^{(l)}_{kj})^{\gamma_{(l)}} \otimes \cdots \otimes (r^{(l)}_{ij} \otimes r^{(l)}_{kj})^{\gamma_{(l)}}
\]

\[
= \left( r^{(l)}_{ij} \right)^{\gamma_{(l)}} \otimes \left( r^{(l)}_{kj} \right)^{\gamma_{(l)}} \otimes \cdots \otimes \left( r^{(l)}_{ij} \right)^{\gamma_{(l)}}
\]

and thus, \( \bar{R} \) is also consistent, which completes the proof of Theorem 5. \( \square \)

As regards group decision making, the analytic hierarchy process (AHP) considers two different approaches: the aggregation of individual judgements (AJ) and the aggregation of individual priorities (AIP). For the RGMM prioritization procedure and the WGMM aggregation procedure, Barzilai and Golany [34] proved that both aggregation approaches (AJ and AIP) provide the same priorities of alternatives. We also can draw an analogous conclusion when using the row geometric mean method (RGMM) as the prioritization procedure and CI-ILOWGCJM as the aggregation procedure.

Definition 10 ([35]). Let \( R^{(l)} \in \mathcal{R} \) be the multiplicative linguistic judgement matrix provided by the \( l \)-th expert when comparing \( n \) alternatives, \( \omega^{(l)} = (\omega_{1}^{(l)}, \omega_{2}^{(l)}, \ldots, \omega_{n}^{(l)})^T \) as its priority vector, \( W^{(l)} = \left( \frac{\omega_{1}^{(l)}}{\omega_{n}^{(l)}} \right) = (w^{(l)}_{ij})_{n \times n} \) as the corresponding characteristic matrix; \( \bar{\omega} = (\bar{\omega}_{1}, \bar{\omega}_{2}, \ldots, \bar{\omega}_{n})^T \) as the priority vector of CI-ILOWGCJM \( \bar{R} \), and \( \bar{W} = \left( \frac{\bar{\omega}_{1}}{\bar{\omega}_{n}} \right) = (\bar{w}_{ij})_{n \times n} \) as the corresponding characteristic matrix of \( \bar{R} \).

Theorem 6. Using the CI-ILOWGCJM as the aggregation procedure, the weighting vector \( \mathbf{y} = (\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(m)})^T \), \( \gamma_{(l)} \geq \gamma_{(l)} \), \( \sum_{l=1}^{m} \gamma_{(l)} = 1 \) and the RGMM as the prioritization procedure, it holds that the AJ and AIP provide the same priorities of alternatives.
**Proof.** Let \( \omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \ldots, \omega_n^{(i)})^T \) be the priority of the individual judgement matrix \( R^{(i)} \) and \( \bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_n)^T \) be the group priorities, then we can obtain

\[
\omega_{\text{AIP}} = \text{I-LOWG}(\{\text{Cl}^{(1)}, \omega^{(1)}\}, \{\text{Cl}^{(2)}, \omega^{(2)}\}, \ldots, \{\text{Cl}^{(m)}, \omega^{(m)}\})
\]

\[= \text{I-LOWG}(\{\text{Cl}^{(1)}, \omega^{(1)}\}, \{\text{Cl}^{(2)}, \omega^{(2)}\}, \ldots, \{\text{Cl}^{(m)}, \omega^{(m)}\})
\]

\[= (\omega^{(1)})^{\gamma_{0(1)}} \circ (\omega^{(2)})^{\gamma_{0(2)}} \circ \cdots \circ (\omega^{(m)})^{\gamma_{0(m)}}
\]

\[
\omega_{\text{AIP}} = (\omega^{(1)})^{\gamma_{0(1)}} \otimes (\omega^{(2)})^{\gamma_{0(2)}} \otimes \cdots \otimes (\omega^{(m)})^{\gamma_{0(m)}} = \prod_{i=1}^{m} (\omega^{(i)})^{\gamma_{0(i)}}
\]

and

\[
\omega_{\text{Al}} = \left( \prod_{j=1}^{n} x_{ij}^{\bar{w}_{ij}} \right)^{1/n} = \left( \prod_{j=1}^{n} \prod_{i=1}^{m} (x_{ij}^{\bar{w}_{ij}})^{\gamma_{0(i)}} \right)^{1/n}
\]

\[= \prod_{i=1}^{m} \left( \prod_{j=1}^{n} (x_{ij}^{\bar{w}_{ij}})^{1/n} \right)^{\gamma_{0(i)}} = \prod_{i=1}^{m} (\omega^{(i)})^{\gamma_{0(i)}}
\]

Thus

\[
\omega_{\text{AIP}} = \omega_{\text{AIP}}
\]

This completes the proof of the **Theorem 6.** □

In a GDM environment, the consensus reaching progress is necessary to obtain a final solution in a certain level of agreement between the experts. Many researches on consensus theory are applied to GDM and use Arrow’s work as a basic rule. According to Arrow’s impossible theorem, it is impossible to aggregate individual preferences into group preference in a completely rational way [23]. Thus, there is one problem that must be solved, how to judge the agreement degree or consensus degree between individual preference and group preference, and how to make the collective priority vector closest to the judgements given by experts. These problems can be defined by compatibility, which is discussed in Ref. [35]. The CI-ILOWG operator aggregates individual preferences into a collective preference in such a way that the greater weight is given the most compatible index one. As a result, it is proved that the CI-ILOWG operator has the desired compatibility, which is able to improve group compatibility.

**Lemma 1 ([36]).** Let \( x_i > 0, \beta_i > 0, i \in N \) and \( \sum_{i=1}^{n} \beta_i = 0 \), then

\[
\prod_{i=1}^{n} x_i^{\beta_i} \leq \sum_{i=1}^{n} \beta_i x_i.
\]  

**Lemma 2 ([37]).** For ordered vector \( x = (x_1, x_2, \ldots, x_n), x_1 \geq x_2 \geq \cdots \geq x_n, \) and weights

\[ w = (w_1, w_2, \ldots, w_n), \]  

then vector \( \alpha_1, \alpha_2, \ldots, \alpha_n \), \( \alpha_i \geq 0 \) \( (i = 1, 2, \ldots, n) \).

If \( w_1 \geq w_2 \geq \cdots \geq w_n \), then

\[
\alpha_1 w_1 x_1 + \alpha_2 w_2 x_2 + \cdots + \alpha_n w_n x_n \geq \frac{\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} (\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n).
\]

If \( w_1 \leq w_2 \leq \cdots \leq w_n \), then

\[
\alpha_1 w_1 x_1 + \alpha_2 w_2 x_2 + \cdots + \alpha_n w_n x_n \leq \frac{\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} (\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)
\]

with equality if and only if \( x_1 = x_2 = \cdots = x_n \).

**Definition 11.** Let \( \text{Cl}(R^{(i)}, W^{(i)}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}^{(i)} w_{ij}^{(i)} \) be the compatibility index of \( R^{(i)} \) and \( W^{(i)} \), \( \text{Cl}(R, W) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} w_{ij} \) be the compatibility index of \( R \) and \( W \), and \( \text{Cl}(\bar{R}, \bar{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{r}_{ij} \bar{w}_{ij} \) be the compatibility index of \( \bar{R} \) and \( \bar{W} \).

From **Theorem 1,** it can be obtained that \( R^{(i)} \) is perfectly compatible with \( \bar{W} \) if \( \text{Cl}(R^{(i)}, \bar{W}) = s_1 \). The smaller the value of \( \text{Cl}(R^{(i)}, W) \), the more compatible the degree and consensus degree of \( R^{(i)} \) and \( W \).

**Theorem 7.** Let \( R^{(1)}, R^{(2)}, \ldots, R^{(m)} \) be the multiplicative linguistic preference relations provided by \( m \) experts when comparing \( n \) alternatives with the corresponding weighting vector \( \gamma = (\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(m)})^T \), \( \gamma_{\sigma(i-1)} \geq \gamma_{\sigma(i)} \), \( \sum_{i=1}^{m} \gamma_{\sigma(i)} = 1. \) Using
the CI-IOWGCJM as the aggregation procedure and the RGMM as the prioritization procedure, it holds that:

\[
\text{Cl}(\tilde{R}, \tilde{W}) \leq \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(l)}, W^{(l)}). 
\]

**Proof.** By Definition 11, we have

\[
\text{Cl}(\tilde{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{r}_{ij} \tilde{w}_{ij} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r_{ij}^{(\sigma(l))} \gamma_{\sigma(l)} \frac{\omega_i}{\omega_j} \right)
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( \omega_i^{(\sigma(l))} \gamma_{\sigma(l)} \frac{\omega_i^{(\sigma(l))}}{\omega_j^{(\sigma(l))}} \right)
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r_{ij}^{(\sigma(l))} \omega_i^{(\sigma(l))} \gamma_{\sigma(l)} \right)
\]

From Lemma 1, it can be obtained that

\[
\prod_{l=1}^{m} \left( r_{ij}^{(\sigma(l))} \omega_i^{(\sigma(l))} \right) \gamma_{\sigma(l)} \leq \sum_{l=1}^{m} \gamma_{\sigma(l)} \left( r_{ij}^{(\sigma(l))} \omega_i^{(\sigma(l))} \right).
\]

Then

\[
\text{Cl}(\tilde{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r_{ij}^{(\sigma(l))} \omega_i^{(\sigma(l))} \right) \gamma_{\sigma(l)}
\]

\[
\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{m} \gamma_{\sigma(l)} \left( r_{ij}^{(\sigma(l))} \omega_i^{(\sigma(l))} \right)
\]

\[
= \sum_{l=1}^{m} \gamma_{\sigma(l)} \text{SD}(R^{(\sigma(l))}, W^{(\sigma(l)))}).
\]

Since \(\text{Cl}(R^{(\sigma(l))}, W^{(\sigma(l))}) \leq \text{Cl}(R^{(\sigma(l+1))}, W^{(\sigma(l+1))})\) and \(\gamma_{\sigma(1)} \geq \gamma_{\sigma(2)} \geq \cdots \geq \gamma_{\sigma(m)}\).

Then, from Lemma 2, we have

\[
\sum_{l=1}^{m} \gamma_{\sigma(l)} \text{Cl}(R^{(\sigma(l))}, W^{(\sigma(l))}) \leq \sum_{l=1}^{m} \gamma_{\sigma(l)} \times \left( \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(\sigma(l))}, W^{(\sigma(l))}) \right)
\]

\[
= \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(\sigma(l))}, W^{(\sigma(l))})
\]

\[
= \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(l)}, W^{(l)}).
\]

Thus

\[
\text{Cl}(\tilde{R}, \tilde{W}) \leq \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(l)}, W^{(l)}).
\]

We complete the proof of the Theorem 7. □

**Corollary 1.** If the individual multiplicative linguistic judgements \(R^{(1)}, R^{(2)}, \ldots, R^{(m)}\) are of acceptable compatibility, then the CI-IOWGCJM \(\tilde{R}\) is also of acceptable compatibility, that is to say,

\[
\text{Cl}(R^{(l)}, W^{(l)}) \leq \tau, \quad \forall l = 1, \ldots, m \Rightarrow \text{Cl}(\tilde{R}, \tilde{W}) \leq \tau,
\]

where \(\tau\) is the threshold for acceptable compatibility.
Theorem 8. Let $R^{(1)}, R^{(2)}, \ldots, R^{(m)}$ be the multiplicative linguistic preference relations provided by $m$ experts when comparing $n$ alternatives with the corresponding weighting vector $\mathbf{y} = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(m)})^T$, $y_{\sigma(i)} \geq y_{\sigma(i-1)}$, $\sum_{i=1}^{m} y_{\sigma(i)} = 1$. Using the CI-ILOWG\textsubscript{CM} as the aggregation procedure and the RGMM as the prioritization procedure, it holds that:

$$\text{Cl}(\tilde{R}, \tilde{W}) \leq \frac{1}{m} \sum_{i=1}^{m} \text{Cl}(R^{(i)}, \tilde{W}). \quad (16)$$

**Proof.** By Definition 11, we have

$$\text{Cl}(\tilde{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{r}_{ij} \tilde{w}_{ij} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} (r^{(\sigma(l))}_{ij})^{y_{\sigma(i)}} \tilde{w}_{ij} \tilde{w}_{ij}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r^{(\sigma(l))}_{ij} \frac{\tilde{w}_{ij}}{\tilde{w}_{ij}} \right)^{y_{\sigma(l)}}$$

from Lemma 1, it can be obtained that

$$\prod_{l=1}^{m} \left( r^{(\sigma(l))}_{ij} \frac{\tilde{w}_{ij}}{\tilde{w}_{ij}} \right)^{y_{\sigma(l)}} \leq \sum_{l=1}^{m} y_{\sigma(l)} \left( r^{(\sigma(l))}_{ij} \frac{\tilde{w}_{ij}}{\tilde{w}_{ij}} \right)^{y_{\sigma(l)}}$$

Then

$$\text{Cl}(\tilde{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r^{(\sigma(l))}_{ij} \frac{\tilde{w}_{ij}}{\tilde{w}_{ij}} \right)^{y_{\sigma(l)}}$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{m} y_{\sigma(l)} \left( r^{(\sigma(l))}_{ij} \frac{\tilde{w}_{ij}}{\tilde{w}_{ij}} \right)^{y_{\sigma(l)}}$$

$$= \sum_{l=1}^{m} y_{\sigma(l)} \text{Cl}(R^{(\sigma(l))}, \tilde{W}).$$

Since $\text{Cl}(R^{(\sigma(l))}, \tilde{W}) \leq \text{Cl}(R^{(\sigma(l-1))}, \tilde{W})$ and $y_{\sigma(1)} \geq y_{\sigma(2)} \geq \cdots \geq y_{\sigma(m)}$.

Then, from Lemma 2, we have

$$\sum_{l=1}^{m} y_{\sigma(l)} \text{Cl}(R^{(\sigma(l))}, \tilde{W}) \leq \sum_{l=1}^{m} y_{\sigma(l)} \otimes \left( \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(\sigma(l))}, \tilde{W}) \right)$$

$$= \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(\sigma(l))}, \tilde{W})$$

$$= \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(l)}, \tilde{W}).$$

Thus

$$\text{Cl}(\tilde{R}, \tilde{W}) \leq \frac{1}{m} \sum_{l=1}^{m} \text{Cl}(R^{(l)}, \tilde{W})$$

which completes the proof of the **Theorem 8.** \(\square\)

**Corollary 2.** If the individual multiplicative linguistic judgements $R^{(1)}, R^{(2)}, \ldots, R^{(m)}$ are of acceptable compatibility, then the CI-ILOWG\textsubscript{CM} $\tilde{R}$ is also acceptable compatibility, that is to say,

$$\text{Cl}(R^{(l)}, \tilde{W}) \leq \tau, \quad \forall l = 1, \ldots, m \Rightarrow \text{Cl}(\tilde{R}, \tilde{W}) \leq \tau, \quad (17)$$

where $\tau$ is the threshold for acceptable compatibility.

4.2. The importance ILOWG (I-ILOWG) operator

In a heterogeneous GDM problem \([38, 39]\), each expert has an importance degree associated with them. Chiclana et al. \([22]\) developed the I-IOWG operator, which used this importance degree variable as the order-inducing variable to induce the ordering of the argument values before their aggregation. Similar to the I-IOWG operator, we present the importance degree ILOWG (I-ILOWG) operator and study its desired properties in the collective multiplicative preference relation.
Definition 12. If a set of experts $E = \{e_1, e_2, \ldots, e_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ by means of multiplicative linguistic preference relations $\{R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)}\}$, $R^{(l)} \in R$, whose associated importance degree $\lambda = (\lambda_1, \ldots, \lambda_l, \ldots, \lambda_m)^T$, $\sum_{l=1}^m \lambda_l = 1$, $0 \leq \lambda_l \leq 1$, then an I-ILOWG operator is reduced to an ILOWG operator in which its order-inducing values are the set of importance degrees.

Definition 13. Let $R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)}$ be the multiplicative linguistic preference relations provided by $m$ experts, then the I-ILOWGCJM $\hat{R} = (\hat{r}_{ij})_{n \times n}$ is expressed as follows:

$$
\hat{R} = I-ILOWG (\{\lambda_1, R^{(1)}\}, \{\lambda_2, R^{(2)}\}, \ldots, \{\lambda_m, R^{(m)}\})
$$

$$
\sum_{\bar{j}} \bar{r}_{ij} = (r_{ij}^{(1)})^\lambda_1 \otimes (r_{ij}^{(2)})^\lambda_2 \otimes \ldots \otimes (r_{ij}^{(m)})^\lambda_m = \prod_{l=1}^m (r_{ij}^{(l)})^\lambda_l,
$$

where $\lambda = (\lambda_1, \ldots, \lambda_l, \ldots, \lambda_m)^T$, $\sum_{l=1}^m \lambda_l = 1$, $0 \leq \lambda_l \leq 1$, is the experts' importance degree.

In GDM models with multiplicative linguistic scale preference assessments, it is usually assumed that the multiplicative linguistic preference relations to express the judgments are reciprocal. The I-ILOWG operator is also able to maintain both the reciprocity and consistency properties in the collective multiplicative linguistic preference relation. To study these desired properties, we derive the following theorems.

Using the RGMM prioritization procedure and the I-ILOWGCJM aggregation procedure, we will prove that both aggregation approaches (AII and AIP) provide the same priorities of alternatives in the following.

Definition 14. Let $R = (r_{ij})_{n \times n}$ be a $n \times n$ multiplicative linguistic preference relation, we denote by $R^\lambda = (r_{ij})^\lambda_{n \times n}$, where $\lambda$ is a real number.

Definition 15. Let $R^{(l)} \in R$ be the multiplicative linguistic judgement matrix provided by the $l$-th expert when comparing $n$ alternatives, $\omega^{(l)} = (\omega_1^{(l)}, \omega_2^{(l)}, \ldots, \omega_n^{(l)})^T$ as its priority vector, $W^{(l)} = \left(\begin{array}{c}
\omega_1^{(l)} \\
\omega_2^{(l)} \\
\vdots \\
\omega_n^{(l)}
\end{array}\right) = (w^{(l)}_{ij})_{n \times n}$ as the corresponding characteristic matrix; $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T$ as the priority vector of I-ILOWGCJM $\hat{R}$, and $\tilde{W} = \left(\tilde{w}_{ij}\right)_{n \times n}$ as the corresponding characteristic matrix of $\hat{R}$.

Definition 16. Let $\text{CI}(R^{(l)}, W^{(l)}) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n r_{ij}^{(l)} w_{ij}$ be the compatibility index of $R^{(l)}$ and $W^{(l)}$, $\text{CI}(\hat{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \bar{r}_{ij} \bar{w}_{ij}$ be the compatibility index of $\hat{R}$ and $\tilde{W}$, and $\text{CI}(R^{(l)}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n r_{ij}^{(l)} \bar{w}_{ij}$ be the compatibility index of $R^{(l)}$ and $\tilde{W}$.

In the following, we shall prove that the I-ILOWGCJM is of acceptable compatibility under the condition that each $R^{(l)} (l = 1, 2, \ldots, m)$ is of acceptable compatibility.

Theorem 9. Let $R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)}$ be the multiplicative linguistic preference relations provided by $m$ experts, whose associated importance degree $\lambda = (\lambda_1, \ldots, \lambda_l, \ldots, \lambda_m)^T$, $\sum_{l=1}^m \lambda_l = 1$, $0 \leq \lambda_l \leq 1$. Using the I-ILOWGCJM as the aggregation procedure and the RGMM as the prioritization procedure, it holds that:

$$
\text{CI}(\hat{R}, \tilde{W}) \leq \sum_{l=1}^m \lambda_l \text{CI}(R^{(l)}, W^{(l)}).
$$

Proof. By Definition 16, we have

$$
\text{CI}(\hat{R}, \tilde{W}) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \bar{r}_{ij} \bar{w}_{ij} = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \prod_{l=1}^m (r_{ij}^{(l)})^\lambda_l \bar{w}_{ij}
$$

$$
= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \prod_{l=1}^m (r_{ij}^{(l)})^\lambda_l \prod_{l=1}^m (\omega_{ij}^{(l)})^\lambda_l
$$

$$
= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \prod_{l=1}^m \left(\frac{r_{ij}^{(l)} \omega_{ij}^{(l)}}{\omega_{ij}^{(l)}}\right)^{\lambda_l}
$$

$$
= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \prod_{l=1}^m \left(\frac{r_{ij}^{(l)} w_{ij}^{(l)}}{w_{ij}^{(l)}}\right)^{\lambda_l}.
$$
From Lemma 1, it can be obtained that
\[ \prod_{i=1}^{m} \left( r_{ij}^{(l)} w_{ij}^{(l)} \right)^{\lambda_i} \leq \sum_{i=1}^{m} \lambda_i \left( r_{ij}^{(l)} w_{ij}^{(l)} \right) . \]

Then
\[ \text{Cl}(\bar{R}, \bar{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( r_{ij}^{(l)} w_{ij}^{(l)} \right)^{\lambda_i} \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{m} \lambda_i \left( r_{ij}^{(l)} w_{ij}^{(l)} \right) = \sum_{i=1}^{m} \lambda_i \text{Cl}(R^{(l)}, W^{(l)}) \]

which completes the proof of Theorem 9. \(\square\)

**Corollary 3.** If the individual multiplicative linguistic judgements \(R^{(1)}, R^{(2)}, \ldots, R^{(m)}\) are of acceptable compatibility, then the I-ILOWGCJM \(R\) is also of acceptable compatibility, that is to say,
\[ \text{Cl}(R^{(l)}, W^{(l)}) \leq \tau, \quad \forall l = 1, \ldots, m \Rightarrow \text{Cl}(\bar{R}, \bar{W}) \leq \tau, \quad (21) \]
where \(\tau\) is the threshold for acceptable compatibility.

**Theorem 10.** Let \(R^{(1)}, \ldots, R^{(l)}, \ldots, R^{(m)}\) be the multiplicative linguistic preference relations provided by \(m\) experts, whose associated importance degree \(\lambda = (\lambda_1, \ldots, \lambda_l, \ldots, \lambda_m)^T\), \(\sum_{l=1}^{m} \lambda_l = 1, 0 \leq \lambda_l \leq 1\). Using the I-ILOWGCJM as the aggregation procedure and the RGMM as the prioritization procedure, it holds that:
\[ \text{Cl}(\bar{R}, \bar{W}) \leq \sum_{l=1}^{m} \lambda_l \text{Cl}(R^{(l)}, \bar{W}). \quad (22) \]

**Proof.** By Definition 16, we can get
\[ \text{Cl}(\bar{R}, \bar{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{r}_{ij} \bar{w}_{ij} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right)^{\lambda_i} \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{m} \lambda_i \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right)^{\lambda_i} . \]

From Lemma 1, it can be obtained that
\[ \prod_{i=1}^{m} \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right)^{\lambda_i} \leq \sum_{i=1}^{m} \lambda_i \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right) . \]

Then
\[ \text{Cl}(\bar{R}, \bar{W}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{m} \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right)^{\lambda_i} \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{m} \lambda_i \left( \frac{r_{ij}^{(l)} \bar{w}_{ij}^{(l)}}{\tilde{\omega}_{ij}^{(l)}} \right) = \sum_{i=1}^{m} \lambda_i \text{Cl}(R^{(l)}, \bar{W}). \]

This completes the proof of Theorem 10. \(\square\)

**Corollary 4.** If the individual multiplicative linguistic judgements \(R^{(1)}, R^{(2)}, \ldots, R^{(m)}\) are of acceptable compatibility, then the I-ILOWGCJM \(R\) is also of acceptable compatibility, that is to say,
\[ \text{Cl}(R^{(l)}, \bar{W}) \leq \tau, \quad \forall l = 1, \ldots, m \Rightarrow \text{Cl}(\bar{R}, \bar{W}) \leq \tau, \quad (23) \]
where \(\tau\) is the threshold for acceptable compatibility.
5. A numerical example

An automotive company desires to select the most appropriate supplier for one of the key elements in its manufacturing process. After pre-evaluation, several suppliers have remained as alternatives for further evaluation. In order to evaluate alternative suppliers, four criteria are considered as:

- \(x_1\): Product quality,
- \(x_2\): Price,
- \(x_3\): Delivery performance,
- \(x_4\): Environment management.

The automotive company has a group of experts form four consultancy departments.

- \(e_1\) is from the financial department,
- \(e_2\) is from the purchasing department,
- \(e_3\) is from the quality inspection department,
- \(e_4\) is from the engineering department.

To determine the importance of the criteria, these experts provide the following multiplicative linguistic preference relations:

\[
R^{(1)} = \begin{bmatrix}
    1 & s_2 & s_3 & s_4 \\
    s_{1/2} & 1 & s_3 & s_4 \\
    s_{1/3} & s_{1/4} & 1 & s_2 \\
    s_{1/5} & s_{1/4} & s_{1/2} & 1
\end{bmatrix};
R^{(2)} = \begin{bmatrix}
    1 & s_3 & s_2 & s_4 \\
    s_{1/3} & 1 & s_4 & s_2 \\
    s_{1/2} & s_{1/4} & 1 & s_1 \\
    s_{1/4} & s_{1/2} & s_{1/2} & 1
\end{bmatrix};
R^{(3)} = \begin{bmatrix}
    s_1 & s_3 & s_2 & s_4 \\
    s_{1/3} & s_1 & s_5 & s_4 \\
    s_{1/2} & s_{1/5} & s_1 & s_2 \\
    s_{1/4} & s_{1/5} & s_{1/2} & 1
\end{bmatrix};
R^{(4)} = \begin{bmatrix}
    s_1 & s_4 & s_5 & s_3 \\
    s_{1/4} & s_1 & s_3 & s_5 \\
    s_{1/5} & s_{1/3} & s_1 & s_2 \\
    s_{1/3} & s_{1/3} & s_{1/2} & 1
\end{bmatrix}.
\]

By using RGMM as the prioritization procedure, we can obtain four characteristic matrices as follows:

\[
W^{(1)} = \begin{bmatrix}
    5.1000 & 5.4593 & 5.4087 & 5.8857 \\
    5.6687 & 5.1000 & 5.2795 & 5.9360 \\
    5.2934 & 5.4387 & 5.1000 & 5.7267 \\
    5.1699 & 5.2541 & 5.5791 & 5.1000
\end{bmatrix};
W^{(2)} = \begin{bmatrix}
    5.1000 & 5.7321 & 5.3102 & 5.4267 \\
    5.5773 & 5.1000 & 5.8072 & 5.5558 \\
    5.3195 & 5.5533 & 5.1000 & 5.8284 \\
    5.2259 & 5.3913 & 5.3536 & 5.1000
\end{bmatrix};
W^{(3)} = \begin{bmatrix}
    5.1000 & 5.3027 & 5.3098 & 5.5663 \\
    5.7676 & 5.1000 & 5.5407 & 5.4279 \\
    5.3021 & 5.3936 & 5.1000 & 5.6818 \\
    5.1797 & 5.2340 & 5.5946 & 5.1000
\end{bmatrix};
W^{(4)} = \begin{bmatrix}
    5.1000 & 5.2724 & 5.4058 & 5.7327 \\
    5.4401 & 5.1000 & 5.0268 & 5.5227 \\
    5.2171 & 5.4034 & 5.1000 & 5.1247 \\
    5.1744 & 5.3964 & 5.8034 & 5.1000
\end{bmatrix}.
\]

Applying \textbf{Definition 7}, we can calculate the compatibility index \(\text{Cl}(R^{(i)}, W^{(l)})\), \(l = 1, 2, 3, 4\):

\[
\text{Cl}(R^{(1)}, W^{(1)}) = 4.6029; \quad \text{Cl}(R^{(2)}, W^{(2)}) = 3.2550;
\]

\[
\text{Cl}(R^{(3)}, W^{(3)}) = 4.6937; \quad \text{Cl}(R^{(4)}, W^{(4)}) = 4.3978
\]

and the judgement matrices \(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}\) and characteristic matrices \(W^{(1)}, W^{(2)}, W^{(3)}, W^{(4)}\) are re-ordered as follows, respectively:

\[
nR^{(1)} = R^{(2)}; \quad nR^{(2)} = R^{(4)}; \quad nR^{(3)} = R^{(1)}; \quad nR^{(4)} = R^{(3)};
nW^{(1)} = W^{(2)}; \quad nW^{(2)} = W^{(4)}; \quad nW^{(3)} = W^{(1)}; \quad nW^{(4)} = W^{(3)}.
\]

Using formula \((11)\) with \(Q(r) = r^{1/2}\), we get the following weights:

\[
\gamma^{(1)} = 0.56; \quad \gamma^{(2)} = 0.18; \quad \gamma^{(3)} = 0.14; \quad \gamma^{(4)} = 0.12.
\]

Then, the CI-ILOWGCGJM \(\bar{R}\) and its corresponding characteristic matrix \(\bar{W}\) are calculated as:

\[
\bar{R} = \begin{bmatrix}
    5.1000 & 5.2951 & 5.4964 & 5.9187 \\
    5.3350 & 5.1000 & 5.7472 & 5.6462 \\
    5.4006 & 5.2669 & 5.1000 & 5.0000 \\
    5.2525 & 5.3779 & 5.5000 & 5.1000
\end{bmatrix};
\bar{W} = \begin{bmatrix}
    5.1000 & 5.129 & 5.4186 & 5.9608 \\
    5.5807 & 5.1000 & 5.9854 & 5.8810 \\
    5.2925 & 5.5037 & 5.1000 & 5.2393 \\
    5.2016 & 5.3471 & 5.4675 & 5.1000
\end{bmatrix}.
\]
By Definition 7, we can obtain that

$$\text{Cl}(\bar{R}, \bar{W}) = 3.5824 < \frac{1}{4} \sum_{i=1}^{4} \text{Cl}(A^{(i)}, W^{(i)}) = 4.2374,$$

which verifies the conclusion of Theorem 7.

Similar to the above calculating procedure, we can also have

$$\text{Cl}(\bar{R}, \bar{W}) = 3.5824 < \frac{1}{4} \sum_{i=1}^{4} \text{Cl}(A^{(i)}, W) = 3.9081.$$

Then, we can find that this result is in accordance with Theorem 8.

By using RGMM, we obtain the global preference degree $z_i (i = 1, 2, 3, 4)$ of ith the criteria over all the other alternatives:

$$z_1 = s_{2.32}; \quad z_2 = s_{1.35}; \quad z_3 = s_{0.68}; \quad z_4 = s_{0.46}.$$

Rank all the criteria $z_i (i = 1, 2, 3, 4)$ in accordance with the descending order of $z_1 > z_2 > z_3 > z_4$, and thus the most important criteria is $x_1$.

If experts have their associated importance degree $\lambda = \{0.1, 0.2, 0.3, 0.4\}$, then we use the experts’ importance degree $\lambda$ to induce the ordering of these multiplicative linguistic preference relations to be aggregated, then we obtain the following collective multiplicative linguistic preference relation $\bar{R} = (\bar{R}_i)_{n \times n}$ and its corresponding characteristic matrix $\bar{W}$:

$$\bar{R} = 1\text{-ILOWGW}_{\lambda}(\{\lambda_1, R^{(1)}\}, \{\lambda_2, R^{(2)}\}, \{\lambda_3, R^{(3)}\}, \{\lambda_4, R^{(4)}\})$$

Applying Eq. (5), we can obtain that

$$\text{Cl}(\bar{R}, \bar{W}) = 3.9380 < \frac{1}{4} \sum_{i=1}^{4} \text{Cl}(A^{(i)}, W^{(i)}) = 4.2785,$$

which verifies the conclusion of Theorem 10.

Similar to the above calculating procedure, we can also have

$$\text{Cl}(\bar{R}, \bar{W}) = 3.9380 < \frac{1}{4} \sum_{i=1}^{4} \text{Cl}(A^{(i)}, \bar{W}) = 4.0823.$$

This confirms the conclusion of Theorem 11.

By using RGMM, we obtain the global preference degree $z_i (i = 1, 2, 3, 4)$ of ith the criteria over all the other alternatives:

$$z_1 = s_{2.43}; \quad z_2 = s_{1.39}; \quad z_3 = s_{0.65}; \quad z_4 = s_{0.45}.$$

Rank all the criteria $z_i (i = 1, 2, 3, 4)$ in accordance with the descending order of $z_1 > z_2 > z_3 > z_4$, and thus the most important criteria is $x_1$.

6. Analysis to the ILOWG operator

The consensus reaching process is necessary to obtain a final solution with a certain level of agreement between the experts in GDM. It has been one of the key issues in prior research, but this work has provided limited insights into the consensus reaching process because it only guarantees that the group consensus degree is bigger than the average of all the individual consensus degrees. Against this background, the CI-ILOWG operator is reasonable to find consensus among a group of decision makers. It is based on the idea that "the more compatible the degree of information provided by the expert, the higher the weighting value that should be placed on that information". However, if the one with greater compatibility is endowed with a lot of weight, then it seems that only this decision maker’s opinion is important. In the above numerical example, if we suppose

$$Q(r) = \begin{cases} 
0, & \text{for } r = 0 \\
1, & \text{for } r > 0.
\end{cases} \tag{24}$$

By Eq. (10), we can get the weights vector of these DMs ($\gamma_{r(1)} = 1$, $\gamma_{r(2)} = 0$, $\gamma_{r(3)} = 0$, $\gamma_{r(4)} = 0$) and group decision making is transformed into just the individual decision making. If we let $Q(r) = r$, then we can get the weights ($\gamma_{r(1)} = 1/3$, $\gamma_{r(2)} = 1/3$, $\gamma_{r(3)} = 1/3$, $\gamma_{r(4)} = 1$). Thus there is no difference in four DMs.
Obviously, the above two cases are extreme and there may exist some irrationality. Thus, we can conclude that the function \( Q \) may affect the reasonability of the CI-ILOWG operator. In our future work, we will present some rules for choosing the function \( Q \) in order to avoid misusing the CI-ILOWG operator.

7. Concluding remarks

In this article we have studied the use of some ILOWG operators in the aggregation of multiplicative linguistic preference relations in GDM problems. In particular, the compatibility index ILOWG (CI-ILOWG) operator and the importance ILOWG (I-ILOWG) operator are introduced. Then, we have shown that the collective preference relations obtained by these cases of ILOWG operators verified the reciprocity and consistency properties. Furthermore, we have proved that the aggregation of individual judgements (AIJ) and the aggregation of individual priorities (AIP) provide the same priorities of alternatives by utilizing the row geometric mean method (RGMM) as a prioritization procedure and the ILOWG operators as an aggregation procedure. Finally, this article has shown that if all the individual decision makers have an acceptable consensus degree, then the collective preference relation is also of an acceptable consensus degree. Moreover, the CI-ILOWGCJM guarantees that the group compatibility degree is bigger than the arithmetic mean of all the individual compatibility degrees. Accordingly, a theoretical basis has been developed for the application of these ILOWG operators in linguistic group decision making. These results are of vital importance to deal with linguistic group decision making problems.

The main novelty of these cases of ILOWG operators is that they support the aggregation of multiplicative linguistic preference relations and allow achieving compatibility solutions with a great level of agreement. But, there may be cases in which experts may not present complete preferences because of not possessing a precise or sufficient level of knowledge of part of the problem. In future works, we will extend these cases of ILOWG operators to the incomplete multiplicative linguistic decision frameworks.

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