Bistable Traveling Waves for Monotone Semiflows with Applications

Jian Fang\textsuperscript{1,2} and Xiao-Qiang Zhao\textsuperscript{2*}
\textsuperscript{1} Department of Mathematics, Harbin Institute of Technology
Harbin 150001, China
\textsuperscript{2} Department of Mathematics and Statistics, Memorial University of Newfoundland
St. John’s, NL A1C 5S7, Canada

Abstract

This paper is devoted to the study of traveling waves for monotone evolution systems of bistable type. Under an abstract setting, we establish the existence of bistable traveling waves for discrete and continuous-time monotone semiflows. This result is then extended to the cases of periodic habitat and weak compactness, respectively. We also apply the developed theory to four classes of evolution systems.

Keywords: Monotone semiflows, traveling waves, bistable dynamics, periodic habitat.

AMS MSC 2010: 37C65, 35C07, 35K55, 35B40.

1 Introduction

In this paper, we study traveling waves for monotone (i.e., order-preserving) semiflows $\{Q_t\}_{t \in \mathcal{T}}$ with the bistability structure on some subsets of the space $\mathcal{C} := C(\mathcal{H}, \mathcal{X})$ consisting of all continuous functions from the habitat $\mathcal{H}(= \mathbb{R} \text{ or } \mathbb{Z})$ to the Banach lattice $\mathcal{X}$, where $\mathcal{T} = \mathbb{Z}^+$ or $\mathbb{R}^+$ is the set of evolution times. Here the bistability structure is generalized from a number of studies for various evolution equations. It means that the restricted semiflow on $\mathcal{X}$ admits two ordered stable equilibria, between which all others are unstable. We focus on the existence of traveling waves connecting these two stable equilibria, which are called bistable traveling waves. This setting allows us to study not only autonomous and time-periodic evolution systems in a homogeneous habitat (media), but also those in a

\*Corresponding author. E-mail address: zhao@mun.ca
periodic habitat. Besides, the obtained results can be extended to the semiflows with weak compactness on some subsets of the space $\mathcal{M}$ consisting of all monotone functions from $\mathbb{R}$ to $\mathcal{X}$.

To explain the concept of the bistability structure, we recall some related works on typical evolution equations. Fife and McLeod \cite{19, 20} proved the existence and global asymptotic stability of monotone traveling waves for the following reaction-diffusion equation:

$$ u_t = u_{xx} + u(1 - u)(u - a), \quad x \in \mathbb{R}, \ t > 0, $$

where $a \in (0, 1)$. Clearly, the restriction of system (1.1) on $\mathcal{X} = \mathbb{R}$ is the ordinary differential equation $u' = u(1 - u)(u - a)$, which admits a unique unstable equilibrium between two ordered and stable ones. The same property is shared by the nonlocal dispersal equation in \cite{3, 4, 46} and the lattice equations in \cite{3, 49, 50}. Chen \cite{13} studied a general nonlocal evolution equation $u_t = A(u(\cdot, t))$, which also possess the above bistability structure. Some related investigations on discrete-time equations can be found in \cite{27, 15}. For the time-periodic reaction-diffusion equation $u_t = u_{xx} + f(t, u)$, the spatially homogeneous equation is a time-periodic ordinary differential equation. In this case, the equilibrium in the bistability structure should be understood as the time-periodic solution. Under such bistability assumption, Alikakos, Bates and Chen \cite{1} obtained the existence of bistable time-periodic traveling waves. Recently, Yagisita \cite{46} studied bistable traveling waves for discrete and continuous-time semiflows on the space consisting of all left-continuous and non-decreasing functions from $\mathbb{R}$ to $\mathcal{X} = \mathbb{R}$ under the assumption that there is exactly one intermediate unstable equilibrium. It should be mentioned that the result in \cite{46} for continuous-time semiflows requires an additional assumption on the existence of a pair of upper and lower solutions.

Note that the restrictions on $\mathcal{X} = \mathbb{R}$ of the afore-mentioned systems are all scalar equations, and hence, there is only one unstable equilibrium in between two stable ones. But in the case where $\mathcal{X} = \mathbb{R}^n$, there may be multiple unstable equilibria. This is one of the main reasons why some ideas and techniques developed for scalar equations can not be easily extended to higher dimensional systems. Volpert \cite{41} established the existence and stability of traveling waves for the bistable reaction-diffusion system $u_t = D \Delta u + f(u)$ by using topological methods, where $D$ is a positive definite diagonal matrix. Fang and Zhao \cite{18} further extended these results to the case where $D$ is semi-positive definite via the vanishing viscosity approach.

Consider the following parabolic equation in a cylindrical domain $\Sigma = \mathbb{R} \times \Omega$:

$$\begin{cases}
  u_t = \Delta u + \alpha(y)u_x + f(u), & x \in \mathbb{R}, y = (y_1, \ldots, y_{n-1}) \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial \Omega \times (0, +\infty),
\end{cases}
$$

where $f$ is of the same type as the nonlinearity in (1.1) and $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^{n-1}$. Obviously, the restriction of the solution semiflow
of (1.2) on $\mathcal{X} = C(\bar{\Omega}, \mathbb{R})$ gives rise to the following $x$-independent system:

$$
\begin{cases}
    u_t = \Delta_y u + f(u), & y \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, +\infty).
\end{cases} \tag{1.3}
$$

One can see from Matano [30] (or Casten and Holland [12]) that any nonconstant steady state of (1.3) is linearly unstable when the domain $\Omega$ is convex. It follows that if $\Omega$ is convex, then (1.2) admits the bistability structure: its $x$-independent system has two (constant) linearly stable steady states, between which all others are linearly unstable. In such a case, Berestycki and Nirenberg [11] obtained the existence and uniqueness of bistable traveling waves. In the case where $\Omega$ is an appropriate dumbbell-shaped domain, Matano [30] constructed a counterexample to show that (1.3) has stable non-constant steady states, and Berestycki and Hamel [9] also proved the nonexistence of traveling waves connecting two stable constant steady states. For bistable traveling waves in time-delayed reaction-diffusion equations, we refer to [31, 35, 29, 36]. For such an equation with time delay $\tau > 0$, one can choose $X = C([-\tau, 0], \mathbb{R})$ so that its solution semiflow has the bistability structure.

Recently, there is an increasing interest in reaction diffusion equations in periodic habitats. A typical example is

$$
u_t = (du_x)_x + f(u), \quad x \in \mathbb{R}, t > 0, \tag{1.4}$$

where $d \in C^1(\mathbb{R}, \mathbb{R})$ is a positive periodic function with period $r > 0$. Define $\mathcal{Y} := C([0, r], \mathbb{R})$ and $C_{\text{per}}(\mathbb{R}, \mathbb{R}) := \{f \in C(\mathbb{R}, \mathbb{R}) : f(x + r) = f(x), \forall x \in \mathbb{R}\}$. It is easy to see that

$$C(\mathbb{R}, \mathbb{R}) = \{f \in C(\mathbb{R}, \mathbb{R}) : f(r(i))(r) = f(r(i + 1))(0), \forall i \in \mathbb{Z}\} := \mathcal{K},$$

and that any element in $C_{\text{per}}(\mathbb{R}, \mathbb{R})$ is a constant function in $\mathcal{K}$. Thus, the solution semiflow of (1.4) on $C(\mathbb{R}, \mathbb{R})$ can be regarded as a conjugate semiflow on $\mathcal{K}$, and hence, the bistability structure should be understood as: the restriction of the solution semiflow of (1.4) on $C_{\text{per}}(\mathbb{R}, \mathbb{R})$ has two ordered $r$-periodic steady states, between which all others are unstable. Assuming that the function $f$ is of bistable type, Xin [42] obtained the existence of spatially periodic (pulsating) traveling wave as long as $d$ is sufficiently close to a positive constant in a certain sense (see also [43, 44]). However, whether the solution semiflow of (1.4) admits the bistability structure remains an open problem. We will give an affirmative answer in section 6.3 and further improve Xin’s existence result. Meanwhile, a counterexample will be constructed to show that the solution semiflow of (1.4) has no bistability structure in the general case of varying $d(x)$. More recently, Chen, Guo and Wu [14] proved the existence, uniqueness and stability of spatially periodic traveling waves for one-dimensional lattice equations in a periodic habitat under the bistability assumption. There are also other types of bistable waves (see, e.g., [7, 33]). For monostable systems in periodic habitats, we refer to [5, 9, 11, 22, 23, 26] and references therein.
In general, there are multiple intermediate unstable equilibria in between two stable ones in the case where the space \(X\) is high dimensional. Meanwhile, it is possible for the given system to have intermediate unstable time-periodic orbits in \(X\). These give more difficulties to the study of bistable semiflows than mostable ones, whose restricted systems on \(X\) have only one unstable and one stable equilibria. To overcome these difficulties, we will show that all these unstable equilibria and all points in these periodic orbits are unordered in \(X\) under some appropriate assumptions. With this in mind, a bistable system can be regarded as the union of two monostable systems although such a union is not unique. From this point of view, we establish a link between monostable subsystems and the bistable system itself, which plays a vital role in the propagation of bistable traveling waves. This link is stated in terms of spreading speeds of monostable subsystems (see assumption (A6)). For spreading speeds of various monostable evolution systems, we refer to [2, 9, 22, 25, 26, 28, 37, 38, 48] and references therein.

In our investigation, we consider seven cases: (I) \(T = \mathbb{Z}^+\) and \(H = \mathbb{R}\); (II) \(T = \mathbb{Z}^+\) and \(H = \mathbb{Z}\); (III) \(T = \mathbb{R}^+\) and \(H = \mathbb{R}\); (IV) \(T = \mathbb{R}^+\) and \(H = \mathbb{Z}\); (V) periodic habitat; (VI) weak compactness; (VII) time periodic. For the case (I), we combine the above observations for general bistable semiflows and Yagisita’s perturbation idea in [46] to prove the existence of traveling waves. For the case (III), we use the bistable traveling waves \(\phi_\pm(x+c_\pm,s)\) of discrete-time semiflows \(\{(Q^n)_s\}_n\geq 0\) to approximate the bistable wave of the continuous-time semiflow \(\{Q_t\}_t\geq 0\). This new approach heavily relies on an estimation of the boundedness of \(\frac{1}{s}c_\pm,s\) as \(s \to 0\), which is proved by the bistability structure of the semiflow (see inequalities (3.9) and (3.10)). It turns out that our result does not require the additional assumption on the existence of a pair of upper and lower solutions as in [46]. In the case (II), both the evolution time \(T\) and the habitat \(H\) are discrete, a traveling wave \(\psi(i+cn)\) of \(\{Q^n\}_n\geq 0\) cannot be well-defined in the usual way because the wave speed \(c\) and hence, the domain of \(\psi\) is unknown. So we define it to be a traveling wave of an associated map \(\tilde{Q}\). However, \(\tilde{Q}\) has much weaker compactness than \(Q\). To overcome this difficulty, we establish a variant of Helly’s theorem for monotone functions from \(\mathbb{R}\) to \(\mathcal{X}\) in the Appendix, which is also of its own interest. This discovery also enables us to study monotone semiflows in a periodic habitat and with weak compactness, respectively. Further, we can deal with the case (IV) by the similar idea as in the case (III) because now traveling waves in the case (II) are defined on \(\mathbb{R}\). Traveling waves for a time-periodic system can be obtained with the help of the discrete-time semiflow generated by the associated Poincaré map. Motivated by the discussions in [26, Section 5], we can regard a semiflow in a periodic habitat as a conjugate semiflow in a homogeneous discrete habitat, and hence, we can employ the arguments for the cases (II) and (IV) to establish the existence of spatially periodic bistable traveling waves.

The rest of this paper is organized as follows. In section 2, we present our main assumptions. Section 3 is focused on discrete-time, continuous-time, and time-periodic compact semiflows on some subsets of \(\mathcal{C}\). In section 4, we extend our results
to compact semiflows in a periodic habitat. In section 5, we further investigate
semiflows with weak compactness. In section 6, we apply the abstract results to four
classes of evolution systems: a time-periodic reaction-diffusion system, a parabolic
system in a cylinder, a parabolic equation with periodic diffusion, and a time-delayed
reaction-diffusion equation. A short appendix section completes the paper.

2 Notations and assumptions

Throughout this paper, we assume that $\mathcal{X}$ is an ordered Banach space with the
norm $\| \cdot \|_{\mathcal{X}}$ and the cone $\mathcal{X}^+$. Further, we assume that $\mathcal{X}$ is also a vector lattice
with the following monotonicity condition:

$$|x|_{\mathcal{X}} \leq |y|_{\mathcal{X}} \Rightarrow \|x\|_{\mathcal{X}} \leq \|y\|_{\mathcal{X}},$$

where $|z|_{\mathcal{X}} := \sup\{z, -z\}$. Such a Banach space is called a Banach lattice. We
use $C(M, \mathbb{R}^d)$ to denote the set of all continuous functions from the compact metric
space $M$ to the $d$-dimensional Euclidean space $\mathbb{R}^d$. We equip $C(M, \mathbb{R}^d)$ with the
maximum norm and the standard cone consisting of all nonnegative functions. Then
$C(M, \mathbb{R}^d)$ is a special Banach lattice, which will be used in this paper. For more
general information about Banach lattices, we refer to the book [32].

Let the spatial habitat $H$ be the real line $\mathbb{R}$ or the lattice
$$r\mathbb{Z} := \{\cdots, -2r, -r, 0, r, 2r, \cdots\}$$
for some positive number $r$. For simplicity, we let $r = 1$. We say a function
$\phi : H \to \mathcal{X}$ is bounded if the set $\{\|\phi(x)\|_{\mathcal{X}} : x \in H\} \subset \mathcal{X}$ is bounded. Throughout
this paper, we always use $B$ to denote the set of all bounded functions from $\mathbb{R}$ to
$\mathcal{X}$, and $C$ to denote the set of all bounded and continuous functions from $H$ to $\mathcal{X}$.
Moreover, any element in $\mathcal{X}$ can be regarded as a constant function in $B$ and $C$.

In this paper, we equip $C$ with the compact open topology, that is, a sequence
$\phi_n$ converges to $\phi$ in $C$ if and only if $\phi_n(x)$ converges to $\phi(x)$ in $\mathcal{X}$ uniformly for $x$
in any bounded subset of $H$. The following norm on $C$ can induce such topology:

$$\|\phi\|_C = \sum_{k=1}^{\infty} \frac{\max_{|x| \leq k} \|\phi(x)\|_{\mathcal{X}}}{2^k}, \quad \forall \phi \in C. \quad (2.1)$$

Clearly, if $H = \mathbb{Z}$, then $\phi_n \to \phi$ with respect to the compact open topology if and
only if $\phi_n(x) \to \phi(x)$ for every $x \in \mathbb{Z}$.

We assume that $\text{Int}(\mathcal{X}^+)$ is not empty. For any $u, v \in \mathcal{X}$, we write $u \geq v$
provided $u - v \in \mathcal{X}^+$, $u > v$ provided $u \geq v$ but $u \neq v$, and $u \gg v$ provided
$u - v \in \text{Int}(\mathcal{X}^+)$. A set $E \subset \mathcal{X}$ is said to be totally unordered if any two elements
(if exist) are unordered. For any $\phi, \psi \in C$, we write $\phi \geq \psi$ provided $\phi(x) \geq \psi(x)$
for all $x \in H$, $\phi > \psi$ provided $\phi \geq \psi$ but $\phi \neq \psi$, and $\phi \gg \psi$ provided $\phi(x) \gg \psi(x)$
for all $x \in H$. For any $\gamma \in \mathcal{X}$ with $\gamma > 0$, we define $\mathcal{X}_\gamma := \{u \in \mathcal{X} : \gamma \geq u \geq 0\}$,
Let $\gamma \in C : \gamma \geq \phi \geq 0$ and $B_\gamma := \{ \phi \in B : \gamma \geq \phi \geq 0 \}$. For any $\phi, \psi \in C$, we write the interval $[\phi, \psi]_C$ to denote the set $\{ w \in C : \phi \leq w \leq \psi \}$, $[[\phi, \psi]]_C$ to denote the set $\{ w \in C : \phi \ll w \ll \psi \}$, and similarly, we can write the intervals $[\phi, \psi]_X$ and $[[\phi, \psi]]_X$ in a similar way.

Let $\beta \in Int(X^+)$ and $Q$ be a map from $C_\beta$ to $C_\beta$. Let $E$ be the set of all fixed points of $Q$ restricted on $X_\beta$.

**Definition 2.1.** For the map $Q : X_\beta \to X_\beta$, a fixed point $\alpha \in E$ is said to be strongly stable from below if there exist a number $\delta > 0$ and a unit vector $e \in Int(X^+)$ such that $Q[\alpha - \eta e] \gg \alpha - \eta e$ for any $\eta \in (0, \delta]$.

Strong instability from below is defined by reversing the inequality (2.2). Similarly, we can define strong stability (instability) from above.

Given $y \in H$, define the translation operator $T_y$ on $B$ by $T_y[\phi](x) = \phi(x - y)$. Assume that 0 and $\beta$ are in $E$. We impose the following hypothesis on $Q$:

- **(A1)** (Translation Invariance) $T_y \circ Q[\phi] = Q \circ T_y[\phi], \forall \phi \in C_\beta, y \in H$.
- **(A2)** (Continuity) $Q : C_\beta \to C_\beta$ is continuous with respect to the compact open topology.
- **(A3)** (Monotonicity) $Q$ is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in $C_\beta$.
- **(A4)** (Compactness) $Q : C_\beta \to C_\beta$ is compact with respect to the compact open topology.
- **(A5)** (Bistability) Two fixed points 0 and $\beta$ are strongly stable from above and below, respectively, for the map $Q : X_\beta \to X_\beta$, and the set $E \setminus \{0, \beta\}$ is totally unordered.

Note that the above bistability assumption is imposed on the spatially homogeneous map $Q : X_\beta \to X_\beta$. We allow the existence of other fixed points on the boundary of $X_\beta$ so that the theory is applicable to two species competitive evolution systems. The non-ordering property of $E \setminus \{0, \beta\}$ can be obtained by the strong instability of all fixed points in this set if the semiflow is eventually strongly monotone. More precisely, a sufficient condition for hypothesis (A5) to hold is:

- **(A5’)** (Bistability) $Q : X_\beta \to X_\beta$ is eventually strongly monotone in the sense that there exists $m_1 \in \mathbb{Z}_+$ such that $Q^m[u] \gg Q^m[v]$ for all $m \geq m_1$ whenever $u > v$ in $X_\beta$. Further, for the map $Q : X_\beta \to X_\beta$, two fixed points 0 and $\beta$ are strongly stable from above and below, respectively, and each $\alpha \in E \setminus \{0, \beta\}$ (if exists) is strongly unstable from both below and above.
Define \( \eta \). Since \( Q \) is strongly unstable from below, we can find \( \delta \) such that \( u_n := u_0 + \delta u_0, e_1 \in [\alpha, \eta) \) and \( Q[u_0] \gg u_0 \). Define the recursion \( u_{n+1} = Q[u_n], n \geq 0 \). Then \( u_n \) is convergent to some \( \alpha \in \mathcal{X} \) with \( \alpha_1 \) due to hypothesis (A4). By the eventual strong monotonicity of \( Q \), we see that

\[
\alpha_1 \subseteq Q^{m_1}[u_{n-m_1}] \ll Q^{m_1}[u_{n+1-m_1}] = u_{n+1} \ll Q^{m_1} = \alpha, \forall n \geq m_1.
\]

Since \( \alpha \) is strongly unstable from below, we can find \( \delta \eta > 0 \) and \( e_\eta \in Int\mathcal{X}^+ \) such that \( Q[\alpha - \delta \eta e_\eta] \ll_\mathcal{X} \alpha - \delta \eta e_\eta, \forall \delta \in (0, \delta_\eta] \). Choose \( n_1 \geq m_1 \) such that \( u_1 \geq \alpha - \delta \eta e_\eta \). Define \( \eta := \sup\{ \delta \in (0, \delta_\eta) : u_1 \leq \alpha - \delta \eta e_\eta \} \). Thus, \( u_{n_1} \ll Q[\alpha - \eta e_\eta] \ll_\mathcal{X} \alpha - \eta e_\eta, \)

a contradiction. \( \square \)

Due to assumption (A5), a bistable system \( \{Q^n\}_{n \geq 0} \) can be regarded as the union of two monostable systems. More precisely, assuming that \( \alpha \in E \setminus \{0, \beta\} \), we have two monostable sub-systems: \( \{Q^n\}_{n \geq 0} \) restricted on \([0, \alpha]_C \) and \([\alpha, \beta]_C \), respectively. With this in mind, next we construct an initial function \( \phi^-_\alpha \) so that we can define the leftward asymptotic speed of propagation of \( \phi^-_\alpha \), and hence, present our last assumption.

Note that in (A5) we do not require \( \alpha \gg 0 \) or \( \alpha \ll \beta \). But (A5) is sufficient to guarantee that \( \alpha \) and \( \beta \) can be separated by two neighborhoods in \([\alpha, \beta]_\mathcal{X} \), and a similar claim is valid for \( 0 \) and \( \alpha \) (see Lemma 3.1). In view of assumption (A5), we can find a positive number \( \delta_\beta > 0 \) and a unit vector \( e_\beta \in int(\mathcal{X}^+) \) such that

\[
Q[\beta - \eta e_\beta] \gg_\mathcal{X} \beta - \eta e_\beta, \quad \forall \eta \in (0, \delta_\beta],
\]  

(2.3)
Define
\[ \theta^- := \sup \{ \theta \in [0, 1] : \theta \alpha + (1 - \theta) \beta \in [\beta - \delta \beta_e, \beta] \}. \]

Let
\[ v^-_\alpha = \theta^- \alpha + (1 - \theta^-) \beta. \] (2.4)

Choose a nondecreasing initial function \( \phi^-_\alpha \in C_\beta \) with the property that
\[ \phi^-_\alpha(x) = \alpha, \forall x \leq -1, \quad \text{and} \quad \phi^-_\alpha(x) = v^-_\alpha, \forall x \geq 0. \] (2.5)

It then follows from assumptions (A1)-(A2) and (A5) that
\[ \lim_{x \to +\infty} Q[\phi^-_\alpha](x) = Q[\phi^-_\alpha(+\infty)](0) = Q[v^-_\alpha] \geq Q[\beta - \delta \beta_e] \gg \beta - \delta \beta_e, \]
and hence, there exists \( \sigma > 0 \) such that
\[ Q[\phi^-_\alpha](x) \gg \beta - \delta \beta_e, \forall x \geq \sigma - 1. \)

Define a sequence \( a_{n,\sigma} \) of points in \( \mathcal{X} \) as follows:
\[ a_{n,\sigma} = Q^n[\phi^-_\alpha](\sigma n), \quad n \geq 1. \]

Then we have
\[ a_{2,\sigma} = Q^2[\phi^-_\alpha](2\sigma) = Q[Q[\phi^-_\alpha](\cdot + \sigma)](\sigma) \geq Q[\phi^-_\alpha](\sigma) = a_{1,\sigma}. \]

By induction, we see that \( a_{n,\sigma} \) is nondecreasing in \( n \). Thus, assumption (A4) implies that \( a_{n,\sigma} \) tends to a fixed point \( e \) with \( e \geq a_{1,\sigma} \gg \beta - \delta \beta_e \). Therefore, \( e = \beta \).

By the above observation, we have
\[ \beta \geq \lim_{n \to \infty, x \geq \sigma n} Q^n[\phi^-_\alpha](x) \geq \lim_{n \to \infty} Q^n[\phi^-_\alpha](\sigma n) = \lim_{n \to \infty} a_{n,\sigma} = \beta, \]
and hence,
\[ (-\infty, -\sigma] \subset \Lambda(\phi^-_\alpha) := \left\{ c \in \mathbb{R} : \lim_{n \to \infty, x \geq -\sigma n} Q^n[\phi^-_\alpha](x) = \beta \right\}. \]

Define
\[ c^+_{\phi^-_\alpha}(\alpha, \beta) := \sup \Lambda(\phi^-_\alpha). \] (2.6)

Clearly, \( c^+_{\phi^-_\alpha}(\alpha, \beta) \in [-\sigma, +\infty] \) and \( (-\infty, c^+_{\phi^-_\alpha}(\alpha, \beta)) \subset \Lambda(\phi^-_\alpha) \). We further claim that \( c^+_{\phi^-_\alpha}(\alpha, \beta) \) is independent of the choice of \( \phi^-_\alpha \) as long as \( \phi^-_\alpha \) has the property (2.5).

Indeed, for any given \( \phi \) with the property (2.5), we have
\[ \phi^-_\alpha(x - 1) \leq \phi(x) \leq \phi^-_\alpha(x + 1), \quad \forall x \in \mathcal{H}. \]

It then follows that for any \( c \in \Lambda(\phi^-_\alpha) \) and \( \epsilon > 0 \),
\[ \beta = \lim_{n \to \infty, x \geq -cn} Q^n[\phi^-_\alpha](x) = \lim_{n \to \infty, x \geq -(c-\epsilon)n} Q^n[\phi^-_\alpha](x - 1) \leq \lim_{n \to \infty, x \geq -(c-\epsilon)n} Q^n[\phi](x) \leq \lim_{n \to \infty, x \geq -(c-\epsilon)n} Q^n[\phi^-_\alpha](x + 1) = \lim_{n \to \infty, x \geq -cn} Q^n[\phi^-_\alpha](x) = \beta, \]
which implies that \( c - \epsilon \in \Lambda(\phi) \) and hence, \( \sup \Lambda(\phi^-) = \sup \Lambda(\phi). \) For convenience, we may call \( c^-_*(\alpha, \beta) \) as the leftward asymptotic speed of propagation of \( \phi^-_\alpha. \)

Following the above procedure, we can find \( \delta_0 > 0, e_0 \in \text{Int}(\mathcal{X}^+) \) such that
\[
Q[\eta e_0] \ll \eta e_0, \forall \eta \in (0, \delta_0].
\] (2.7)

Here we emphasis that \( \delta_0, e_0 \) above and \( \delta_\beta, e_\beta \) will play a vital role in the whole paper because they describe the local stability of fixed points 0 and \( \beta. \) Similarly, we can define \( \theta^+ := \sup \{ \theta \in [0, 1] : \theta \alpha \in [0, \delta_0 e_0]_{\mathcal{X}} \} \) and \( v^+_\alpha := \theta^+ \alpha. \) Let \( \phi^+_\alpha \in C_\beta \) be a nondecreasing initial function with the property that
\[
\phi^+_\alpha(x) = \alpha, \forall x \geq 1, \quad \text{and} \quad \phi^+_\alpha(x) = v^+_\alpha, \forall x \leq 0.
\]

Due to the same reason, we can define the number
\[
c^+*(0, \alpha) := \sup \left\{ c \in \mathbb{R} : \lim_{n \to \infty, x \leq cn} Q^n[\phi^+_\alpha](x) = 0 \right\},
\] (2.8)

which is called the rightward asymptotic speed of propagation of \( \phi^+_\alpha. \) As showed above, these two speeds are bounded below, but may be plus infinity. To better understand these two spreading speeds, we use Figure 2 (Left) to explain them.

Now we are ready to state our last assumption on \( Q: \)

(A6) (Counter-propagation) For each \( \alpha \in E \setminus \{0, \beta\}, \) \( c^-_*(\alpha, \beta) + c^+*(0, \alpha) > 0. \)

Assumption (A6) assures that two initial functions in the left hand side of Figure 2 will eventually propagate oppositely although one of these two speeds may be negative. It is interesting to note that assumption (A6) is nearly necessary for the propagation of a bistable traveling wave. Indeed, if a monotone evolution system admits a bistable traveling wave, then it is usually unique (up to translation) and globally attractive (see, e.g., Remark 6.2). This implies that the solution starting from the initial data \( \frac{1}{2}(\phi^+_\alpha + \phi^-_\alpha) \) converges to a phase shift of the bistable wave. If \( c^-_*(\alpha, \beta) + c^+*(0, \alpha) < 0, \) then the comparison principle would force the solutions starting from \( \phi^-_\alpha \) to split the bistable wave.

Comparing with the definition of spreading speeds (short for asymptotic speeds of spread/propagation) for monostable semiflows (see, e.g., [2, 26]), one can find
that the leftward spreading speed of the monostable subsystem \( Q^n \) restricted on \([\alpha, \beta]_C\) is shared by a large class of initial functions, and in many applications, it equals \( c^*_-(\alpha, \beta) \). A similar observation holds for \( c^*_+(0, \alpha) \). Thus, for a specific bistable system, the assumption (A6) can be verified by using the properties of spreading speeds for monostable subsystems.

**Remark 2.1.** If we consider the non-increasing traveling waves, then we can similarly define the numbers \( c^*_+(\alpha, \beta) \) and \( c^*_-(0, \alpha) \) (See Figure 2(Right)). As such, (A6) should be stated as \( c^*_+(\alpha, \beta) + c^*_-(0, \alpha) > 0 \).

# 3 Semiflows in a homogeneous habitat

We say a habitat is homogeneous for the semiflow \( \{Q_t\}_{t \in T} \) on a metric space \( E \subset C \) if

\[
Q_t[\phi](x - y) = Q_t[\phi(x - y)](x), \quad \forall \phi \in E, x, y \in H, t \in T. 
\]

In this section, we will establish the existence of bistable traveling waves for the semiflow \( \{Q_t\}_{t \in T} \) on \( E \) in the following order: discrete-time semiflows in a continuous habitat, discrete-time semiflows in a discrete habitat, time-periodic semiflows, continuous-time semiflows in a continuous habitat, and continuous-time semiflows in a discrete habitat.

## 3.1 Discrete-time semiflows in a continuous habitat

In this case, time \( T \) is discrete and habitat \( H \) is continuous: \( T = \mathbb{Z}^+ \) and \( H = \mathbb{R} \). For convenience, we use \( Q \) to denote \( Q_1 \), and consider the semiflow \( \{Q^n\}_{n \geq 0} \), where \( Q^n \) is the \( n \)-th iteration of \( Q \).

**Definition 3.1.** \( \psi(x + cn) \) with \( \psi \in C \) is said to be a traveling wave with speed \( c \in \mathbb{R} \) of the discrete semiflow \( \{Q^n\}_{n \geq 0} \) if \( Q^n[\psi](x) = \psi(x + cn) \), \( \forall x \in \mathbb{R}, n \geq 0 \). We say that \( \psi \) connects 0 to \( \beta \) if \( \psi(-\infty) := \lim_{x \to -\infty} \psi(x) = 0 \) and \( \psi(+\infty) := \lim_{x \to +\infty} \psi(x) = \beta \).

We first show that 0 and \( \beta \) are two isolated fixed points of \( Q \) in \( X_\beta \) if (A5) holds.

**Lemma 3.1.** Let \( \delta_0, \epsilon_0 \) and \( \delta_\beta, \epsilon_\beta \) be chosen such that (2.7) and (2.3) hold, respectively. Then \( E \cap X_{\delta_0 \epsilon_0} = \{0\} \) and \( E \cap [\beta - \delta_\beta \epsilon_\beta, \beta]_X = \{\beta\} \).

**Proof.** Assume, for the sake of contradiction, that \( 0 \neq \alpha \in E \cap X_{\delta_0 \epsilon_0} \). Define the number \( \bar{\delta} \in (0, \delta_0] \) by

\[
\bar{\delta} := \inf\{\delta \in (0, \delta_0] : \alpha \in [0, \delta \epsilon_0]_X\}. 
\]

Then it follows that \( \alpha \leq \bar{\delta} \epsilon_0 \) but \( \alpha \not\in [0, \bar{\delta} \epsilon_0]_X \). However, by the monotonicity of \( Q \) and the fact that 0 is strongly stable, we have

\[
\alpha = Q[\alpha] \leq Q[\bar{\delta} \epsilon_0] \ll \bar{\delta} \epsilon_0.
\]
This contradicts $\alpha \notin [0, \delta e_0]_\chi$. And hence, $E \cap \mathcal{X}_{\delta e_0} = \{0\}$. Similarly, we have $E \cap [\beta - \delta \epsilon_\beta, \beta]_\chi = \{\beta\}$. \hfill $\square$

Choose $\delta > 0$ such that

$$\delta < \min\{\delta_0, \delta_\beta\} \quad \text{and} \quad \delta e_0 \ll \beta - \delta \epsilon_\beta. \quad (3.1)$$

Assume that $\psi$ and $\bar{\psi}$ are two nondecreasing functions in $C(\mathbb{R}, \mathcal{X}_\beta)$ with the properties that

$$\psi(x) = \begin{cases} 0, & x \leq 0 \\ \beta - \delta \epsilon_\beta, & x \geq 1 \end{cases} \quad \text{and} \quad \bar{\psi}(x) = \begin{cases} \delta e_0, & x \leq -1 \\ \beta, & x \geq 0. \end{cases}$$

Clearly $\bar{\psi} \leq \bar{\psi}$. And we have the following observation.

**Lemma 3.2.** Assume that $Q$ satisfies (A1)-(A3) and (A5). Then there exists a positive rational number $\bar{c}$ such that for any $c \geq \bar{c}$, we have

$$Q[\bar{\psi}](x) \geq \bar{\psi}(x - c) \quad \text{and} \quad Q[\bar{\psi}](x) \leq \bar{\psi}(x + c) \quad \text{for any} \ x \in \mathbb{R}.$$

**Proof.** Assume that $x_n \to +\infty$ be an increasing sequence in $\mathbb{R}$. Then the sequence $\psi_n := \psi(\cdot + x_n)$ converges to $\beta - \delta \epsilon_\beta$ in $\mathcal{C}_\beta$ since $\psi(x) = \beta - \delta \epsilon_\beta$, $\forall x \geq 1$. It then follows from (A1)-(A2) and (A5) that

$$Q[\psi](+\infty) = \lim_{n \to \infty} Q[\psi](x_n) = \lim_{n \to \infty} Q[\psi(\cdot + x_n)](0) = Q[\beta - \delta \epsilon_\beta] \gg \beta - \delta \epsilon_\beta.$$

Therefore, there exists a positive $y_0 \in \mathbb{R}$ such that $Q[\psi](y_0) \geq \beta - \delta \epsilon_\beta$. Note that $Q[\psi](x)$ is nondecreasing in $x$. Then for any $c \geq y_0$ we have

$$Q[\psi](x) \geq Q[\psi](y_0) \geq \beta - \delta \epsilon_\beta \geq \psi(x - c), \forall x \geq y_0$$

and

$$Q[\psi](x) \geq 0 = \psi(0) \geq \bar{\psi}(x - y_0) \geq \bar{\psi}(x - c), \forall x < y_0,$$

which means $Q[\psi](x) \geq \bar{\psi}(x - c), \forall x \geq y_0$. Similarly, we have

$$Q[\bar{\psi}](+\infty) = \lim_{n \to \infty} Q[\bar{\psi}](x_n) = \lim_{n \to \infty} Q[\bar{\psi}(\cdot - x_n)](0) = Q[\delta e_0] \ll \delta e_0 = \bar{\psi}(\infty),$$

and hence, there exists $z_0 > 0$ such that $Q[\bar{\psi}](x) \leq \bar{\psi}(x + c), \forall c \geq z_0$. Choosing $\bar{c} = \max\{y_0, z_0\}$, we complete the proof. \hfill $\square$

Let $\kappa_n := \frac{n + \bar{c}}{n}$. Clearly, $\kappa_n, \forall n \geq 1$, is a rational number. For any $\xi \in \mathbb{R}$, define the map $A_\xi : \mathcal{B} \to \mathcal{B}$ by $A_\xi[\phi](x) = \phi(\xi x), \forall x \in \mathbb{R}$. Define $\psi_n, \bar{\psi}_n \in \mathcal{C}_\beta$ by

$$\psi_n(x) = \psi \left( x - (n + \bar{c}) \right) \quad \text{and} \quad \bar{\psi}_n(x) = \bar{\psi} \left( x + (n + \bar{c}) \right).$$

**Lemma 3.3.** Assume that $Q$ satisfies (A1)-(A5). Then for each $n \in \mathbb{N}$, $G_n := Q \circ A_{\kappa_n}$ has a fixed point $\phi_n$ in $\mathcal{C}_\beta$ such that $\phi_n$ is nondecreasing and $\underline{\psi}_n \leq \phi_n \leq \bar{\psi}_n$. 

11
Proof. We first show that \( \psi_n \leq G_n[\psi_n] \). Indeed, when \( x < n \) we have
\[
\psi_n(x + \bar{c}) \leq \psi_n(n + \bar{c}) = \psi(0) = 0 \leq A_{\kappa n}[\psi_n](x);
\]
when \( x \geq n \) we have
\[
A_{\kappa n}[\psi_n](x) = \psi_n(\kappa_n x) = \psi_n(x + \frac{\bar{c}}{n} x) \geq \psi_n(x + \bar{c}),
\]
and hence, \( \psi_n(x + \bar{c}) \leq A_{\kappa n}[\psi_n](x) \) for all \( x \in \mathbb{R} \). Consequently, by the monotonicity of \( Q \) and \( \psi(x) \leq Q[\psi](x + \bar{c}) \) (see Lemma 3.2) we obtain
\[
\psi_n(x) \leq Q[\psi_n](x + \bar{c}) \leq Q \circ A_{\kappa n}[\psi_n](x) = G_n[\psi_n](x).
\]
Similarly, we have \( \bar{\psi}_n \geq G_n[\bar{\psi}_n] \). It then follows that
\[
\psi_n \leq G^k_n[\psi_n] \leq G^k_n[\bar{\psi}_n] \leq \bar{\psi}_n, \quad \forall k \in \mathbb{N}.
\]
(3.2)
For any \( k \geq 1 \), we have
\[
G^k_n[\psi_n] = G_n \circ G^{k-1}_n[\psi_n] \in G_n[C_\beta].
\]
(3.3)
Since \( G_n \) is order preserving and \( \psi_n(x) \) is nondecreasing in \( x \), we know that \( G^k_n[\psi_n](x) \) is nondecreasing both in \( k \) and \( x \). Recall that \( G_n \) is compact due to assumption (A4). It then follows that \( G^k_n[\psi_n] \) converges in \( C_\beta \). Denote the limit by \( \phi_n \). By inequality (3.2), we also get \( \psi_n \leq \phi_n \leq \bar{\psi}_n \). Moreover, \( \phi_n(x) \) is also nondecreasing due to Proposition 7.1(2). And obviously,
\[
\phi_n = \lim_{k \to \infty} G^{k+1}_n[\psi_n] = G_n[\lim_{k \to \infty} G^k_n[\psi_n]] = G_n[\phi_n].
\]
This completes the proof.

The following lemma reveals a relation between the wave speeds of monostable traveling waves in the sub-monostable systems and the numbers defined in (2.6) and (2.8).

**Lemma 3.4.** Let \( c^*_-(\alpha, \beta) \) and \( c^*_+(0, \alpha) \) be defined as in (2.6) and (2.8). Assume that \( Q \) satisfies (A3). Then the following statements are valid:

1. If \( \psi(x + ct) \) is a monotone traveling wave connecting \( \alpha \) to \( \beta \) of the discrete semiflow \( \{Q^n\}_{n \geq 1} \), then the speed \( c \geq c^*_-(\alpha, \beta) \).
2. If \( \psi(x + ct) \) is a monotone traveling wave connecting \( 0 \) to \( \alpha \) of the discrete semiflow \( \{Q^n\}_{n \geq 1} \), then the speed \( c \leq -c^*_+(0, \alpha) \).
Proof. We only prove the statement (1) since the proof for (2) is similar. In view of Lemma 3.1, we see that $\alpha$ and $\beta$ can be separated by balls $N(\alpha, \gamma)$ and $N(\beta, \gamma)$ with radius $\gamma < \delta_\beta/2$ in the metric space $[\alpha, \beta]_X$. Then $\alpha < u, \forall u \in N(\beta, \gamma)$. We write $u \in N(\beta, \gamma)$ as the form $u = \beta + \nu$. Recall the definition of $v_\alpha^-$ in (2.4), it then follows that

$$v_\alpha^- = \theta^- \alpha + (1 - \theta^-)\beta < \theta^- (\beta + \nu) + (1 - \theta^-)\beta = \beta + \theta^- \nu,$$

which implies that

$$v_\alpha^- < w, \quad \forall w \in N(\beta, \theta^- \gamma).$$

Since $\psi(-\infty) = \alpha$ and $\psi(+\infty) = \beta$, there must exist a nondecreasing initial function $\phi_\alpha^-$ with property (2.5) such that $\phi_\alpha^- \leq \psi$. Assume, for the sake of contradiction, that $c < c^*_-(\alpha, \beta)$. Choose a rational number $\frac{q}{p} \in (c, c^*_-(\alpha, \beta))$ with $p, q \in \mathbb{Z}$. It then follows from (2.6) that

$$\beta = \lim_{n \to \infty} Q^{pn}[\phi_\alpha^-](-\frac{q}{p} + pn) \leq \lim_{n \to \infty} Q^{pn}[\psi](-qn)$$

$$= \lim_{n \to \infty} \psi(-qn + cpn) = \psi(-\infty) = 0,$$

a contradiction. Thus, we have $c \geq c^*_-(\alpha, \beta)$. \hfill $\square$

Now we are ready to prove the main result of this subsection.

**Theorem 3.1.** Assume that $Q$ satisfies (A1)-(A6). Then there exists $c \in \mathbb{R}$ such that the discrete semiflow $\{Q^n\}_{n \geq 1}$ admits a non-decreasing traveling wave with speed $c$ and connecting 0 to $\beta$.

**Proof.** We spend three steps to complete the proof. Firstly, we construct $\phi_+, \phi_- \in C_\beta$, $c_+ \leq c_- \in \mathbb{R}$ such that

$$Q[\phi_+(x)] = \phi_+(x + c_+) \quad \text{and} \quad Q[\phi_-](x) = \phi_-(x + c_-)$$

with

$$\phi_-(0) \in (0, \delta e_0]_X \quad \text{and} \quad \phi_+(0) \in [\beta - \delta e_\beta, \beta)_X.$$

Indeed, let $\phi_n$ be obtained in Lemma 3.3. Since $0 \ll \psi(-1) = \delta e_0 \ll \psi(1) = \beta - \delta e_\beta \ll \beta$ and $\bar{\psi}_n \leq \phi_n \leq \bar{\psi}_n$, we have

$$\bar{\psi}(-1) = \bar{\psi}_n(-1 - (n + \bar{c})) \geq \phi_n(-1 - (n + \bar{c}))$$

and

$$\bar{\psi}(1) = \bar{\psi}_n(1 + (n + \bar{c})) \leq \phi_n(1 + (n + \bar{c})).$$

Now we define $a_n, b_n \in \mathbb{R}$ as follows:

$$a_n := \sup_{x \in \mathbb{R}} \{\phi_n(x) \in [0, \delta e_0]_X\}, \quad b_n := \inf_{x \in \mathbb{R}} \{\phi_n(x) \in [\beta - \delta e_\beta, \beta]_X\}.$$
It then follows that

\[-1 - (n + \bar{c}) \leq a_n \leq b_n \leq 1 + (n + \bar{c})\]

and

\[\phi_n(a_n) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \phi_n(b_n).\]

Define \(\phi_{-n}(x) := \phi_n(x + a_n)\) and \(\phi_{+n}(x) := \phi_n(x + b_n)\). Then

\[\phi_{-n} = \phi_n(\cdot + a_n) = G_n[\phi_n](\cdot + a_n) = Q[\phi_n(\kappa_n)(\cdot + a_n)](0) = Q[\phi_n(\kappa_n(\cdot + a_n))] \in Q[C_B].\]

Similarly, \(\phi_{+n} = Q[\phi_n(\kappa_n(\cdot + b_n))] \in Q[C_B]\). Thus, there exists a subindex (still denoted by \(n\)), two nondecreasing functions \(\phi_-, \phi_+ \in C_B\) and \(\xi_-, \xi_+ \in [-1, 1]\) with \(\xi_\leq \leq \xi_+\) such that

\[\lim_{n \to \infty} \frac{a_n}{n} = \xi_-, \quad \lim_{n \to \infty} \frac{b_n}{n} = \xi_+, \quad \lim_{n \to \infty} \phi_{-n} = \phi_- \quad \text{and} \quad \lim_{n \to \infty} \phi_{+n} = \phi_+.\]

Obviously, \(\phi_-(0) = \lim_{n \to \infty} \phi_n(a_n)\) and \(\phi_+(0) = \lim_{n \to \infty} \phi_n(b_n)\). By the definitions of \(a_n\) and \(b_n\), we immediately have \(\phi_-(0) \neq 0\) and \(\phi_+(0) \neq \beta\), and hence \(0 < \phi_-(0) \leq \psi(-1) = \delta e_0\) and \(\beta - \delta e_\beta = \psi(1) \leq \phi_+(0) < \beta\). Define \(c_- := -\varepsilon \xi_-\) and \(c_+ := -\varepsilon \xi_+\). Obviously, \(c_- \geq c_+\) because \(\xi_- \leq \xi_+\). Now we want to only prove \(Q[\phi_-](x) = \phi_-(x + c_-)\) because the proof of the other one is similar. Note that the following limit is uniform for \(x\) in any bounded subset \(M \subset \mathbb{R}\)

\[\lim_{n \to \infty} \kappa_n(x + a_n) - a_n = \lim_{n \to \infty} \left( x + \bar{c} \cdot \frac{x + a_n}{n} \right) = x - c_-\]

It then follows that for any \(x \in \mathbb{R}\), we have

\[
\phi_-(x + c_-) = \lim_{n \to \infty} \phi_{-n}(x + c_-) = \lim_{n \to \infty} \phi_n(x + c_- + a_n) = \lim_{n \to \infty} G_n[\phi_n](x + c_- + a_n) \\
= \lim_{n \to \infty} Q[\phi_n(\kappa_n)(\cdot + a_n)](x + c_-) = \lim_{n \to \infty} Q[\phi_n(\kappa_n(\cdot + a_n))] (x + c_-) = Q[\phi_-](x),
\]

where the last equality is obtained from Proposition 7.2(2) and the continuity of \(Q\).

Secondly, we prove that \(\phi_\pm(x)\) obtained in the first step have the following properties:

(i) \(\phi_-(\infty) = 0\) and \(\phi_+(\infty) = \beta\);

(ii) \(\phi_-(\infty)\) and \(\phi_+(\infty)\) are ordered.

Indeed, let \(x_n \to +\infty\) be an increasing sequence in \(\mathbb{R}\). Note that \(\phi_-(x_n) = Q[\phi_-(\cdot - c_- + x_n)](0) \in Q[C_B](0)\), which is precompact in \(X_B\). It then follows that there exists a subindex \(\{n_l\}\) and \(v \in X_B\) such that \(\lim_{l \to \infty} \phi_-(x_{n_l}) = v\), which, together with the fact that \(\phi_-\) is nondecreasing and proposition 7.2(1), implies that \(\phi_-(\infty) :=\)
lim_{x \to +\infty} \phi_-(x) = v. Besides, from (3.4) we see that \( \phi_+ (+\infty) \in X_\beta \) is a fixed point of \( Q \). Similar results hold for \( \phi_- (-\infty) \) and \( \phi_+ (\pm \infty) \). Recall that \( \phi_- (-\infty) \leq \phi_- (0) \leq \delta_0 \) and \( \phi_+ (+\infty) \geq \phi_+ (0) \geq \beta - \delta_\beta \), which, together with the choice of \( \delta \), implies that \( \phi_- (-\infty) = 0 \) and \( \phi_+ (+\infty) = \beta \). Further, since any two real numbers are ordered, we see that there exist sequences

\[
\{n\}_{n \geq 0} \supset \{n_{1m}\}_{m \geq 1} \supset \{n_{2m}\}_{m \geq 2} \supset \cdots \supset \{n_{km}\}_{m \geq 1} \ldots
\]

such that for each \( k \geq 1 \),

\[
k + a_{n_{km}} \leq -k + b_{n_{km}}, \forall m \geq 1 \quad \text{or} \quad k + a_{n_{km}} \geq -k + b_{n_{km}}, \forall m \geq 1.
\]

Define \( \Gamma_1 := \{k \in \mathbb{N} : k + a_{n_{km}} \leq -k + b_{n_{km}}, \forall m \geq 1\} \) and \( \Gamma_2 := \mathbb{N} \setminus \Gamma_1 \). Then either \( \Gamma_1 \) or \( \Gamma_2 \) has infinitely many elements. If \( \Gamma_1 \) does, then there holds

\[
\phi_{-n_{km}}(k) = \phi_{n_{km}}(k + a_{n_{km}}) \leq \phi_{n_{km}}(-k + b_{n_{km}}) = \phi_{+n_{km}}(k), \forall k \in \Gamma_1, m \in \mathbb{N}.
\]

This implies that \( \phi_- (k) \leq \phi_+ (-k), \forall k \in \Gamma_1 \), and hence, \( \phi_- (+\infty) \leq \phi_+ (-\infty) \). If \( \Gamma_2 \) has infinitely many elements, then we have \( \phi_- (+\infty) \geq \phi_+ (-\infty) \) by a similar argument. Thus, \( \phi_- (+\infty) \) and \( \phi_+ (-\infty) \) must be ordered in \( X_\beta \).

Finally, we prove that either \( \phi_- \) or \( \phi_+ \) connects 0 to \( \beta \). Indeed, we have shown in the second step that \( \phi_- (+\infty) \) and \( \phi_+ (-\infty) \) are ordered. It then follows from the bistability assumption (A5) that there are only three possibilities:

(i) \( \beta = \phi_- (+\infty) \geq \phi_+ (-\infty) \);

(ii) \( \phi_- (+\infty) \geq \phi_+ (-\infty) = 0 \);

(iii) \( \phi_- (+\infty) = \alpha = \phi_+ (-\infty) \) for some \( \alpha \in E \setminus \{0, \beta\} \).

We further claim that the possibility (iii) cannot happen. Otherwise, Lemma 3.4 implies that \( c_+ \geq c^*_+(\alpha, \beta) \) and \( c_- \leq c^*_+(0, \alpha) \). Since \( c_- \geq c_+ \), it then follows that

\[
0 \geq c_+ + (-c_-) \geq c^*_+(\alpha, \beta) + c^*_+(0, \alpha),
\]

which contradicts assumption (A6). Thus, either (i) or (ii) holds, and hence, we complete the proof.

\[\square\]

### 3.2 Discrete-time semiflows in a discrete habitat

In this case, both time \( T \) and habitat \( H \) are discrete: \( T = \mathbb{Z}^+ \) and \( H = \mathbb{Z} \). Without confusion, we consider the semiflow \( \{Q^n\}_{n \geq 0} \) in a metric space \( E \subset C \). Since the habitat is discrete, we cannot use the definition of traveling waves with a unknown speed as in Definition 3.1. This is because the wave profile \( \psi(x) \) may not be well-defined for all \( x \in \mathbb{R} \). So we start with the modification of the definition of traveling waves in a discrete habitat.
Definition 3.2. $\psi(x + cn)$ with $\psi \in \mathcal{B}$ is said to be a traveling wave with speed $c \in \mathbb{R}$ of the discrete semiflow $\{Q^n\}_{n \geq 0}$ if there exists a countable set $\Gamma \subset \mathbb{R}$ such that $Q^n[\psi(\cdot + x)](i) = \psi(i + x + c), \forall i \in \mathbb{Z}, x \in \mathbb{R} \setminus \Gamma$.

By Definition 3.2 and Proposition 7.8, it follows that there exists $x_0 \in \mathbb{R}$ such that $Q^n[\psi(\cdot + x_0)](i) = \psi(i + x_0 + cn), \forall i \in \mathbb{Z}, n \geq 0$. Define $\phi(x) := \psi(x + x_0), \forall x \in \mathbb{R}$. Then, with a little abuse of notation, we have $Q^n[\phi](i) = \phi(i + cn), \forall i \in \mathbb{Z}, n \geq 0$. Such a definition of traveling waves is motivated by the idea employed in the proof of Theorem 3.2.

Let $\beta \gg 0$ be a fixed point of $Q$. Define $\bar{Q} : \mathcal{B}_\beta \rightarrow \mathcal{B}_\beta$ by

$$
\bar{Q}[\phi](x) = Q[\phi(\cdot + x)](0), \quad \forall x \in \mathbb{R}.
$$

Then we see from [25, Lemma 2.1] that $\bar{Q}$ satisfies (A1)-(A3) and (A5) with $Q = \bar{Q}$ and $\mathcal{C}_\beta = \mathcal{B}_\beta$ if $Q$ itself satisfies (A1)-(A3) and (A5). Further, if $Q$ satisfies (A4), then the set $Q[\mathcal{B}_\beta](x) \subset X_\beta$ is precompact for any $x \in \mathbb{R}$.

For $\bar{Q} : \mathcal{B}_\beta \rightarrow \mathcal{B}_\beta$, we have similar results as in Lemma 3.2 and 3.3.

Lemma 3.5. Assume that $Q$ satisfies (A1)-(A3) and (A5). Then there exists a positive rational number $\bar{c}$ such that for any $c \geq \bar{c}$, we have

$$
\bar{Q}[\psi_i](x) \geq \psi(x - c) \quad \text{and} \quad \bar{Q}[\psi_j](x) \leq \psi(x + c) \quad \text{for any} \ x \in \mathbb{R}.
$$

Lemma 3.6. Assume that $Q$ satisfies (A1)-(A5). Then for each $n \in \mathbb{N}$, $\bar{G}_n := \bar{Q} \circ A_{\kappa_n}$ has a fixed point $\bar{\phi}_n$ in $\mathcal{B}_\beta$ such that $\bar{\phi}_n$ is nondecreasing and $\bar{\psi}_n \leq \bar{\phi}_n \leq \bar{\psi}_n$.

Proof. By the same arguments as in the proof of Lemma 3.3, we can obtain a similar inequality as (3.2):

$$
\bar{\psi}_n \leq \bar{G}_n[\bar{\psi}_n] \leq \bar{G}_n[\bar{\psi}_n] \leq \bar{\psi}_n, \quad \forall k \in \mathbb{N}.
$$

Define $w_{n,1} := \bar{\psi}_n$ and $w_{n,k+1} := \bar{G}_n[w_{n,k}], k \geq 1$. Then

$$
w_{n,k+1}(x) = \bar{Q} \circ A_{\kappa_n}[w_{n,k}](x) = \bar{Q}[w_{n,k}(\kappa_n \cdot + x)](x) = Q[w_{n,k}(\kappa_n \cdot + x)](0). \quad (3.5)
$$

Note that $Q[\mathcal{C}_\beta]$ is compact and $w_{n,k}$ is nondecreasing in $k$. It then follows that for any fixed $x \in \mathbb{R}$, $w_{n,k}(x)$ converges in $X_\beta$. Denote the limit by $\bar{\phi}_n(x)$. Then $\bar{\phi}_n(x)$ is nondecreasing in $x \in \mathbb{R}$ and $\bar{\psi}_n \leq \bar{\phi}_n \leq \bar{\psi}_n$. Taking $k \rightarrow \infty$ in (3.5), we arrive at $\bar{\phi}_n(x) = Q[\bar{\phi}_n(\kappa_n \cdot + x)](0)$. Consequently,

$$
\bar{\phi}_n = \bar{Q}[\bar{\phi}_n(\kappa_n \cdot + x)] = \bar{Q} \circ A_{\kappa_n}[\bar{\phi}_n] = \bar{G}_n[\bar{\phi}_n].
$$

This completes the proof.

To overcome the difficulty due to the lack of compactness for $\bar{Q}$, we will use the properties of monotone functions established in the Appendix to show the convergence of a sequence in $\bar{Q}[\mathcal{B}_\beta]$. 

16
Theorem 3.2. Assume that $\mathcal{X} = C(M, \mathbb{R}^d)$ and $Q$ satisfies (A1)-(A6). Then there exists $c \in \mathbb{R}$ such that the semiflow $\{Q^n\}_{n \geq 1}$ on $C$ admits a nondecreasing traveling wave $\psi(x + cn)$ with speed $c$ and connecting 0 to $\beta$. Further, $\psi$ is either left or right continuous.

Proof. As in the proof of Theorem 3.1 we define

$$\bar{a}_n := \sup_{x \in \mathbb{R}} \{ \phi_n(x) \in [0, \delta e_0] x \}, \quad \bar{b}_n := \inf_{x \in \mathbb{R}} \{ \phi_n(x) \in [\beta - \delta e_\beta, \beta] x \}. $$

Then $-1 - (n + \tilde{c}) \leq \bar{a}_n \leq \bar{b}_n \leq 1 + (n + \tilde{c})$. Note that for any $x \in \mathbb{R}$, we have

$$\bar{\phi}_n(x) = \bar{G}_n[\bar{\phi}_n](x) = Q[\bar{\phi}_n(\kappa_n)](x) = Q[\bar{\phi}_n(\kappa_n(\cdot + x))](0) \in Q[C_{\beta}](0).$$

Since $Q[C_{\beta}](0)$ is precompact in $\mathcal{X}_\beta$, it then follows that for any $x \in \mathbb{R}$, $\bar{\phi}_n(x^-) := \lim_{t \to x} \phi_n(y)$ and $\bar{\phi}_n(x^+) := \lim_{t \to x} \phi_n(y)$ both exist. And hence, by the definitions of $\bar{a}_n$ and $\bar{b}_n$, we have

$$\bar{\phi}_n(\bar{a}_n^-) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \bar{\phi}_n(\bar{b}_n^+),$$

but

$$\bar{\phi}_n(\bar{a}_n^+) \notin [0, \delta e_0] x \quad \text{and} \quad \bar{\phi}_n(\bar{b}_n^-) \notin [\beta - \delta e_\beta, \beta] x.$$

Define $\bar{\phi}_{-n}(x) := \bar{\phi}_n(x + \bar{a}_n)$ and $\bar{\phi}_{+n}(x) := \bar{\phi}_n(x + \bar{b}_n)$. Then

$$\bar{\phi}_{-n}(0^-) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \bar{\phi}_{+n}(0^+),$$

but

$$\bar{\phi}_{-n}(0^+) \notin [0, \delta e_0] x \quad \text{and} \quad \bar{\phi}_{+n}(0^-) \notin [\beta - \delta e_\beta, \beta] x.$$

Since $\bar{\phi}_n = \bar{G}_n[\bar{\phi}_n]$, we have

$$\bar{\phi}_{-n}(x) = \bar{G}_n[\bar{\phi}_n](x + \bar{a}_n) = Q[\bar{\phi}_n(\kappa_n(\cdot + \bar{a}_n + x))](0) \in Q[C_{\beta}](0).$$

Similarly, $\bar{\phi}_{+n}(x) = Q[\bar{\phi}_n(\kappa_n(\cdot + \bar{b}_n + x))](0) \in Q[C_{\beta}](0)$. Let $Q$ be the set of all rational numbers, and $\{x_l\}_{l \geq 1} \subset Q$ be an increasing sequence converging to $x$. Using $\bar{\phi}_n = \bar{G}_n[\bar{\phi}_n]$ again, we see that for any $i \in \mathbb{Z}$ and $l \geq 1$,

$$\bar{\phi}_n(\kappa_n(i + \bar{a}_n + x_l)) = Q[\bar{\phi}_n(\kappa_n(\cdot + \kappa_n(i + \bar{a}_n + x_l)))](0) \in Q[C_{\beta}](0).$$

Similarly, $\bar{\phi}_n(\kappa_n(i + \bar{b}_n + x_l)) \in Q[C_{\beta}](0)$. Note that $Q[C_{\beta}](0)$ is precompact in $\mathcal{X}_\beta$ and that $Q$ is countable. It then follows that there exists a subindex (still denoted by $\{n\}$) and $\xi_- \leq \xi_+ \in \mathbb{R}$ such that $\lim_{n \to \infty} \bar{a}_n / n = \xi_-$, $\lim_{n \to \infty} \bar{b}_n / n = \xi_+$ and that for any $x \in Q$, $i \in \mathbb{Z}$ and $l \geq 1$, sequences $\bar{\phi}_{\pm n}(x)$, $\bar{\phi}_n(\kappa_n(i + \bar{a}_n + x_l))$ and $\bar{\phi}_n(\kappa_n(i + \bar{b}_n + x_l))$ converge in $\mathcal{X}_\beta$. And hence, the following limits

$$\lim_{l \to \infty} \lim_{n \to \infty} \bar{\phi}_{-n}(x_l) = \lim_{l \to \infty} \lim_{n \to \infty} Q[\bar{\phi}_n(\kappa_n(\cdot + \bar{a}_n + x_l))](0)$$

and

$$\lim_{l \to \infty} \lim_{n \to \infty} \bar{\phi}_{+n}(x_l) = \lim_{l \to \infty} \lim_{n \to \infty} Q[\bar{\phi}_n(\kappa_n(\cdot + \bar{b}_n + x_l))](0).$$
both exist. This means the limits
\[
\lim_{y \in \mathbb{Q}, y \leq x} \lim_{n \to \infty} \tilde{\phi}_{x,n}(y) \quad \text{and} \quad \lim_{y \in \mathbb{Q}, y \geq x} \lim_{n \to \infty} \tilde{\phi}_{x,n}(y)
\]
exist for all \(x \in \mathbb{R}\). Define
\[
\hat{\phi}_-(x) := \begin{cases} 
\lim_{n \to \infty} \tilde{\phi}_{-,n}(x), & x \in \mathbb{Q} \\
\lim_{y \in \mathbb{Q}, y \leq x} \lim_{n \to \infty} \tilde{\phi}_{-,n}(x), & x \in \mathbb{R} \setminus \mathbb{Q},
\end{cases}
\]
and
\[
\hat{\phi}_+(x) := \begin{cases} 
\lim_{n \to \infty} \tilde{\phi}_{+,n}(x), & x \in \mathbb{Q} \\
\lim_{y \in \mathbb{Q}, y \geq x} \lim_{n \to \infty} \tilde{\phi}_{+,n}(x), & x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]
Clearly, \(\hat{\phi}_\pm\) are nondecreasing functions in \(\mathcal{B}_\beta\) and for any \(x \in \mathbb{R} \setminus \mathbb{Q}\), \(\hat{\phi}_\pm(x^\pm)\) all exist. Hence, we see from Theorem 7.1 that there exists a countable subset \(\Gamma_1 \subseteq \mathbb{R}\) such that \(\tilde{\phi}_{\pm,n}(x)\) converges to \(\hat{\phi}_\pm(x)\) for all \(x \in \mathbb{R} \setminus \Gamma_1\). Define
\[
\tilde{\phi}_-(x) := \lim_{y \in \mathbb{Q}, y \leq x} \lim_{n \to \infty} \tilde{\phi}_{-,n}(y), \quad \forall x \in \mathbb{R}.
\]
and
\[
\tilde{\phi}_+(x) := \lim_{y \in \mathbb{Q}, y \geq x} \lim_{n \to \infty} \tilde{\phi}_{+,n}(y), \quad \forall x \in \mathbb{R}.
\]
Thus, \(\tilde{\phi}_-(x)\) is left continuous and \(\tilde{\phi}_+(x)\) is right continuous. Note that \(\hat{\phi}_\pm(x) = \tilde{\phi}_\pm(x)\) for all \(x \in \mathbb{R} \setminus \mathbb{Q}\). It then follows that \(\tilde{\phi}_{\pm,n}(x)\) converges to \(\hat{\phi}_\pm(x)\) for \(x \in \mathbb{R} \setminus \Gamma_2\), where \(\Gamma_2 := \mathbb{Q} \cup \Gamma_1\) is also countable.

Let \(y_k \in \mathbb{R} \setminus \Gamma_2\) be an increasing sequence converging to 0 and \(z_k \in \mathbb{R} \setminus \Gamma_2\) be an increasing sequence converging to 1, respectively. Note that
\[
\tilde{\phi}_-(0) = \lim_{k \to \infty} \tilde{\phi}_-(y_k) = \lim_{k \to \infty} \lim_{n \to \infty} \tilde{\phi}_{-,n}(y_k) \leq \delta e_0,
\]
and
\[
\tilde{\phi}_-(1) = \lim_{k \to \infty} \tilde{\phi}_-(z_k) = \lim_{k \to \infty} \lim_{n \to \infty} \tilde{\phi}_{-,n}(z_k) \not\in [0, \delta e_0], x.
\]
Similarly, we have \(\hat{\phi}_\pm(0) \geq \beta - \delta e_\beta\) but \(\hat{\phi}_+(1) \not\in [\beta - \delta e_\beta, \beta], x\). Define \(c_- := -\xi_+\) and \(c_+ := -\xi_+\). Obviously, \(c_- \geq c_+\) since \(\xi_- \leq \xi_+\). Now we want to prove \(Q[\tilde{\phi}_-(\cdot + x)](0) = \tilde{\phi}_-(x + c_-), \forall x \in \mathbb{R} \setminus \Gamma_2\). Note that
\[
\lim_{n \to \infty} \kappa_n(x + \tilde{a}_n) - \tilde{a}_n = \lim_{n \to \infty} \left( x + \tilde{c} \cdot \frac{x + \tilde{a}_n}{n} \right) = x - c_-.
\]
It then follows that
\[
\hat{\phi}_-(x + c_-) = \lim_{n \to \infty} \hat{\phi}_{-,n}(x + c_-) = \lim_{n \to \infty} \hat{\phi}_{n}(x + c_- + \tilde{a}_n) = \lim_{n \to \infty} \tilde{G}_n[\hat{\phi}_n](x + c_- + \tilde{a}_n) = \lim_{n \to \infty} \tilde{Q}[\tilde{\phi}_n(\kappa_n \cdot + \tilde{a}_n)](x + c_-) = \lim_{n \to \infty} \tilde{Q}[\tilde{\phi}_n(\kappa_n \cdot + \tilde{a}_n) - \tilde{a}_n](x + c_-) = \lim_{n \to \infty} Q[\hat{\phi}_{-,n}(\kappa_n \cdot + x + c_- + \tilde{a}_n) - \tilde{a}_n](0).
\]
In view of Proposition 3.3, we obtain that \( \tilde{\phi}_-(x + c) = Q(\tilde{\phi}_-(\cdot + x))(0) \) \( x \in \mathbb{R} \setminus \Gamma_2 \). A similar result also holds for \( \tilde{\phi}_+ \).

Now, the same argument as in the proof of Theorem 3.1 completes the proof. \( \blacksquare \)

### 3.3 Time-periodic semiflows

Let \( \omega \in \mathcal{T} \) be a positive number, where \( \mathcal{T} = \mathbb{R}^+ \) or \( \mathbb{Z}^+ \). Recall that a family of mappings \( \{Q_t\}_{t \in \mathcal{T}} \) is said to be an \( \omega \)-time periodic semiflow on a metric space \( \mathcal{E} \subset \mathcal{C} \) provided that it has the following properties:

1. \( Q_0[\phi] = \phi, \forall \phi \in \mathcal{E}. \)
2. \( Q_t \circ Q_\omega[\phi] = Q_{t+\omega}[\phi], \forall t \geq 0, \phi \in \mathcal{E}. \)
3. \( Q_t[\phi] \) is continuous jointly in \( (t, \phi) \) on \( [0, \infty) \times \mathcal{E}. \)

The mapping \( Q_\omega \) is called the Poincaré map associated with this periodic semiflow.

**Definition 3.3.** (i) In the case where \( \mathcal{H} = \mathbb{R} \), \( U(t, x + ct) \) is said to be an \( \omega \)-time periodic traveling wave with speed \( c \) of the semiflow \( \{Q_t\}_{t \in \mathcal{T}} \) if \( Q_t[U(0, \cdot)](x) = U(t, x + ct) \) and \( U(t, x) = U(t + \omega, x) \) for all \( t \in \mathcal{T}, x \in \mathbb{R} \).

(ii) In the case where \( \mathcal{H} = \mathbb{Z} \), \( U(t, x + ct) \) is said to be an \( \omega \)-time periodic traveling wave with speed \( c \) of the semiflow \( \{Q_t\}_{t \in \mathcal{T}} \) if there exists a countable subset \( \Gamma \subset \mathbb{R} \) such that \( Q_t[U(0, \cdot + x)](0) = U(t, x + ct) \) for all \( t \in \mathcal{T}, x \in \mathbb{R} \) and \( U(t, x) = U(t + \omega, x) \) for all \( t \in \mathcal{T}, x \in \mathbb{R} \setminus \Gamma \).

**Theorem 3.3.** Let \( \beta(t) \) be a strongly positive \( \omega \)-time periodic orbit of \( \{Q_t\}_{t \in \mathcal{T}} \) restricted on \( \mathcal{X} \). Assume that \( Q := Q_\omega \) satisfies hypotheses (A1)-(A6) with \( \beta = \beta(0) \). Then \( \{Q_t\}_{t \in \mathcal{T}} \) admits a traveling wave \( U(t, x + ct) \) with \( U(t, -\infty) = 0 \) and \( U(t, +\infty) = \beta(t) \) uniformly for \( t \in \mathcal{T} \). Furthermore, \( U(t, x) \) is nondecreasing in \( x \in \mathbb{R} \).

**Proof.** Case 1. \( \mathcal{H} = \mathbb{R} \). Since the map \( Q_\omega \) satisfies (A1)-(A6), there exits \( c \in \mathbb{R} \) and a nondecreasing function \( \phi \in \mathcal{C} \) connecting \( 0 \) to \( \beta(0) \) such that \( Q_\omega[\phi](x) = \phi(x + c\omega) \).

Clearly, \( T_{c\omega}Q_\omega[\phi] = \phi \). Define \( U(t, x) := T_{ct}Q_t[\phi](x) \). Then we have \( U(t, x + ct) = Q_t[\phi](x) = Q_t[U(0, \cdot)](x) \), and

\[
U(t + \omega, x) = T_{ct+c\omega}Q_{t+c\omega}[\phi](x) = T_{ct}Q_{t+c\omega}Q_\omega[\phi](x) = T_{ct}Q_t[\phi](x) = U(t, x).
\]

Note that \( Q_t[\beta(0)] = \beta(t) \) and that \( \phi \) is nondecreasing and connecting \( 0 \) to \( \beta(0) \). It then follows that \( U(t, -\infty) = 0 \) and \( U(t, +\infty) = \beta(t) \).

Case 2. \( \mathcal{H} = \mathbb{Z} \). Since the map \( Q_\omega \) satisfies (A1)-(A6), there exits \( c \in \mathbb{R} \), a countable subset \( \Gamma \subset \mathbb{R} \) and a nondecreasing function \( \phi \in \mathcal{B} \) connecting \( 0 \) to \( \beta(0) \) such that \( Q_\omega[\phi](x) = \phi(x + c\omega), \forall x \in \mathbb{R} \setminus \Gamma \). Clearly, \( T_{c\omega}Q_\omega[\phi](x) = \phi(x), \forall x \in \mathbb{R} \setminus \Gamma \). Define \( U(t, x) := T_{ct}Q_t[\phi](x) \). Thus, we have

\[
U(t, x + ct) = Q_t[\phi](x) = Q_t[U(0, \cdot)](x) = Q_t[U(0, \cdot + x)](0), \forall x \in \mathbb{R},
\]
and
\[ U(t + \omega, x) = T_{ct + cw} \tilde{Q}_{t + \omega}[\phi](x) = T_{ct} \tilde{Q}_t T_{cw} \tilde{Q}_\omega[\phi](x) = T_{ct} \tilde{Q}_t[\phi](x) = U(t, x), \]
for all \( x \in \mathbb{R} \setminus \Gamma \). Note that \( Q_t[\beta(0)] = \beta(t) \) and that \( \phi \) is nondecreasing and connecting \( 0 \) to \( \beta(0) \). It then follows that \( U(t, -\infty) = 0 \) and \( U(t, +\infty) = \beta(t) \).

**3.4 Continuous-time semiflows in a continuous habitat**

In this subsection, we consider continuous-time semiflows in the continuous habitat \( \mathcal{H} = \mathbb{R} \). Recall that a family of mappings \( \{Q_t\}_{t \geq 0} \) is said to be a semiflow on a metric space \( E \subset C \) provided that \( Q_t : \mathcal{E} \to \mathcal{E} \) satisfies the following properties:

1. \( Q_0[\phi] = \phi, \forall \phi \in \mathcal{E} \).
2. \( Q_t \circ Q_s[\phi] = Q_{t+s}[\phi], \forall t, s \geq 0, \phi \in \mathcal{E} \).
3. \( Q_t[\phi] \) is continuous jointly in \((t, \phi)\) on \([0, \infty) \times \mathcal{E} \).

Before moving to the study of traveling waves of the semiflow \( \{Q_t\}_{t \geq 0} \), we first investigate the spatially homogeneous system, that is, the system restricted on \( \mathcal{X} \). Let \( \beta \gg 0 \) be an equilibrium in \( \mathcal{X} \). For each \( t > 0 \), we use \( \Sigma_t \) to denote the set of all fixed points of the map \( Q_t \) restricted on \( \mathcal{X} \). Clearly, the equilibrium set of the semiflow is \( \Sigma := \cap_{t > 0} \Sigma_t \), which is a subset of \( \Sigma_t \) for any \( t > 0 \). The subsequent results indicates that the instability of intermediate equilibria of the semiflow implies the nonordering property of all intermediate fixed points of each time-\( t \) map.

**Proposition 3.1.** For any given \( t > 0 \), if the map \( Q_t \) satisfies the bistability assumption \((A5')\) with \( E = \Sigma \), then \( Q_t \) satisfies \((A5)\) with \( E = \Sigma_t \).

**Proof.** Let \( t_0 > 0 \) be given. We first show that any two points \( u \in \Sigma \setminus \{0, \beta\} \) and \( v \in \Sigma_{t_0} \setminus \{0, \beta\} \) are unordered. Assume, for the sake of contradiction, that \( u \) and \( v \) are ordered. Without loss of generality, we also assume that \( u < v \). Then the eventual strong monotonicity implies that \( u \ll v \). Since \( u \) is strongly unstable from above, there exist a unit vector \( e \in Int(\mathcal{X}^+) \) and a number \( \delta > 0 \) such that \( Q_{t_0}[u + \delta e] \gg u + \delta e \) with \( u + \delta e \notin [u, v]_\mathcal{X} \). From \( [34] \) Theorem 1.2.1], we see that \( (Q_{t_0})^n[u + \delta e] \) is eventually strongly increasing and converges to some \( \alpha \in \Sigma \). Note that \( \alpha \in [u, v]_\mathcal{X} \) is strongly unstable from below. Hence, by the same arguments as in the proof of Proposition 2.1, we obtain a contradiction.

Next we show the set \( \Sigma_{t_0} \setminus \Sigma \) is unordered. For this purpose, we see from the first step that it suffices to prove that for any two ordered elements \( u < v \) in \( \Sigma_{t_0} \setminus \Sigma \), \( [u, v]_\mathcal{X} \cap \Sigma \neq \emptyset \). Indeed, by the eventual strong monotonicity, we have \( u \ll v \). Then, we can choose a sequence \( \{u_n\} \) on the segment connecting \( u \) and \( v \) such that \( u \ll u_n \ll u_{n+1} \ll v, \forall n \geq 1 \). By \( [34] \) Theorem 1.3.7], it follows that \( \omega(u) \leq \omega(u_n) \leq \omega(u_{n+1}) \leq \omega(v), \forall n \geq 1 \). Clearly, we have \( \omega(u) = \{Q_t u : t \in [0, t_0]\} \) and \( \omega(v) = \{Q_t v : t \in [0, t_0]\} \), and hence \( u \leq \omega(u_n) \leq v, \forall n \geq 1 \).
Note that $\bigcup_{n \geq 1} \omega(u_n)$ is contained in the compact set $\overline{Q_{t_0}[X_\beta]}$. In the compact metric space consisting of all nonempty compact subsets of $Q_{t_0}[X_\beta]$ with Hausdorff distance $d_H$, the sequence $\{\omega(u_n) : n \geq 1\}$ has a convergent subsequence. Without loss of generality, we assume that for some nonempty compact set $\overline{x} \subset Q_{t_0}[X_\beta]$, 

\[ \lim_{n \to \infty} d_H(\omega(u_n), \overline{x}) = 0. \]

Since each $\omega(u_n)$ is invariant for the semiflow $\{Q_t\}_{t \geq 0}$, so is the compact set $\overline{x}$, that is, $Q_t\overline{x} = \overline{x}$, $\forall t \geq 0$. For any given $x, y \in \omega$, there exist two sequences of points $x_n, y_n \in \omega(u_n)$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Since $\omega(u_n) \leq \omega(u_{n+1})$, we have $x_n \leq y_{n+1} + y_n \leq x_{n+1}, \forall n \geq 1$. Letting $n \to \infty$, we then have $x \leq y$ and $y \leq x$, and hence $x = y$. This implies that $\overline{x}$ is a singleton, that is, $\overline{x} = \{\alpha\}$. By the invariance of $\overline{x}$ for the semiflow, we see that $\alpha$ is an equilibrium. Since $u \leq \omega(u_n) \leq v$, $\forall n \geq 1$, it follows that $\alpha \in [u, v]_X$. \hfill $\square$

For a continuous-time semiflow $\{Q_t\}_{t \geq 0}$, we need the following definition of traveling waves.

**Definition 3.4.** $\psi(x+ct)$ with $\psi \in \mathcal{C}$ is said to be a traveling wave with speed $c \in \mathbb{R}$ of the continuous-time semiflow $\{Q_t\}_{t \geq 0}$ if $Q_t[\psi](x) = \psi(x+ct), \forall x \in \mathbb{R}, t \geq 0$. We say that $\psi$ connects $0$ to $\beta$ if $\psi(-\infty) = 0$ and $\psi(+\infty) = \beta$.

**Theorem 3.4.** Assume that for each $t > 0$, the map $Q_t$ satisfies assumptions (A1)-(A3)-(A5) with $E = \Sigma_1$, and the time-one map $Q_1$ satisfies (A6) with $E = \Sigma$. Then there exists $c \in \mathbb{R}$ such that $\{Q_t\}_{t \geq 0}$ admits a non-decreasing traveling wave with speed $c$ and connecting to $0$ to $\beta$.

**Proof.** Let $e_0, e_\beta$ and $\delta$ be chosen as in [2.7] [2.3] and [3.1] respectively. We proceed with three steps.

Firstly, we show that there exists $s_k \downarrow 0$ such that each discrete semiflow $\{Q^n_{s_k}\}_{n \geq 0}$ admits two nondecreasing traveling waves $\psi_{\pm,s_k}(x + c_{\pm,s_k}t)$ with $c_{-,s_k} \geq c_{+,s_k}$ and $\psi_{\pm,s_k}$ has the following properties:

\[ 0 < \psi_{-,s_k}(0) \leq \delta e_0 \quad \text{and} \quad \beta - \delta e_{\beta} \leq \psi_{+,s_k}(0) < \beta, \]

but

\[ \psi_{-,s_k}(0) \not\in [0, \delta e_0]_X \quad \text{and} \quad \psi_{+,s_k}(0) \not\in [(\beta - \delta e_{\beta}), \beta]_X. \]

Indeed, since for each $s > 0$ the map $Q_s$ satisfies (A1)-(A5), from the first two steps of the proof for Theorem 3.1 we see that for the discrete semiflow $\{(Q_s^n)\}_{n \geq 0}$, there exists two nondecreasing traveling waves $\phi_{\pm,s}(x + c_{\pm,s}t)$ with the following properties:

1. $\phi_{-,s}$ connects $0$ to some $\alpha_{-,s} \in E_s \setminus \{0\}$ and $\phi_{+,s}$ connects some $\alpha_{+,s} \in E_s \setminus \{\beta\}$ to $\beta$;

2. $\alpha_{-,s}$ and $\alpha_{+,s}$ are ordered and $c_{-,s} \geq c_{+,s}$.

By a similar argument as in [47] Theorem 1.3.7], it then follows that both $\alpha_{\pm,s}$ have a subsequence $\alpha_{\pm,s_k}$ which tends to an equilibrium of the semiflow as $s_k \to 0$, say
the limit \( \alpha_- \) and \( \alpha_+ \), respectively. Since \( \alpha_{-,s} \) and \( \alpha_{+,s} \) are ordered, it follows from Proposition 3.1 that there are only three possibilities for the relation of \( \alpha_- \) and \( \alpha_+ \):

(i) \( \beta = \alpha_- \geq \alpha_+ \); (ii) \( \alpha_- \geq \alpha_+ = 0 \); and (iii) \( \alpha_- = \alpha_+ \in E \setminus \{0, \beta\} \).

If \( \alpha_- = \beta \), then for sufficiently large \( k \) we can define

\[
a_{sk} := \sup\{x \in \mathbb{R} : \phi_{-,sk}(x) \in [0, \delta e_0] X\}, \quad b_{sk} := \inf\{x \in \mathbb{R} : \phi_{-,sk}(x) \in [\beta - \delta e_\beta, \beta] X\}.
\]

And hence, \( \psi_{-,sk}(x) := \psi_s(x + a_{sk}) \) and \( \psi_{+,sk}(x) := \psi_s(x + b_{sk}) \) are the required traveling waves. If \( \alpha_+ = 0 \), then for sufficient large \( k \) we can define

\[
a_{sk} := \sup\{x \in \mathbb{R} : \phi_{+,sk}(x) \in [0, \delta e_0] X\}, \quad b_{sk} := \inf\{x \in \mathbb{R} : \phi_{+,sk}(x) \in [\beta - \delta e_\beta, \beta] X\}.
\]

And hence, \( \psi_{-,sk}(x) := \phi_{-,sk}(x + a_{sk}) \) and \( \psi_{+,sk}(x) := \phi_{-,sk}(x + b_{sk}) \) are the required traveling waves.

Secondly, we show that there exists a subindex, still denoted by \( s_k \), such that \( \psi_{\pm,sk} \to \psi_\pm \) in \( C_\beta \) and \( \frac{1}{s_k} c_{\pm,sk} \to c_\pm \in \mathbb{R} \). Indeed, for each \( s_k > 0 \), there exists an integer \( m_k \) such that \( m_k s_k > 2 \). Then

\[
\psi_{-,sk} = T_{m_k c_{sk}} \circ Q_{m_k s_k} [\psi_{-,sk}]_E = Q_2 \circ Q_{m_k s_k - 2} \circ T_{m_k c_{sk}} [\psi_{-,sk}]_E \in Q_1 \circ Q_1 [C_\beta]. \tag{3.6}
\]

Clearly, the compactness of \( Q_1 \) implies that the set \( Q_1 \circ Q_1 [C_\beta] \) is precompact in \( C_\beta \). Thus, there exists subsequence, still denoted by \( s_{k_i} \), and nonincreasing functions \( \psi_-, \psi_+ \in C_\beta \) with \( 0 < \psi_-(0) \leq \delta e_0 \) and \( \beta - \delta e_\beta \leq \psi_+(0) < \beta \) such that \( \psi_{-,s_{k_i}} \to \psi_- \) and \( \psi_{+,s_{k_i}} \to \psi_+ \) in \( C_\beta \). Also we claim that \( \psi_{\pm,s_{k_i}}(\pm \infty) \) all exist. Indeed, from (3.6) we see that there exists \( \phi_{sk} \in C_\beta \) such that \( Q_1 \circ Q_1 [\phi_{sk}] \to \psi_- \). Note that \( \{Q_1[\phi_{sk}]\}_{k \geq 1} \) also has a convergent subsequence with the limit \( \phi \in C_\beta \). And hence, by the uniqueness of limit we have \( Q_1[\phi] = \psi_- \). Note that \( \psi_-(k) = Q_1[\phi](0) = Q_1[\phi(\cdot + k)](0) \) and \( \{Q_1[\phi(\cdot + k)]\}_{k \geq 1} \) has a convergent subsequence. It then follows that \( \psi_-(\pm \infty) \) exist because \( \psi_- \) is nonincreasing. Similarly, \( \psi_+(\pm \infty) \) exist. Also, we have

\[
\psi_-(\pm \infty) \leq \psi_-(0) \leq \delta e_0 \quad \text{and} \quad \psi_+(\pm \infty) \geq \psi_+(0) \geq \beta - \delta e_\beta, \tag{3.7}
\]

but

\[
\psi_-(0) \notin [0, \delta e_0] X \quad \text{and} \quad \psi_+(0) \notin [\beta - \delta e_\beta, \beta] X.
\]

Consequently, by the monotonicity of \( \psi_\pm \), we have

\[
\psi_-(x) \notin [0, \delta e_0] X, \forall x > 0 \quad \text{and} \quad \psi_+(x) \notin [\beta - \delta e_\beta, \beta] X, \forall x < 0. \tag{3.8}
\]
Since $\psi_-$ and $\psi_+$ are the limits of the sequence of monotone functions with different translations, respectively, we can employ the same arguments as in the second step of the proof of Theorem 3.1 to show that $\psi_-(+\infty)$ and $\psi_+(-\infty)$ are ordered.

To prove that $\frac{1}{s_k}c_{\pm,s_k}$ have convergent subsequences, we only need to prove that $\frac{1}{s_k}c_{-,s_k}$ is bounded above and $\frac{1}{s_k}c_{+,s_k}$ is bounded below because $c_{-,s_k} \geq c_{+,s_k}$. Assume, for the sake of contradiction, that some subsequence, still say $\frac{1}{s_k}c_{-,s_k}$, tends to $+\infty$. Note that for each $s > 0$ there exists $n_s \in \mathbb{Z}^+$ such that the integer part of $\frac{1}{s}$, denoted by $\langle \frac{1}{s} \rangle$, equals $n_s$ and $\frac{1}{n_s+1} < s \leq \frac{1}{n_s}$. Hence, $s \langle \frac{1}{s} \rangle \to 1$ as $s \to 0$. It then follows that

$$\lim_{k \to \infty} \frac{1}{s_k}c_{-,s_k} = \lim_{k \to \infty} \frac{1}{s_k}c_{-,s_k} \times s_k\left(\frac{1}{s_k}\right) = \lim_{k \to \infty} \frac{1}{s_k}c_{-,s_k} = +\infty.$$  

Thus, using the first observation in (3.8), we have

$$Q_1[\delta e_0] \geq Q_1[\psi_-(\infty)] = Q_1[\psi_-(\infty)](0) = \lim_{x \to -\infty} Q_1[\psi_-(\cdot + x)](0)$$  

$$= \lim_{x \to -\infty} Q_1[\psi_-](x) = \lim_{x \to -\infty} \lim_{k \to \infty} (Q_{s_k})^{\langle \frac{1}{s_k} \rangle}[\psi_-,s_k](x)$$  

$$= \lim_{y \to -\infty} \lim_{k \to \infty} \psi_-(y) = \psi_-(\infty) \notin [0,\delta e_0] \times X,$$

which contradicts the fact that $Q_1[\delta e_0] \ll \delta e_0$. Similarly, if $\frac{1}{s_k}c_{+,s_k} \to -\infty$, then the second observation in (3.8) implies that

$$Q_1[\beta - \delta e_\beta] \leq Q_1[\psi_+(\infty)] = Q_1[\psi_+(\infty)](0) = \lim_{x \to +\infty} Q_1[\psi_+(\cdot + x)](0)$$  

$$= \lim_{x \to +\infty} Q_1[\psi_+](x) = \lim_{x \to +\infty} \lim_{k \to \infty} (Q_{s_k})^{\langle \frac{1}{s_k} \rangle}[\psi_+,s_k](x)$$  

$$= \lim_{y \to +\infty} \lim_{k \to \infty} \psi_+(y) = \psi_+(\infty) \notin [\beta - \delta e_\beta, \beta],$$

which contradicts the fact that $Q_1[\beta - \delta e_\beta] \gg \beta - \delta e_\beta$. Consequently, $\frac{1}{s_k}c_{\pm,s_k}$ are bounded.

Finally, we show that either $\psi_-(x+c_-t)$ or $\psi_+(x+c_+t)$ established in the second step is a traveling wave connecting 0 to $\beta$. Indeed, for any $t > 0$, there exists $m_k \in \mathbb{Z}$ and $r_k \in [0,s_k)$ such that $t = m_k s_k - r_k$. Clearly, $r_k \to 0$ as $k \to \infty$. Then we have

$$Q_t[\psi_\pm] = \lim_{k \to \infty} Q_t[r_k]\psi_\pm(s_k) = \lim_{k \to \infty} Q_{m_k s_k}[\psi_\pm(s_k)] = \lim_{k \to \infty} \psi_\pm(s_k)(\cdot + m_k c_\pm s_k)$$  

$$= \lim_{k \to \infty} \psi_\pm(s_k)((\cdot + t + r_k)\frac{1}{s_k}c_\pm s_k) = \psi_\pm((\cdot + c_\pm t),$$

where the last equality follows from Proposition (7.2)(2). From the equality $Q_t[\psi_\pm] = \psi_\pm((\cdot + ct), \forall t \geq 0$, we see that $\psi((\pm \infty)$ are equilibria. Recall that $\psi_-(\infty) \leq \delta e_0 \leq \psi_+(\infty)$. Since $\psi_-$ and $\psi_+$ are monotone functions with different translations, we can employ the same arguments as in the second step of the proof of Theorem 3.1 to show that $\psi_-(+\infty)$ and $\psi_+(-\infty)$ are ordered.
ψ(+) and ψ(+∞) ≥ β − δεβ ≥ ψ(−∞). It then follows that ψ(−∞) = 0, ψ(+∞) = β, and there are only three possibilities for ψ(+) and ψ(−∞):

(i) β = ψ(+) > ψ(−∞);
(ii) ψ(+) > ψ(−∞) = 0;
(iii) ψ(+) = α = ψ(−∞) for some α ∈ Σ \ {0, β}.

Since the time-one map Q1 satisfies (A6) with E = Σ, we can employ the same arguments as in the proof of Lemma 3.4 to exclude the possibility (iii). Thus, either (i) or (ii) holds, and hence, we complete the proof.

3.5 Continuous-time semiflows in a discrete habitat

In this case, time T = R+ and habitat H = Z. Let β ≫ 0 be an equilibrium of the semiflow \{Qt\}_{t≥0}. We start with the definition of traveling waves for this case.

**Definition 3.5.** ψ(i + ct) with ψ ∈ Bβ is said to be a traveling wave with speed c ∈ R of the continuous-time semiflow \{Qt\}_{t≥0} if Qt[ψ](i) = ψ(i + ct), ∀i ∈ Z, t ≥ 0. Clearly, ψ is continuous if c ≠ 0.

For each t > 0, define \(\tilde{Q}_t : B_β \rightarrow B_β\) by \(\tilde{Q}_t[φ](x) = Q_t[φ(· + x)](0)\). Then it is easy to see the following result holds.

**Lemma 3.7.** \(\{\tilde{Q}_t\}_{t≥0}\) has the following properties:

(i) \(\tilde{Q}_0[φ] = φ, ∀φ ∈ B\).

(ii) \(\tilde{Q}_t \circ \tilde{Q}_s[φ] = \tilde{Q}_{t+s}[φ], ∀t, s ≥ 0, φ ∈ B\).

(iii) For fixed \(x ∈ \mathbb{R}\), if \(t_n → t\) and \(φ_n(i + x) → φ(i + x)\) in \(\mathcal{X}\) for any \(i ∈ \mathbb{Z}\), then \(\tilde{Q}_{t_n}[φ_n](x) → \tilde{Q}_t[φ](x)\) in \(\mathcal{X}\).

We combine the ideas in the proofs of Theorems 3.2 and 3.4 to prove the following result for continuous-time semiflows in a discrete habitat Z.

**Theorem 3.5.** Let \(\mathcal{X} = C(M, \mathbb{R}^d)\). Assume that for each t > 0, the map Qt satisfies (A1), (A3)-(A5) with E = Σt, and that the time-one map Q1 satisfies (A6) with E = Σ. Then there exists c ∈ R such that \(\{Qt\}_{t≥0}\) admits a non-decreasing traveling wave with speed c and connecting 0 to β.

**Proof.** Let δ, ε0, εβ be chosen as in (3.1), (2.7) and (2.3). We proceed with three steps.

Firstly, since for any s > 0 the map Qs satisfies assumptions (A1)-(A5), it then follows from the proof of Theorems 3.2 and 3.4 that there exists \(s_k \downarrow 0\) such
that \( \{(Q_{s_k})^n\}_{n \geq 0} \) admits two nondecreasing traveling waves \( \tilde{\psi}_{\pm, s_k}(x + c_{\pm, s_k}n) \) with 
\( c_{-, s_k} \leq c_{+, s_k} \), that is, there exists countable subset \( \Theta_k \) such that
\[
\tilde{Q}_{s_k}[\tilde{\psi}_{\pm, s_k}](x) = \tilde{\psi}_{\pm, s_k}(x + c_{\pm, s_k}), \quad \forall x \in \mathbb{R} \setminus \Theta_k.
\]
Furthermore, \( \tilde{\psi}_{-, s_k} \) is left continuous and \( \tilde{\psi}_{+, s_k} \) is right continuous with the following properties:
\[
0 < \tilde{\psi}_{-, s_k}(0) \leq \delta e_0 \quad \text{and} \quad \beta - \delta e_0 \leq \tilde{\psi}_{+, s_k}(0) < \beta,
\]
but
\[
\tilde{\psi}_{-, s_k}(0) \not\in [0, \delta e_0]_X \quad \text{and} \quad \tilde{\psi}_{+, s_k}(0) \not\in [(\beta - \delta e_0, \beta)]_X.
\]
Secondly, we show that for the above sequence \( s_k \), there exist a countable set \( \tilde{\Gamma} \subset \mathbb{R} \) and a subsequence, still denoted by \( s_k \), such that \( \frac{1}{s_k}c_{\pm, s_k} \to c_{\pm} \in \mathbb{R} \) and that \( \tilde{\psi}_{\pm, s_k}(x) \) converges in \( X \) for all \( x \in \mathbb{R} \setminus \tilde{\Gamma} \). Indeed, let \( \Theta = \cup_{k=1}^{\infty} \Theta_k \). Hence, \( \Theta \) is countable and
\[
\tilde{Q}_{s_k}[\tilde{\psi}_{\pm, s_k}](x) = \tilde{\psi}_{\pm, s_k}(x + c_{\pm, s_k}), \quad \forall k \geq 1, x \in \mathbb{R} \setminus \Theta.
\]
From Proposition 7.4, we see that there exists another countably dense set \( \Gamma \subset \mathbb{R} \) such that \( \Gamma \cap \Theta = \emptyset \). By the same arguments as in the proof of Theorem 3.2, we can show that
\[
\tilde{\psi}_{-}(x) := \lim_{y \in \Gamma, y \uparrow x} \lim_{k \to \infty} \tilde{\psi}_{-, s_k}(y), \quad \forall x \in \mathbb{R},
\]
and
\[
\tilde{\psi}_{+}(x) := \lim_{y \in \Gamma, y \downarrow x} \lim_{k \to \infty} \tilde{\psi}_{+, s_k}(y), \quad \forall x \in \mathbb{R},
\]
are well-defined and all \( \tilde{\psi}_{\pm}(\pm \infty) \) exist. Furthermore, \( \tilde{\psi}_{-}(+\infty) \) and \( \tilde{\psi}_{+}(-\infty) \) are ordered in \( X \) and
\[
\tilde{\psi}_{-}(-\infty) \leq \tilde{\psi}_{-}(0) \leq \delta e_0 \quad \text{and} \quad \tilde{\psi}_{+}(+\infty) \geq \tilde{\psi}_{+}(0) \geq \beta - \delta e_0,
\]
but
\[
\tilde{\psi}_{-}(0) \not\in [0, \delta e_0]_X \quad \text{and} \quad \tilde{\psi}_{+}(0) \not\in [(\beta - \delta e_0, \beta)]_X.
\]
Further, \( \tilde{\psi}_{\pm}(x^{\pm}) \) exist for all \( x \in \mathbb{R} \setminus \Gamma \). Hence, it follows from Theorem 7.1 that there exists a countable subset \( \tilde{\Gamma} \) of \( \mathbb{R} \) such that
\[
\tilde{\psi}_{\pm, s_k}(x) \to \tilde{\psi}_{\pm}(x), \quad \forall x \in \mathbb{R} \setminus \tilde{\Gamma}.
\]
(3.11)
By similar arguments as in the second step of the proof of Theorem 3.4, we can show that \( \frac{1}{s_k}c_{\pm, s_k} \) are bounded.

Finally, we prove that either \( \tilde{\psi}_{-}(x + c_{-}t) \) or \( \tilde{\psi}_{+}(x + c_{+}t) \) is a nondecreasing traveling wave connecting 0 to \( \beta \). Indeed, from (3.11) and Proposition 7.3 we see that there exists a countable subset \( \Gamma_1 \) of \( \mathbb{R} \) such that
\[
\tilde{\psi}_{\pm, s_k}(i + x) \to \tilde{\psi}_{\pm}(i + x), \quad \forall i \in \mathbb{Z}, x \in \mathbb{R} \setminus \Gamma_1.
\]
Hence, for any \( x \in \mathbb{R} \setminus \Gamma_1 \) and \( t > 0 \), we have
\[
\tilde{Q}_t[\tilde{\psi}_-](x) = Q_t[\tilde{\psi}_-(\cdot + x)](0) = \lim_{k \to \infty} Q_{t+r_k}[\tilde{\psi}_-,-s_k(\cdot + x)](0) = \lim_{k \to \infty} Q_{m_k,s_k}[\tilde{\psi}_-,-s_k](0) = \lim_{k \to \infty} \tilde{Q}_{m_k,s_k}[\tilde{\psi}_-,-s_k](x) = \lim_{k \to \infty} \tilde{\psi}_-,-s_k(x + m_k c,-s_k) = \lim_{k \to \infty} \tilde{\psi}_-,-s_k(x + (t + r_k) \frac{1}{s_k} c,-s_k).
\]

(3.12)

In the case where \( c_- = 0 \), we can choose \( x_0 \) such that
\[
Q_t[\tilde{\psi}_-(x_0 + \cdot)](i) = \lim_{k \to \infty} \tilde{\psi}_-,-s_k(x + (t + r_k) \frac{1}{s_k} c,-s_k) = \tilde{\psi}_-(x_0 + i), \forall i \in \mathbb{Z}.
\]

In the case where \( c_- \neq 0 \), we know that there exists a countable subset \( \Gamma_2 \) of \( \mathbb{R} \) such that
\[
\tilde{Q}_t[\tilde{\psi}_-](x) = \lim_{k \to \infty} \tilde{\psi}_-,-s_k(x + (t + r_k) \frac{1}{s_k} c,-s_k) = \tilde{\psi}_-(x + c_- t), \forall x \notin \Gamma_1, x + c_- t \in \Gamma_2.
\]

Without loss of generality, we assume that \( c_- > 0 \). For any \( y \in \mathbb{R} \), we can choose \( x_0 \in \mathbb{R} \) and \( t_0 \geq 0 \) such that \( x_0 + c_- t_0 = y \) and \( \tilde{\psi}_-(x) \) is continuous at \( x = x_0 + i \) for all \( i \in \mathbb{Z} \). Now one can find \( x_{\pm,k} \in \mathbb{R} \setminus \Gamma_1 \) and \( t_{\pm,k} \to t_0 \) with \( y_{\pm,k} := x_{\pm,k} + c_- t_{\pm,k} \in \mathbb{R} \setminus \Gamma_2 \) such that \( y_{-,k} \uparrow y \) and \( y_{+,k} \downarrow y \). Note that
\[
\tilde{\psi}_-(y^-) := \lim_{k \to \infty} \tilde{\psi}_-(y_{-,k}) = \lim_{k \to \infty} Q_{t_-,-k}[\tilde{\psi}_-(\cdot + x_{-,k})](0) = Q_{t_0}[\tilde{\psi}_-(\cdot + x_0)](0)
\]
and
\[
\tilde{\psi}_-(y^+) := \lim_{k \to \infty} \tilde{\psi}_-(y_{+,k}) = \lim_{k \to \infty} Q_{t_+,+k}[\tilde{\psi}_-(\cdot + x_{+,k})](0) = Q_{t_0}[\tilde{\psi}_-(\cdot + x_0)](0).
\]

Thus, \( \tilde{\psi}_-(x) \) is continuous in \( x \in \mathbb{R} \). And hence, again by Proposition [7,5] and the equality [3,12], we have \( \tilde{Q}_t[\tilde{\psi}_-](x) = \tilde{\psi}_-(x + c_- t) \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \). Therefore, \( \tilde{\psi}_-(x + c_- t) \) is a traveling wave connecting \( 0 \) to some \( \alpha_- \in \Sigma \setminus \{0\} \). Similarly, we can construct the traveling wave \( \tilde{\psi}_+(x + c_+ t) \) connecting some \( \alpha_+ \in \Sigma \setminus \{\beta\} \). Besides, \( \alpha_- \) and \( \alpha_+ \) are ordered. Now the rest of the proof is essentially the same as in the proof of Theorem [3,4].

\[\square\]

4 Semiflows in a periodic habitat

A typical example of evolution systems in a periodic habitat is
\[
u_t = (d(x)u_x)_x + f(u), \quad t > 0, x \in \mathbb{R},
\]
where \( d(x) \) is a positive periodic function of \( x \in \mathbb{R} \). Under the assumption that \( f \) has exactly three ordered zeros \( 0 < a < 1 \) and \( f'(0) < 0, f'(a) > 0, f'(1) < 0 \), Xin
employed perturbation methods to obtain the existence of spatially periodic traveling wave \( V(x + ct, x) \) with \( V(-\infty, \cdot) = 0 \) and \( V(+\infty, \cdot) = 1 \) provided that \( d(x) \) is sufficiently closed to a positive constant in certain sense (see also [41]). For a general positive periodic function \( d(x) \), the existence of such a traveling wave remains open. We will revisit this problem in subsection 6.3.

A map \( Q : \mathcal{E} \to \mathcal{E} \subset \mathcal{C} \) is said to be spatially periodic with a positive period \( r \in \mathcal{H} \) if \( Q \circ T_r = T_r \circ Q \), where \( T_r \) is the \( r \)-translation operator. Similarly, a semiflow \( \{Q_t\}_{t \in \mathbb{T}} \) on \( \mathcal{E} \subset \mathcal{C} \) is said to be spatially periodic with a positive period \( r \in \mathcal{H} \) if \( Q_t \circ T_r = T_r \circ Q_t \) for all \( t \geq 0 \).

**Definition 4.1.**

(i) An \( r \)-periodic function \( \beta(x) \) is said to be an \( r \)-periodic steady state of the map \( Q \) (semiflow \( \{Q_t\}_{t \in \mathbb{T}} \)) if \( Q[\beta] = \beta(Q_t[\beta] = \beta, \forall t \in \mathbb{T}) \).

(ii) \( V(x + ct, x) \) is said to be a spatially periodic traveling wave with speed \( c \) of the semiflow \( \{Q_t\}_{t \in \mathbb{T}} \) if \( Q_t[V(\cdot, \cdot)](x) = V(x + ct, x) \) and \( V(\cdot, x) \) is \( r \)-periodic in \( x \). Besides, we say that \( V(\xi, x) \) connects 0 to \( \beta(x) \) if \( \lim_{\xi \to -\infty}\|V(\xi, x)\|_X = 0 \) and \( \lim_{\xi \to +\infty}\|V(\xi, x) - \beta(x)\|_X = 0 \) uniformly for \( x \in \mathcal{H} \).

Motivated by [26] Section 5], we can regard a spatially periodic semiflow on \( \mathcal{E} \subset \mathcal{C} \) as a spatially homogeneous semiflow on another phase space. For any positive \( h \in \mathcal{H} \), define \( [0, h]_\mathcal{H} := \{l \in \mathcal{H} : 0 \leq l \leq h\} \). We use \( \mathcal{Y} \) to denote \( C([0, \mathcal{H}], \mathcal{X}) \) and \( \mathcal{S} \) to denote the set of all bounded functions from \( r\mathcal{H} \) to \( \mathcal{Y} \). Clearly, \( \mathcal{Y} \) can be regarded as a subspace of \( \mathcal{S} \). Let \( \mathcal{Y}^+ = C([0, \mathcal{H}], \mathcal{X}^+) \) and \( \mathcal{S}^+ \) be the set of all bounded functions from \( r\mathcal{H} \) to \( \mathcal{Y}^+ \). We equip \( \mathcal{Y} \) with the norm \( \|u\|_\mathcal{Y} = \max\{\|u(x)\|_X : x \in [0, \mathcal{H}]\} \) and \( \mathcal{S} \) with the compact open topology. Thus, \( \mathcal{Y} \) is a Banach lattice with the norm \( \|\cdot\|_\mathcal{Y} \) and the cone \( \mathcal{Y}^+ \).

Let

\[
\mathcal{K} := \{f \in \mathcal{S} : f(r)(r) = f(r(i + 1))(0), \forall i \in \mathbb{Z}\}.
\]

It is easy to see that

\[
\mathcal{K} \cap \mathcal{Y} = \{f \in \mathcal{S} : f(r(i + 1)) = f(r(i + 1))(0), \forall i, j \in \mathbb{Z}\}.
\]

For any \( \phi \in \mathcal{C} \), define \( \tilde{\phi} \in \mathcal{S} \) by

\[
\tilde{\phi}(r)(y) = \phi(r + y), \ \forall i \in \mathbb{Z}, \ y \in [0, \mathcal{H}].
\]

Then we have the following observation.

**Lemma 4.1.** For any \( f \in \mathcal{K} \), there exists a unique \( \tilde{f} \in \mathcal{C} \) such that \( \tilde{f} = f \). Further, if \( f \in \mathcal{K} \cap \mathcal{Y} \), then \( \tilde{f} \) is \( r \)-periodic.

**Proof.** For any \( x \in \mathcal{H} \), we can find \( i \in \mathbb{Z} \) and \( y \in [0, \mathcal{H}] \) such that \( x = ri + y \). It is easy to see that such decomposition of \( x \) is unique when \( x \in \mathcal{H} \setminus r\mathcal{H} \) and is in two possible ways when \( x \in r\mathcal{H} \). More precisely, when \( x \in r\mathcal{H} \), it can be decomposed into either \( x = r(i+1) + 0 \) or \( x = ri + r \) for some \( i \in \mathbb{Z} \). Note that \( f(r(i+1))(0) = f(r(i))(0) \).

It then follows that \( \phi_f(x) = \phi_f(ri + y) := f(r)(y) \) is a well-defined function in \( \mathcal{C} \). Clearly, \( \phi_f = f \). If \( f(r(i)) = u, \forall i \in \mathbb{Z} \), then \( \phi_f(r(i)) = u, \forall i \in \mathbb{Z} \), which implies that \( \phi_f \) is \( r \)-periodic.
If we define $F : C \to K$ by $F(\phi) = \tilde{\phi}$, then $F$ is a homeomorphism between $C$ and $K$. Let $\beta(x)$ be a strongly positive $r$-periodic steady state of the semiflow $\{Q_t\}_{t \geq 0}$. With a little abuse of notation, we use $C_\beta$ to denote the set $\{\phi \in C : 0 \leq \phi \leq \beta\}$. Now we can define a semiflow $\{P_t\}_{t \in T}$ on $K_{\tilde{\beta}} := \{f \in K : 0 \leq f \leq \tilde{\beta}\}$ by

$$P_t[f] = F \circ Q_t[\phi_f], \quad \forall f \in K_{\tilde{\beta}}, \; t \in T. \quad (4.2)$$

Clearly, $P_t \circ F = F \circ Q_t$, $\forall t \in T$, which implies that semiflows $\{Q_t\}_{t \in T}$ and $\{P_t\}_{t \in T}$ are topologically conjugate. Moreover, $\{P_t\}_{t \in T}$ is spatially homogeneous and $\beta$ is its equilibrium. Thus, we see that the semiflow $\{Q_t\}_{t \in T}$ on $C_\beta$ has a spatially $r$-periodic traveling wave if the semiflow $\{P_t\}_{t \in T}$ on $K_{\tilde{\beta}}$ has a traveling wave. Before stating the main result, we first introduce the bistability assumption. Let $\beta(x) \gg 0$ be an $r$-periodic steady state of the semiflow $\{Q_t\}_{t \in T}$ on $C_\beta$. Assume that $0$ is a trivial steady state. Define

$$\Pi_\beta := \{\phi \in C : \phi(x) = \phi(x + r), \; 0 \leq \phi(x) \leq \beta(x), \; \forall x \in H\}.$$

As in Definition 2.1, we can define the strong stability of periodic steady states for a map $Q$ in the space of periodic functions.

**Definition 4.2.** A steady state $\alpha \in \Pi_\beta$ is said to be strongly stable from below for the map $Q : \Pi_\beta \to \Pi_\beta$ if there exist a positive number $\delta_\alpha^+$ and a strongly positive element $e_\alpha^+ \in \Pi_\beta$ such that

$$Q[\alpha - \eta e_\alpha^+] \gg \alpha - \eta e_\alpha^+, \; \forall \eta \in (0, \delta_\alpha^+]. \quad (4.3)$$

The strong instability from below is defined by reversing the inequality (4.3). Similarly, we can define strong stability (instability) from above.

We need the following bistability assumption on the spatially $r$-periodic map $Q$.

**(A5″) (it Bistability)** $0$ and $\beta \gg 0$ are two strongly stable $r$-periodic steady states from above and below, respectively, for $Q : \Pi_\beta \to \Pi_\beta$, and the set of all intermediate $r$-periodic steady states are totally unordered in $\Pi_\beta$.

We note that a sufficient condition for the non-ordering property of all intermediate $r$-periodic steady states is: $Q : \Pi_\beta \to \Pi_\beta$ is eventually strongly monotone and all intermediate fixed points are strongly unstable from both above and below.

**Theorem 4.1.** Let $X = C(M, \mathbb{R}^d)$. Assume that for any $t > 0$, the map $Q_t$ satisfies (A2)-(A4) and the bistability assumption (A5″). Further, assume that the map $P_1 := FQ_1F^{-1}$ satisfies assumption (A6) with $C$ and $\beta$ replaced by $K$ and $\tilde{\beta}$, respectively. Then the spatially $r$-periodic semiflow $\{Q_t\}_{t \in T}$ admits an $r$-periodic traveling wave $V(x, x + ct)$. Besides, $V(x, \xi)$ is nondecreasing in $\xi$ and connecting $0$ to $\beta(x)$.
Proof. Let \( t \geq 0 \) be fixed and \( P_t \) be defined as in (4.2). Then it is easy to see that \( P_t \) satisfies (A1)-(A5) with \( C_\beta \) replaced by \( K_\beta \). From Theorems 3.2 and 3.5 we see that \( \{ P_t \}_{t \in T} \) admits a traveling waves \( U(x + ct) \) with \( U \) connecting 0 to \( \beta \). By the definitions of traveling waves in a discrete habitat (see Definitions 3.2 and 3.5), we can find \( x_0 \in \mathbb{R} \) such that \( g := U(\cdot + x_0) \in K_\beta \) and \( P_t[g](ri) = U(ri + ct + x_0), \forall i \in \mathbb{Z} \). By Lemma 4.1 we can find \( \psi, h_t \in \mathcal{C} \) such that \( \tilde{\psi} = g \) and \( \tilde{h}_t = U(\cdot + ct + x_0) \), and hence, \( P_t[\tilde{\psi}] = \tilde{h}_t \). By the topological conjugacy of \( Q_t \) and \( P_t \), we have \( Q_t[\psi] = h_t \).

Note that \( \tilde{\psi} = g = U(\cdot + x_0) \). It then follows from Lemma 4.1 that \( \psi = h_0 \).

If \( c = 0 \), then we obtain \( Q_t[\psi] = h_t \equiv h_0 = \psi \), which implies that \( \psi \) is a traveling wave with speed zero. If \( c \neq 0 \), then we define \( V(\xi, x) := h_{\frac{\xi}{c}}(x) \). Consequently,

\[
V(x + ct, x) = h_t(x) = Q_t[\psi](x) = Q_t[h_0](x) = Q_t[V(\cdot, \cdot)](x), \forall x \in \mathbb{H}, t \geq 0.
\]

This completes the proof. \( \square \)

To finish this section, we remark that the bistability structure can be obtained for equation (4.1) under appropriate conditions so that the existence result in [42, 44] is improved (see the details in subsection 6.3). Further, Theorem 4.1 with \( \mathbb{H} = \mathbb{Z} \) and \( \mathcal{X} = \mathbb{R} \) can be used to rediscover the existence result in [14] for one dimensional lattice equation under the bistability assumption.

5 Semiflows with weak compactness

In assumption (A4) of section 2, we assume that \( Q : C_\beta \rightarrow C_\beta \) is compact with respect to the compact open topology. In this section, we establish the existence of bistable waves under some weaker compactness assumptions.

Let \( \tau > 0 \) be a fixed number. It is well known that the time-\( t \) solution map of time-delayed reaction-diffusion equations such as

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u(t, x), u(t - \tau, x)), \tag{5.1}
\]

is compact with respect to the compact open topology if and only if \( t > \tau \), where the phase space \( \mathcal{C} \) is chosen as \( C(\mathbb{R}, C([-\tau, 0], \mathbb{R})) \). The first purpose of this section is to show that our results are still valid for this kind of evolution equations by introducing an alternative assumption (A4').

In order to state this assumption, we need some notations for time-delayed evolution systems. Let \( \tau \in T \) be a positive number, \( \mathcal{F} \) be a Banach lattice with the positive cone \( \mathcal{F}^+ \) having non-empty interior, \( \beta \in Int(\mathcal{F}^+) \), and \( X_\beta = C([-\tau, 0], \mathcal{F}_\beta) \). For any \( \phi \in C_\beta \), we can regard it as an element in \( C([-\tau, 0] \times \mathcal{H}, \mathcal{F}^+) \). For any subset \( B \) of \([-\tau, 0] \times \mathcal{H} \), we define \( \phi|_B \) as the restriction of \( \phi \) on \( B \).

(A4') (Compactness) There exists \( s \in (0, \tau] \) such that

(i) \( Q[\phi](\theta, x) = \phi(\theta + s, x) \) whenever \( \theta + s \leq 0 \).
(ii) For any $\epsilon \in (0, s)$, the set $Q[C_\beta]|_{\tau - s + \epsilon, 0} \times \mathcal{H}$ is precompact.

(iii) For any subset $\mathcal{J} \subset C_\beta$ with $\mathcal{J}(0, \cdot) \subset C(\mathcal{H}, \mathcal{Y}_\beta)$ being precompact, the set $Q[\mathcal{J}]|_{\tau - s, 0} \times \mathcal{H}$ is precompact.

This assumption was motivated by [25, Assumption (A6')]. Let us use equation (5.1) to explain (A4'). For any $t > \tau$, one can directly verify that the solution map $Q_t$ satisfies (A4) by rewriting (5.1) as an integral form (see, e.g., [40]); and for any $t \in (0, \tau]$, one can show that $Q_t$ satisfies (A4') (i) and (ii) by the same arguments. For (A4') (iii), we provide a proof below.

Let $T(0) = I$, and for any $t \in (0, \tau]$, let $T(t)$ be the time-$t$ map of the heat equation $u_t = \Delta u$. Then (5.1) can be written as the following form:

$$u(t, x; \phi) = T(t)\phi(x) + \int_{0}^{t} T(t - s)f(u(s, u(s - \tau)))(x)ds,$$

and hence, $Q_t[\phi](\theta, x) = u(t + \theta, x)$. Note that for any $\phi \in C_\beta$, $T(t)\phi \to \phi$ with respect to the compact open topology as $t \to 0$. It then follows from the triangular inequality and the absolute continuity of integrals that for any compact subset $\mathcal{H}_1 \subset \mathbb{R}$, the set $Q_t[\mathcal{J}]|_{\tau - t, 0} \times \mathcal{H}_1$ is equi-continuous, and hence, $Q[\mathcal{J}]|_{\tau - t, 0} \times \mathbb{R}$ is precompact in $C_\beta$.

**Lemma 5.1.** Let $A_{\xi}, \xi \geq 1$, be defined as in section 3 and $\beta \in Int(\mathcal{F}^+)$. Assume that $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies (A4'). Then there exists an integer $m_0$ such that $\cup_{\xi \in [1, 1 + \theta]}(Q \circ A_{\xi})^{m_0}[C_\beta] \subset C_\beta$ is precompact when $\mathcal{H} = \mathbb{R}$, and $\cup_{\xi \in [1, 1 + \theta]}(Q \circ A_{\xi})^{m_0}[B_\beta](x) \subset X_\beta$ is precompact for any $x \in \mathbb{R}$ when $\mathcal{H} = \mathbb{Z}$.

**Proof.** We only prove the case where $\mathcal{H} = \mathbb{R}$ since the proof for $\mathcal{H} = \mathbb{Z}$ is essentially similar. Let $s$ and $\tau$ be defined in (A4'). For such $s$ and $\tau$, there exists $m_0 \in \mathbb{N}$ such that $s \in \left(\frac{1}{m_0 + 1}\tau, \frac{1}{m_0}\tau\right)$. By assumption (A4')(i), we see that for any $\xi \geq 1$ and $\phi_0 \in C_\beta$,

$$\phi_1^\xi(\theta, x) := Q \circ A_{\xi}[\phi_0](\theta, x) = \begin{cases} \phi_0(\theta + s, \xi x), & \theta + s \leq 0 \\ Q[\phi_0(\xi \cdot)](\theta, x), & \theta + s > 0, \end{cases}$$

This implies that for any $\xi \geq 1$ and $\epsilon < s - \frac{1}{m_0 + 1}\tau$,

$$\cup_{\xi \in [1, 1 + \theta]}Q \circ A_{\xi}[C_\beta]|_{\tau - s + \epsilon, 0} \times \mathbb{R} \subset Q[C_\beta]|_{\tau - s + \epsilon, 0} \times \mathbb{R}.$$

Since $Q[C_\beta]|_{\tau - s + \epsilon, 0} \times \mathbb{R}$ is precompact, as assumed in (A4'(ii)), it then follows that $\cup_{\xi \in [1, 1 + \theta]}Q \circ A_{\xi}[C_\beta](0, \cdot) \subset C(\mathbb{R}, \mathcal{Y}_\beta)$ is precompact. By (A4')(iii) and similar arguments as above, we have

$$\phi_2^\xi(\theta, x) := Q \circ A_{\xi}[\phi_1^\xi](\theta, x) = \begin{cases} \phi_1^\xi(\theta + s, \xi^2 x), & \theta + 2s \leq 0 \\ Q[\phi_0(\xi \cdot)](\theta + s, \xi x), & 0 < \theta + 2s \leq s \\ Q[\phi_1^\xi(\xi \cdot)](\theta, x), & \theta + s > 0, \end{cases}$$
This implies that $\cup_{\xi \in [1, 2]} (Q \circ A_\xi)^2[C_\beta]([-2s + \epsilon, 0] \times \mathbb{R})$ is precompact. Consequently, $\cup_{\xi \in [1, 2]} (Q \circ A_\xi)^2[C_\beta](0, \cdot) \subset C(\mathbb{R}, Y_\beta)$ is compact. By induction, we have

$$\phi_{m_0 + 1}^\xi(\theta, x) := Q \circ A[\phi_{m_0}^\xi](\theta, x) = \begin{cases} \phi_{m_0}^\xi(\theta + s, \xi x), & \theta + s \leq 0 \\ Q[\phi_{m_0}^\xi](\theta, x), & \theta + s > 0 \end{cases}$$

$$= \ldots$$

$$= \begin{cases} Q[\phi_0(\xi^s)](\theta + (m_0 + 1)s, \xi^{m_0} x), & 0 < \theta + (m_0 + 1)s \leq s \\ Q[\phi_{m_0}^\xi(\xi)](\theta + m_0 s, \xi^{m_0 - 1} x), & 0 < \theta + m_0 s \leq s \\ \ldots \\ Q[\phi_{m_0 - 1}^\xi(\xi)](\theta + s, \xi x), & 0 < \theta + s \leq s \\ Q[\phi_{m_0}^\xi](\theta, x), & \theta + s > 0. \end{cases}$$

This implies that $\cup_{\xi \in [1, 2]} (Q \circ A_\xi)^m_0[C_\beta]$ is precompact in $C_\beta$. \hfill \Box

**Theorem 5.1.** All results in Theorems 3.1-3.5 and 4.1 are valid if we replace (A4) with (A4').

**Proof.** Following the proof of these theorems, we only need to modify the parts where we use the compactness assumption (A4). At these parts, by Lemma 5.1 we can easily complete the proof. \hfill \Box

Note that the solution maps of the integro-differential equation

$$u_t = J * u - u + f(u)$$

satisfy neither (A4) nor (A4'). The second purpose of this section is to modify our developed theory in such a way that it applies to these integro-differential systems.

Let $\mathcal{M}$ denote the set of all nondecreasing functions from $\mathbb{R}$ to $\mathcal{X}$ and $\beta \in \mathcal{X}^+$. We equip $\mathcal{M}$ with the compact open topology. Assume that $Q$ maps $\mathcal{M}_\beta$ to $\mathcal{M}_\beta$. Let $E$ denote the set of fixed point of $Q$ restricted on $\mathcal{X}_\beta$. Suppose that 0 and $\beta$ are in $E$. We impose the following assumptions on $Q$:

(B1) *(Translation Invariance)* $T_y \circ Q[\phi] = Q \circ T_y[\phi], \forall \phi \in \mathcal{M}_\beta, y \in \mathbb{R}$.

(B2) *(Continuity)* $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ is continuous in the sense that if $\phi_n \to \phi$ in $\mathcal{M}_\beta$, then $Q[\phi_n](x) \to Q[\phi](x)$ in $\mathcal{X}_\beta$ for almost all $x \in \mathbb{R}$.

(B3) *(Monotonicity)* $Q$ is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in $\mathcal{M}_\beta$.

(B4) *(Weak Compactness)* For any fixed $x \in \mathbb{R}$, the set $Q[\mathcal{M}_\beta](x)$ is precompact in $\mathcal{X}_\beta$.

(B5) *(Bistability)* Fixed points 0 and $\beta$ are strongly stable from above and below, respectively, for the map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, and the set $E \setminus \{0, \beta\} \subset \mathcal{X}_\beta$ is totally unordered.
(B6) \emph{(Counter-propagation)} For each $\alpha \in E \setminus \{0, \beta\}$, $c^-_\alpha(\alpha, \beta) + c^+_\alpha(0, \alpha) > 0$.

Comparing assumptions (A1)-(A6) and (B1)-(B6), one can find that the assumptions of translation invariance, monotonicity, bistability and counter-propagation are the same. The difference lies in the assumptions of continuity and compactness. Clearly, compactness assumption (B4) is much weaker than (A4).

**Theorem 5.2.** Let $\mathcal{X} = C(M, \mathbb{R}^d)$ and assume that $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ satisfies (B1)-(B6). Then there exists $c \in \mathbb{R}$ and $\psi \in \mathcal{M}_\beta$ connecting $0$ to $\beta$ such that $Q[\psi](x) = \psi(x + c)$ for all $x \in \mathbb{R}$.

**Proof.** Combining the proofs of Theorems 3.1 and 3.2, we can obtain the result. More precisely, one can repeat the proof of Theorem 3.1 except for the parts where the compactness assumption (A4) are used. For these parts, one use the idea in Theorem 3.2 where $\tilde{Q}$ has the same compactness property as $Q$. \hfill $\square$

In the rest of this section, we say $\{Q_t\}_{t \geq 0}$ is a semiflow on $\mathcal{M}_\beta$ provided that $Q_0 = I$; $Q_t \circ Q_s = Q_{t+s}$; $\forall t, s > 0$; and $Q_t[\phi_n](x) \to Q_t[\phi](x)$ in $\mathcal{X}_\beta$ for almost all $x \in \mathbb{R}$ whenever $t_n \to t$ and $\phi_n \to \phi$ in $\mathcal{M}_\beta$.

**Theorem 5.3.** Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that $\{Q_t\}_{t \geq 0}$ is a semiflow on $\mathcal{M}_\beta$, and for any $t > 0$, the map $Q_t$ satisfies (B1) and (B3)-(B6). Then there exist $c \in \mathbb{R}$ and $\psi \in \mathcal{M}_\beta$ connecting $0$ to $\beta$ such that $Q_t[\psi](x) = \psi(x + ct)$ for all $x \in \mathbb{R}$.

**Proof.** As in the proof of Theorem 5.2, we can prove the conclusion by combining the proofs of Theorems 3.4 and 3.5. \hfill $\square$

Similarly, we can define $\omega$-time periodic semiflows on $\mathcal{M}_\beta$ and then obtain the following result.

**Theorem 5.4.** Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that $\{Q_t\}_{t \geq 0}$ is an $\omega$-time periodic semiflow on $\mathcal{M}_\beta$. Let $\beta(t)$ be a strongly positive periodic solution of $\{Q_t\}_{t \geq 0}$ restricted on $\mathcal{X}_\beta$. Further, assume that the Poincaré map $Q_\omega$ satisfies (B1) and (B3)-(B6) with $\beta = \beta(0)$. Then there exist $c \in \mathbb{R}$ and $\phi(t, x)$ with $\phi(t, -\infty) = 0$ and $\phi(t, +\infty) = \beta(t)$ such that $Q_t[\psi](x) = \psi(t, x + ct)$ for all $x \in \mathbb{R}$. Besides, $\phi(t, \cdot) \in \mathcal{M}_\beta$ and $\phi(t, \cdot)$ is $\omega$-periodic in $t \geq 0$.

## 6 Applications

In this section, we apply the obtained abstract results to four kinds of monotone evolution systems: a time-periodic reaction-diffusion system, a parabolic system in a cylinder, a parabolic equation with variable diffusion, and a nonlocal and time-delayed reaction-diffusion equation.
6.1 A time-periodic reaction-diffusion system

Consider the time-periodic reaction-diffusion system

$$\frac{\partial u}{\partial t} = A\Delta u + f(t,u), \quad x \in \mathbb{R},$$

where $u = (u_1, \ldots, u_n)^T$, $A = \text{diag}\{d_1, \ldots, d_n\}$ with each $d_i > 0$ and $f = (f_1, \ldots, f_n)^T$ is $\omega$-periodic in $t \geq 0$ (i.e., $f(t, \cdot) = f(t + \omega, \cdot)$). The existence of periodic bistable traveling waves of (6.1) with $n = 1$ was proved in [1]. Here we generalize this result to the case $n \geq 1$.

Let $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. In order to apply Theorem 3.3 to system (6.1), we choose $C := C(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{X} := \mathbb{R}^n$, and $\mathcal{E} \subset C$ to be the set of all bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$. Using the solution maps $\{T(t)\}_{t \geq 0}$ of the heat equation $\frac{\partial u}{\partial t} = \Delta u$, we write (6.1) as the following integral form:

$$u(t; \phi) = T(t)\phi + \int_0^t T(t-s)f(s,u(s;\phi))ds.$$  \hspace{1cm} (6.2)

Define $Q_1[\phi] := u(t; \phi), \forall \phi \in \mathcal{E}$. Let 0 and $\beta \gg 0$ be two fixed points of the Poincaré map $Q_\omega$ in $\mathcal{X}$, and let $E$ be the set of all spatially homogeneous fixed points of $Q_\omega$ in $\mathcal{X}_s$. We impose the following assumptions:

(C1) The Jacobian matrix $D_u f(t,u)$ is cooperative and irreducible for all $t \geq 0$ and $u \geq 0$.

(C2) The spatially homogeneous system $u' = f(t,u)$ is of bistable type, that is, 0 and $\beta$ are two stable fixed points of $Q_\omega$ in the sense that $s(\frac{d}{du}Q_\omega[0]) < 0$ and $s(\frac{d}{du}Q_\omega[\beta]) < 0$, and any $\alpha \in E \setminus \{0, \beta\}$ is an unstable in the sense that $s(\frac{d}{du}Q_\omega[\alpha]) > 0$, where $s(M)$ is the stability modulus of the matrix $M$ defined by $s(M) = \max\{\text{Re} \lambda : \lambda \text{ is an eigenvalue}\}$.

**Theorem 6.1.** Assume that (C1)-(C2) hold, and let $\beta(t)$ be the periodic solution of $u' = f(t,u)$ with $\beta(0) = \beta$. Then there exists $c \in \mathbb{R}$ such that (6.1) admits a time-periodic traveling wave $U(t,x+ct)$ connecting 0 to $\beta(t)$.

**Proof.** It is easy to see that the discrete semiflow $\{Q_\omega^n\}_{n \geq 1}$ on $C_\beta$ satisfies (A1)-(A5) with $Q = Q_\omega$. Next we show that (A6) holds with $Q = Q_\omega$.

Note that for any $\alpha \in E \setminus \{0, \beta\}$, $\{Q_\omega^n\}_{n \geq 1} : [\alpha, \beta]_c \to [\alpha, \beta]_c$ performs a monostable dynamics, where $\alpha$ is unstable and $\beta$ is stable. By the theory developed in [25], it follows that $Q_\omega$ admits leftward and rightward spreading speeds $c_-(\alpha, \beta)$ and $c_+(\alpha, \beta)$. Since $Q_\omega$ is reflectively invariant, we further have $c_+(\alpha, \beta) = c_+(\alpha, \beta) := c^*(\alpha, \beta)$, which is called the spreading speed of this monostable subsystem. Note that $\{Q_\omega^n\}_{n \geq 1} : [0, \alpha]_c \to [0, \alpha]_c$ also performs a monostable dynamics, where 0 is stable and $\alpha$ is unstable. Similarly, this monostable subsystem also admits a spreading speed $c^*(0, \alpha)$. Let $M_t$ be the solution map of the linearized system of (6.1) at the periodic solution $\alpha(t) := Q_t[\alpha]$:

$$\frac{\partial u}{\partial t} = A\Delta u + D_u f(t, \alpha(t))u.$$  \hspace{1cm} (6.3)
By a similar argument as in the proof of [39, Lemma 4.1], we see that for each $t > 0$, there exists a strongly positive vector $\eta \in \mathbb{R}^n$ such that
\[ Q_t[u] \geq M_t[u] \quad \text{whenever} \quad u \in [\alpha, \alpha + \eta]_\mathcal{C} \]  
(6.4)
and
\[ Q_t[u] \leq M_t[u] \quad \text{whenever} \quad u \in [\alpha - \eta, \alpha]_\mathcal{C}. \]  
(6.5)

Let $\rho(\mu)$ be the principle Floquet multiplier of the following linear periodic cooperative and irreducible system
\[
\frac{dv}{dt} = [\mu A + D_u f(t, \alpha(t))]v. \tag{6.6}
\]

Let $v(t, w)$ be the solution of (6.6) satisfying $v(0, w) = w \in \mathbb{R}^n$. It is easy to see that $u(t, x) = e^{-\mu x}v(t, w)$ is the solution of linear periodic system (6.3). Define
\[
\Phi(\mu) := \ln \rho(\mu)/\mu.
\]
From [25, Theorem 3.10] and inequalities (6.4)-(6.5), we then have
\[
c^*(\alpha, \beta) \geq \inf_{\mu > 0} \Phi(\mu) \quad \text{and} \quad c^*(0, \alpha) \geq \inf_{\mu > 0} \Phi(\mu). \tag{6.7}
\]

Now we prove that $\Phi(+\infty) = +\infty$. Let $\lambda(\mu) = \frac{1}{\omega} \ln \rho(\mu)$. By the Floquet theory, it then follows that there exists a positive $\omega$-periodic function $\xi(t) := (\xi_1(t), \ldots, \xi_n(t))^T$ such that $v(t) := e^{\lambda(\mu)t}\xi(t)$ is a solution of (6.3). In particular, we have
\[
\xi_1'(t) = (\mu^2 - \lambda(\mu))\xi_1(t) + \sum_{i=1}^n \frac{\partial}{\partial u_i} f_1(t, \alpha(t))\xi_i(t).
\]

Dividing $\xi_1(t)$ in both sides and integrating the above equality from 0 to $\omega$ gives
\[
0 = (\mu^2 - \lambda(\mu))\omega + \int_0^\omega \sum_{i=1}^n \frac{\partial}{\partial u_i} f_1(t, \alpha(t))\xi_i(t)/\xi_1(t)dt, \quad \forall \mu > 0.
\]

Since the matrix $D_u f(t, \alpha(t))$ is cooperative and $\xi(t)$ is positive, we obtain
\[
0 \geq (\mu^2 - \lambda(\mu))\omega + \int_0^\omega \frac{\partial}{\partial u_1} f_1(t, \alpha(t))dt.
\]

This implies that
\[
\Phi(\mu) = \frac{\omega \lambda(\mu)}{\mu} \geq \mu \omega + \int_0^\omega \frac{\partial}{\partial u_1} f_1(t, \alpha(t))dt,
\]
and hence, $\Phi(+\infty) = +\infty$. By [25, Lemma 3.8], we then have $\inf_{\mu > 0} \Phi(\mu) > 0$. Thus, the assumption (A6) with $Q = Q_\omega$ holds. Consequently, Theorem 3.3 completes the proof. \qed
6.2 A reaction-diffusion-advection system in a cylinder

In this subsection, we consider the following system

\[
\begin{aligned}
  \frac{\partial u}{\partial t} &= A \frac{\partial^2 u}{\partial x^2} + B \Delta_y u + E(y) \frac{\partial u}{\partial x} + f(u), & x &\in \mathbb{R}, y &\in \Omega \subset \mathbb{R}^{m-1}, t > 0, \\
  \frac{\partial u}{\partial \nu} &= 0, & \text{on } (0, +\infty) \times \mathbb{R} \times \partial \Omega,
\end{aligned}
\]  

(6.8)

where \(A, B\) are positively definite diagonal \(n \times n\) matrix, \(E\) is diagonal matrix of smooth functions of \(y\), \(\Omega\) is a bounded and convex open subset in \(\mathbb{R}^{m-1}\) with smooth boundary \(\partial \Omega\), \(\Delta_y = \sum_{i=1}^{m-1} \frac{\partial^2}{\partial y_i^2}\), and \(\nu\) is the outer unit normal vector to \(\partial \Omega \times \mathbb{R}\).

The existence of bistable traveling waves for (6.8) with \(n = 1\) was obtained in [11]. Here we extend this result to the case \(n \geq 2\). Assume that \(f \in C^1(\mathbb{R}^n, \mathbb{R}^n)\) satisfies the following two conditions:

(D1) The Jacobian matrix \(Df(u)\) is cooperative and irreducible for all \(u \geq 0\).

(D2) \(f\) is of bistable type in the sense that it has exactly three ordered zeros:

\(0 < a < \beta\) and \(s(Df(0)) < 0, s(Df(a)) > 0, s(Df(\beta)) < 0\).

**Theorem 6.2.** Assume that (D1)-(D2) hold. Then there exists \(c \in \mathbb{R}\) such that system (6.8) admits a traveling wave connecting \(0\) to \(\beta\) with speed \(c\).

**Proof.** In order to employ Theorem [34] we choose \(X = C(\bar{\Omega}, \mathbb{R}^n)\) and \(C := C(\mathbb{R}, X)\) with the standard cones \(X^+\) and \(C^+\), respectively. Let \(G(t, x, y, w)\) be the Green function of the linear equation

\[
\begin{aligned}
  \frac{\partial u}{\partial t} &= A \frac{\partial^2 u}{\partial x^2} + B \Delta_y u + E(y) \frac{\partial u}{\partial x}, & x &\in \mathbb{R}, y &\in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} &= 0, & \text{on } (0, +\infty) \times \mathbb{R} \times \partial \Omega.
\end{aligned}
\]  

(6.9)

Then the solution of (6.9) with initial value \(u(0, \cdot) = \phi(\cdot) \in C\) can be expressed as

\[u(t, x, y; \phi) = \int_{\mathbb{R}} \int_{\Omega} G(t, x - z, y, w) \phi(z, w) dw dz.\]

Define \(T(t)\phi = u(t, \cdot; \phi), \forall \phi \in C_\beta\). Using the constant variation formula, we write (6.8) subject to \(u(0, \cdot) = \phi(\cdot) \in C_\beta\) as an integral equation

\[u(t, x, y; \phi) = T(t)[\phi](x, y) + \int_0^t T(t - s)f(u(s, x, y))ds.\]  

(6.10)

By the linear operators theory, we see that for any \(\phi \in C_\beta\), system (6.8) has a unique solution \(u(t; \phi)\) with \(u(0; \phi) = \phi\), which exists globally on \([0, +\infty)\). Define \(Q_t[\phi] := u(t, \phi)\). Then \(\{Q_t\}_{t \geq 0}\) is a subhomogeneous semiflow on \(C_\beta\) (see [25] Section 5.3). Also, assumption (D1) assures that the semiflow \(\{Q_t\}_{t \geq 0}\) restricted on \(X_\beta\) is strongly monotone (see [34]). Further, it is easy to see that \(Q_t, t \geq 0\), satisfies
assumption (A1)-(A4). Since the domain $\Omega$ is convex, it follows from the result in [24] that any non-constant steady state of the $x$-independent system

$$
\begin{align*}
\frac{\partial u}{\partial t} &= B\Delta_y u + f(u), \quad y \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \partial \Omega
\end{align*}
$$

is linearly unstable. This then implies that $Q_t$ satisfies (A5′). Now it remains to show that (A6) holds for $Q_1$.

For each $x$-independent steady state $\alpha = \alpha(y)$ in $[0, \beta]$ in $\mathbb{R}$, system (6.8) performs a monostable dynamics on $[\alpha, \beta]$. To better understand the dynamics of this subsystem, we make a transform $g(u, y) := f(u + \alpha(y))$. Then its dynamics is equivalent to that of the following system on $[0, \beta - \alpha]$:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= A\frac{\partial^2 u}{\partial x^2} + B\Delta_y u + E(y)\frac{\partial u}{\partial x} + g(u, y), \quad x \in \mathbb{R}, y \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \mathbb{R} \times \partial \Omega.
\end{align*}
$$

System (6.11) has exactly two $x$-independent steady state $S_1 := 0$ and $S_2 := \beta - \alpha \gg 0$. By the theory developed in [26], it follows that (6.11) has a leftward spreading speed $\hat{c}^*$ in a strong sense. Let $c^*(\alpha, \beta)$ be defined as in (2.6) with $Q = Q_1$. We then have $c^*(\alpha, \beta) \geq \hat{c}^*$.

To verify (A6) for $Q_1$, we first estimate the speed $\hat{c}^*$. Consider the linearized system of (6.11) at equilibrium $S_1$:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= A\frac{\partial^2 u}{\partial x^2} + B\Delta_y u + E(y)\frac{\partial u}{\partial x} + g(0, y), \quad x \in \mathbb{R}, y \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \mathbb{R} \times \partial \Omega.
\end{align*}
$$

Suppose $u(t, x, y) := e^{\mu x} \eta(t, y)$ is a solution of (6.12), then $\eta(t, y)$ satisfies the $\mu$-parameterized linear parabolic equation

$$
\begin{align*}
\frac{\partial v}{\partial t} &= B\Delta_y v + \left[\mu^2 A + \mu E(y) + \frac{\partial g(0, y)}{\partial u}\right] v, \quad y \in \Omega, t > 0, \\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \partial \Omega.
\end{align*}
$$

Let $\lambda^+(\mu)$ be the principle eigenvalue of the elliptic problem:

$$
\begin{align*}
\lambda v &= B\Delta_y v + \left[\mu^2 A + \mu E(y) + \frac{\partial g(0, y)}{\partial u}\right] v, \quad y \in \Omega, \\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

By the theory in [25, Section 3], it follows that $\hat{c}^* \geq \inf_{\mu > 0} \frac{\lambda^+(\mu)}{\mu}$, and $\lambda^+(\mu)$ is convex. Then it is easy to see from (6.14) that $\lim_{\mu \to +\infty} \frac{\lambda^+(\mu)}{\mu} = +\infty$ and $\lim_{\mu \to 0^+} \frac{\lambda^+(\mu)}{\mu} = +\infty$, and hence, $\frac{\lambda^+(\mu)}{\mu}$ attains its infimum at some $\mu_1 \in (0, +\infty)$.

Similarly, system (6.8) performs a monostable dynamics on $[0, \alpha]$. To better understand the dynamics of this subsystem, we make a transform $h(u, y) :=
\[-f(\alpha(y) - u)\]. Then its dynamics is equivalent to that of the following system on 
\([0, \beta - \alpha]c\):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Au_{xx} + B\Delta_y u + E(y)\frac{\partial u}{\partial x} + h(u, y), \quad x \in \mathbb{R}, \ y \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \mathbb{R} \times \partial \Omega.
\end{align*}
\]

By the same arguments, such system have a rightward spreading speed \(\hat{c}^*_+\), and we have \(\hat{c}^*_+ \geq c^*_+(0, \alpha)\). Also, by the same procedure as above, we define \(\lambda^- (\mu)\) as the principle eigenvalue of the following elliptic problem:

\[
\begin{align*}
\lambda v &= B\Delta_y v + [\mu^2 A + \mu E(y) + \frac{\partial g(0, y)}{\partial u}]v, \ y \in \Omega, \\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

It then follows that \(\hat{c}^*_+ \geq \inf_{\mu > 0} \frac{\lambda^-(\mu)}{\mu}\), and \(\frac{\lambda^-(\mu)}{\mu}\) attains its infimum at some \(\mu_2 \in (0, +\infty)\). Clearly, \(\lambda^-(\mu) = \lambda^+(-\mu)\).

From assumption (D2), we see that \(S_1 (S_2)\) is a linearly unstable (stable) steady state of the \(x\)-independent system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= B\Delta_y u + g(u, y), \ y \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } (0, +\infty) \times \partial \Omega.
\end{align*}
\]

More precisely, letting \(\lambda_0\) be the principle eigenvalue of the following elliptic problem:

\[
\begin{align*}
\lambda u &= B\Delta_y u + u\frac{\partial g(0, y)}{\partial u}, \ y \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

then \(\lambda_0 > 0\). Obviously, equations (6.14) and (6.15) with \(\mu = 0\) both become equation (6.17), and hence, \(\lambda^+(0) = \lambda_0 = \lambda^-(0) > 0\).

With the information above, now we can show that \(Q_1\) satisfies (A6). Let \(\theta = \frac{\mu_2}{\mu_1 + \mu_2}\). Note that \(\theta \mu_1 + (1 - \theta)(-\mu_2) = 0\). It then follows that

\[
c^*_+(\alpha, \beta) + c^*_+(0, \alpha) \geq \frac{\lambda^+(\mu_1)}{\mu_1} + \frac{\lambda^+(-\mu_2)}{\mu_2} = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} [\theta \lambda^+(\mu_1) + (1 - \theta)\lambda^+(-\mu_2)] \geq \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \lambda^+(\theta \mu_1 + (1 - \theta)(-\mu_2)) = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \lambda^+(0) = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \lambda_0 > 0.
\]

Consequently, Theorem 3.4 completes the proof. \(\square\)
6.3 A parabolic equation with periodic diffusion

In this subsection, we study the existence of spatially periodic traveling waves of the parabolic equation

\[ u_t = (d(x)u_x)_x + f(u), \quad t > 0, \, x \in \mathbb{R}, \quad (6.19) \]

where \( f(u) = u(1-u)(u-a), \, a \in (0,1), \) and \( d(x) \) is a positive, \( C^1 \)-continuous, and \( r \)-periodic function on \( \mathbb{R} \) for some real number \( r > 0 \).

For any \( \phi \in C(\mathbb{R}, [0,1]) \), equation (6.19) admits a unique solution \( u(t; \phi) \) with \( u(0; \phi) = \phi \). Define \( Q_t : C([0,1]) \rightarrow C([0,1]) \) by \( Q_t[\phi] = u(t; \phi) \). It then follows that \( \{Q_t\}_{t \geq 0} \) is a continuous, compact and monotone semiflow on \( C([0,1]) \) equipped with the compact open topology. Let \( C_{\text{per}} ([0,1]) \) be the set of all continuous and \( r \)-periodic functions from \( [0,1] \) to \( \mathbb{R} \). Then the semiflow \( \{Q_t\}_{t \geq 0} \) restricted on \( C_{\text{per}} ([0,1]) \) is strongly monotone. Choosing \( \mathcal{H} = \mathbb{R} \) and \( \mathcal{X} = \mathbb{R} \) in Theorem 4.4, one can easily verify that \( \{Q_t\}_{t \geq 0} \) satisfies assumptions (A2)-(A4). If (6.19) admits the bistability structure, then Proposition 3.1 implies (A5''), and a similar argument as in the previous section shows that (A6) also holds. Thus, we focus on finding sufficient conditions on \( d(x) \) under which (6.19) admits the bistability structure.

Let \( \bar{u} \) be an \( r \)-periodic steady state of (6.19). As in [8], we define \( \lambda_1(\bar{u}, d) \) as the largest number such that there exists a function \( \phi > 0 \) which satisfies

\[
\begin{cases}
(d\phi_x)_x + f'(\bar{u})\phi = \lambda_1(\bar{u}, d)\phi, \quad x \in \mathbb{R} \\
\phi \text{ is } r\text{-periodic and } \|\phi\|_\infty = 1.
\end{cases}
\quad (6.20)
\]

We call \( \lambda_1(\bar{u}, d) \) the principle eigenvalue of \( \bar{u} \), and \( \phi \) the corresponding eigenfunction. We say \( \bar{u} \) is linearly unstable if \( \lambda_1(\bar{u}, d) > 0 \), and linearly stable if \( \lambda_1(\bar{u}, d) < 0 \). Define

\[ \mathcal{C}^1_{\text{per}} := \{ \psi \in C^1([0,1]) : \psi(x) = \psi(x + r), \forall x \in \mathbb{R} \} \]

with the \( C^0 \)-norm induced topology. We say \( \psi \in \mathcal{C}^1_{\text{per}} \) has the property (P) if every possible non-constant \( r \)-periodic steady state of (6.19) with \( d = \psi \) is linearly unstable, that is, if the equation (6.19) with \( d = \psi \) does not admit any non-constant \( r \)-periodic steady state \( \bar{u} \) such that \( \lambda_1(\bar{u}, \psi) \leq 0 \). Define

\[ Y := \{ \psi \in \mathcal{C}^1_{\text{per}} : \psi(x) > 0 \text{ and } \psi \text{ has the property (P)} \}. \]

Lemma 6.1. Any positive constant function is in \( Y \).

Proof. Let \( d(x) \equiv \bar{d} \) be given. If (6.19) has no non-constant \( r \)-periodic steady state, we are done. Let \( \bar{u} \) be a non-constant \( r \)-periodic steady state of (6.19). We need to prove \( \lambda_1(\bar{u}, \bar{d}) > 0 \). Assume, for the sake of contradiction, that \( \lambda_1(\bar{u}, \bar{d}) \leq 0 \). Let \( \phi \) be the positive eigenfunction associated with \( \lambda_1(\bar{u}, \bar{d}) \). Define \( M := \max_{0 \leq x \leq r} \left\{ \frac{\bar{u}_x}{\phi} \right\} \).
and $\psi(x, t) := e^{-\gamma t} (|\bar{u}_x|^2 - M^2 \phi)$. It is easy to see that $\psi(t, x) \leq 0$ for all $x$ and $t$.

Let $\xi := |\bar{u}_x|^2$ and $\eta := M^2 \phi$. Then we have

$$\xi_x = (|\bar{u}_x|^2 \phi^{-1})_x = 2\bar{u}_x \bar{u}_{xx} \phi^{-1} - |\bar{u}_x|^2 \phi^{-2} \phi_x,$$

and

$$\xi_{xx} = 2\phi^{-3}[\bar{u}_{xx} \phi - \bar{u}_x \phi_x]^2 + \phi^{-3}[2\bar{u}_x \bar{u}_{xxx} \phi^2 - |\bar{u}_x|^2 \phi \phi_{xx}].$$

Note that

$$0 = [\ddot{\bar{u}}_{xx} + f(\bar{u})]_x = \ddot{\bar{u}}_{xxx} + f'(\bar{u}) \bar{u}_x.$$

It then follows that

$$e^{\gamma t} (\psi_t - \ddot{\psi}_{xx} + [\gamma - f'(\bar{u})] \psi)$$

$$= -[\dddot{\bar{u}}_{xx} + f'(\bar{u})] \xi + \lambda_1(\bar{u}, \bar{d}) \eta$$

$$= -2\phi^{-3}[\bar{u}_{xx} \phi - \bar{u}_x \phi_x]^2 - \phi^{-3}[2\bar{u}_x \bar{u}_{xxx} \phi^2 - |\bar{u}_x|^2 \phi \phi_{xx}] - f'(\bar{u})|\bar{u}_x|^2 \phi^{-1} + \lambda_1(\bar{u}, \bar{d}) \eta$$

$$\leq \ddot{\bar{u}}_{xx} \phi^{-2} \phi_{xx} + f'(\bar{u}) \phi^{-1} |\bar{u}_x|^{-2} - 2f'(\bar{u})|\bar{u}_x|^2 \phi^{-1} - 2d\phi^{-1} \bar{u}_x \bar{u}_{xxx} + \lambda_1(\bar{u}, \bar{d}) \eta$$

$$= \lambda_1(\bar{u}, \bar{d}) \xi + \lambda_1(\bar{u}, \bar{d}) \eta - 2\bar{u}_x \phi^{-1} [f'(\bar{u}) \bar{u}_x + \dddot{\bar{u}}_{xxx}]$$

$$= \lambda_1(\bar{u}, \bar{d}) [\xi + \eta].$$

Hence, $\psi_t - \ddot{\psi}_{xx} + [\gamma - f'(\bar{u})] \psi \leq 0$ because $\lambda_1(\bar{u}, \bar{d}) \leq 0$.

Since $\bar{u}$ is not a constant and $\psi(t, x)$ is $r$-periodic in $x \in \mathbb{R}$, we can choose $x_0$ such that $\psi(x_0, t) = \psi(x_0 + r, t) = \min_{x \in \mathbb{R}} \psi(x, t) < 0$, and hence, $\psi_x|_{x=x_0} = \psi_x|_{x=x_0+r} = 0$. Thus, $\psi(t, x)$ with $x \in [x_0, x_0 + r]$ satisfies the following equation

$$\begin{cases}
\psi_t - \ddot{\psi}_{xx} + [\gamma - f'(\bar{u})] \psi \leq 0, x \in (x_0, x_0 + r), \\
\psi_x|_{x=x_0} = \psi_x|_{x=x_0+r} = 0,
\end{cases}
$$

(6.21)

and $\psi(t, x)$ attains its maximum 0 at $(x^*, t)$ with $x^* \in (x_0, x_0 + r)$. By the strong maximum principle, we see that $\psi(t, x) \equiv 0$, which implies that $\bar{u}_x/\phi$ is a constant. Since $\bar{u}_x/\phi$ is $r$-periodic, it then follows that $\bar{u}_x \equiv 0$, and hence, $\bar{u}$ is a constant, a contradiction. $\square$

Remark 6.1. By the proof above, it follows that the conclusion of Lemma 6.1 is valid for any $f \in C^1$.

Lemma 6.2. $Y$ is open in $C^1_{\text{per}}$.

Proof. Clearly, Lemma 6.1 implies that $Y \neq \emptyset$. Let $d^* \in Y$ be given. We need to show that $d^*$ is an interior point of $Y$. Assume, for the sake of contradiction, that there is a sequence of points $d_n \in C^1_{\text{per}} \setminus Y$ such that $d_n \rightarrow d^*$ in $C^1_{\text{per}}$ as $n \rightarrow \infty$. Then (6.19) with $d = d_n$ admits a non-constant $r$-periodic steady state $u_n$ with the principle eigenvalue $\lambda_1(u_n, d_n) \leq 0$. Using the transformation $v_n = d_n(\eta_n)x$, we see that $(u_n, v_n)$ is a periodic solution of the following ordinary differential system:

$$\begin{cases}
(u_n)_x = v_n/d_n, \\
v_n)_x = -f(u_n).
\end{cases}
$$

(6.22)
By elementary phase plane arguments, it then follows that
\[ 0 \leq \inf_{x \in \mathbb{R}} u_n(x) \leq a \leq \sup_{x \in \mathbb{R}} u_n(x) \leq 1, \quad \forall n \geq 1, \ x \in \mathbb{R}. \]  
(6.23)

Thus, the sequence of functions \((u_n)_x, (v_n)_x\) is uniformly bounded and equicontinuous, and hence, \((u_n, v_n)\) has a uniformly convergent subsequence, still denoted by \((u_n, v_n)\). Let \((u^*, v^*)\) be the limiting function of \((u_n, v_n)\). Then \(u^*\) is an \(r\)-periodic steady state of (6.19) with \(d = d^*\). It is easy to see from (6.23) that \(u^*\) is not the constant function 0 or 1.

Let \(\phi_n\) be the positive eigenfunction associated with \(\lambda_1(u_n, d_n)\). Then
\[ (d_n(\phi_n)_x + f'(u_n(x))\phi_n = \lambda_1(u_n, d_n)\phi_n. \]  
(6.24)

Dividing both sides of (6.24) by \(\phi_n\) and integrating from 0 to \(r\), we obtain
\[ \int_0^r \frac{d_n[(\phi_n)_x]^2}{\phi_n^2} dx + \int_0^r f'(u_n(x))dx = \lambda_1(u_n, d_n)r \leq 0. \]  
(6.25)

Since \(f \in C^1\) and \(f'(a) > 0\), we see that \(u^*\) cannot be the constant \(a\). Otherwise, the uniform convergence of \(u_n\) to \(a\) implies that \(f'(u_n(x)) > 0\) for all \(x \in [0, r]\) and sufficiently large \(n\), which contradicts (6.25). Thus, \(u^*\) is a non-constant \(r\)-periodic function. Since \(d^* \in Y\), we have \(\lambda_1(u^*, d^*) > 0\).

Note that \(u_n \rightarrow u^*\) in \(C([0, r])\) and \(d_n \rightarrow d^*\) in \(C^1_{\text{per}}\). By the variational characterization of the principal eigenvalue \(\lambda_1(u_n, d_n)\) (see, e.g., Eq. (5.2) of [8]), it then follows that \(0 \geq \lambda_1(u_n, d_n) \rightarrow \lambda_1(u^*, d^*) > 0\), a contradiction. \(\square\)

The following counter-example shows that the parabolic equation (6.19) admits no bistability structure in the general case of periodic function \(d(x)\).

**Lemma 6.3.** Let either \(f(u) = u(1 - u^2)\), or \(f(u) = u(1 - u)(u - 1/2)\). Then there exists a positive function \(d \in C^1_{\text{per}}\) such that (6.19) admits a pair of linearly stable, non-constant, and \(r\)-periodic steady states.

**Proof.** We only consider the case where \(f(u) = u(1 - u^2)\) since the other one can be obtained under appropriate scalings. Our proof is based on the main result in [21, Theorem 3]. Without loss of generality, we assume that \(r = 4\). In what follows, we use some notations of [21].

Let \(l \in (0, 1)\) be fixed and \(c^0\) be the step function on \([-1, 1]\) defined by
\[ c^0(x) = \begin{cases} 1, & x \in [-1, -l] \cup (l, 1], \\ 0, & x \in (-l, l]. \end{cases} \]  
(6.26)

Define \(D := \{(x, y) : x = \pm l, y \in [0, 1]\} \cup \text{graph of } c^0\). By [21, Theorem 3], it then follows that for any positive even function \(c \in C^1([-1, 1], \mathbb{R}^+)\) which is sufficiently
closed to \( c^0 \) (in the sense that the distance between \( D \) and the graph of \( c \) is small enough), the following Neumann boundary problem

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
  u_t = (cu)_x + u(1-u^2), \ x \in (-1,1) \\
  u_x(t, \pm 1) = 0
\end{array}
\right.
\end{aligned}
\]  

\tag{6.27}

admits an odd increasing steady state \( u_c \) which is linearly stable. That is, there exist \( \lambda_1 < 0 \) and \( \phi > 0 \) such that

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
  (c\phi)_x + f'(u_c)\phi = \lambda_1 \phi, \ x \in (-1,1) \\
  \phi_{x}(\pm 1) = 0
\end{array}
\right.
\end{aligned}
\]  

\tag{6.28}

In particular, we can choose \( c \) such that \( c_x(-1) = c_x(1) = 0 \). Since \( c \) is even and \( f \) is odd, we see that \( v_c(x) := u_c(−x) \) is also a steady state, and \( \lambda_1 \) is the corresponding eigenvalue with the positive eigenfunction \( \phi(−x) \).

Now we can construct a linearly stable 4-periodic steady state of (6.19). Define two 4-periodic functions:

\[
\begin{aligned}
\tilde{d}(x) &= \begin{cases} c(x), & x \in [-1,1] \\ c(2-x), & x \in (1,3) \end{cases} \quad \text{and} \quad w_1(x) = \begin{cases} u_c(x), & x \in [-1,1] \\ u_c(2-x), & x \in (1,3) \end{cases}
\end{aligned}
\]

Then \( w_1(x) \) is a 4-periodic steady state of (6.19) with \( d = \tilde{d} \). Let the positive 4-periodic function \( \rho(x) \) be defined by

\[
\rho(x) = \begin{cases} \phi(x), & x \in [-1,1] \\ \phi(2-x), & x \in (1,3) \end{cases}
\]

It follows that \( \lambda_1 \) and \( \rho \) solve the following eigenvalue problem

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
  (\tilde{d}\rho_x)_x + f'(w_1)\rho = \lambda_1 \rho, \ x \in \mathbb{R} \\
  \rho \text{ is } r\text{-periodic.}
\end{array}
\right.
\end{aligned}
\]

This implies that \( w_1 \) is a linearly stable periodic steady state of (6.19) with \( d = \tilde{d} \). Similarly, so is \( w_2(x) := w_1(x + 2) \).

As a consequence of Theorem 4.1, together with Lemmas 6.1 and 6.2, we have the following result on the existence of bistable traveling waves for (6.19).

**Theorem 6.3.** Let \( \tilde{d} \) be a given positive constant. Then there exists \( \delta_0 > 0 \) such that for any \( d \in C^1_{per} \) with \( \|d - \tilde{d}\|_{C^0} < \delta_0 \), (6.19) admits a spatially periodic traveling wave solution \( u(t,x) := V(x + ct,x) \) with some speed \( c \in \mathbb{R} \) and connecting \( 0 \) to \( 1 \). Besides, \( V(\xi,x) \) is nondecreasing in \( \xi \).

We remark that Theorem 6.3 is a \( C^0 \)-perturbation result in \( C^1_{per} \), and hence, it improves the existence result in [44, Theorem 3.1], where the \( H^s \)-perturbation is used for some \( s > 2 \).
6.4 A nonlocal and time-delayed reaction-diffusion equation

Let $\tau > 0$ be a fixed real number. Choose $\mathcal{X} := C([-\tau, 0], \mathbb{R}), \mathcal{Y} := C(\mathbb{R}, \mathbb{R})$ and $\mathcal{C} := C([-\tau, 0], \mathcal{Y})$. We equip $\mathcal{X}$ with the maximum norm, $\mathcal{Y}$ and $\mathcal{C}$ with the similar norms as in (2.1). Define $\mathcal{Y}_+ := C(\mathbb{R}, \mathbb{R}_+)$. Let $d$ be the metric in $\mathcal{C}(\mathcal{Y})$ induced by the norm. We are interested in bistable traveling waves of the following nonlocal and time-delayed reaction-diffusion equation:

\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} + f(u_t)(x), \quad t > 0, x \in \mathbb{R} \\
u_0 &= \phi \in \mathcal{C}, \quad \theta \in [-\tau, 0],
\end{aligned}
\]  

(6.29)

where $f : \mathcal{C} \to \mathcal{Y}$ is Lipschitz continuous and for each $t \geq 0$, $u_t \in \mathcal{C}$ is defined by

\[
u_t(\theta, x) := u(t + \theta, x), \quad \forall \theta \in [-\tau, 0], x \in \mathbb{R}.
\]

If the functional $f$ takes the form $f(\phi)(x) = F(\phi(0, x), \phi(-\tau, x))$, then (6.29) becomes a local and time-delayed reaction-diffusion equation:

\[
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + F(u(t,x), u(t - \tau, x)).
\]  

(6.30)

The bistable traveling waves of (6.30) were studied in [31]. If $f(\phi)(x) = -d\phi(0, x) + \int_{\mathbb{R}} b(\phi(-\tau, y))k(x - y)dy$, then (6.29) becomes a nonlocal and time-delayed reaction-diffusion equation:

\[
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - du(t,x) + \int_{\mathbb{R}} b(u(t - \tau, y))k(x - y)dy.
\]  

(6.31)

The existence, uniqueness and stability of bistable waves of (6.31) were established in [29].

Note that $\mathbb{R}$ can be regarded as a subspace of $\mathcal{X}$, and the latter can also be regarded as a subspace of $\mathcal{C}$. Define $\bar{f} : \mathcal{X} \to \mathbb{R}$ by $\bar{f}(\varphi) = f(\varphi)$ and $\hat{f} : \mathbb{R} \to \mathbb{R}$ by $\hat{f}(\xi) = f(\xi)$. In order to obtain the existence of bistable waves for system (6.29), we impose the following assumptions on the functional $f$:

(E1) $0 < \alpha < \beta$ are three equilibria and there are no other equilibria between 0 and $\beta$.

(E2) The functional $f : \mathcal{C}_\beta \to \mathcal{Y}$ is quasi-monotone in the sense that

\[
\lim_{h \to 0^+} \frac{1}{h} d([\phi(0) - \psi(0)] + h[f(\phi) - f(\psi)]; \mathcal{Y}_+) = 0 \quad \text{whenever } \phi \geq \psi \text{ in } \mathcal{C}_\beta.
\]

(E3) Equilibria 0 and $\beta$ are stable, and $\alpha$ is unstable in the sense that $\hat{f}'(0) < 0, \hat{f}'(\alpha) > 0$ and $\hat{f}'(\beta) < 0$. 

42
By assumptions (E2) and (E4), it follows that system (6.33) admits a principle eigenvalue $\bar{\lambda}$, where $\eta(\varphi)$ is a positive Borel measure on $[-\tau, 0]$ and $\eta(\varphi)([-\tau, -\tau + \epsilon]) > 0$ for all small $\epsilon > 0$.

(E5) For any small number $\epsilon > 0$, there exists a number $\delta \in (0, \beta)$ and a linear operator $L_\epsilon : C_\beta \to \mathcal{Y}$ such that $L_\epsilon \phi \to \text{D}f(\alpha)\phi, \forall \phi \in C_\beta$, as $\epsilon \to 0$ and that $f(\alpha + \phi) \geq L_\epsilon(\phi)$, and $f(\alpha - \phi) \leq -L_\epsilon(\phi)$, $\forall \phi \in C_\delta$.

Using the solution maps $\{T(t)\}_{t \geq 0}$ generated by the heat equation \( \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} \), we write system (6.29) as the integral form

\[
\begin{aligned}
&u(t, \cdot) = T(t)\phi(0, \cdot) + \int_0^t T(t-r) f(u_r(\cdot, \cdot))dr, \quad t > 0 \\
&u(\theta, \cdot) = \phi(\theta, \cdot), \quad \theta \in [-\tau, 0].
\end{aligned}
\] (6.32)

Note that traveling waves of system (6.32) are those of system (6.29). It then remains to show (6.32) admits a bistable traveling wave.

**Theorem 6.4.** Under assumption (E1)-(E5), system (6.29) admits a nondecreasing traveling wave $\phi(x + ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = \beta$.

**Proof.** From assumptions (E1)-(E2), we see that system (6.32) generates a monotone semiflow $\{Q_t\}_{t \geq 0}$ on $C_\beta$ with

\[
Q_t[\phi](\theta, x) = u_t(\theta, x; \phi), \quad \forall (\theta, x) \in [-\tau, 0] \times \mathbb{R},
\]

where $u(t, x; \phi)$ is the unique solution of system (6.32) satisfying $u_0(\cdot, \cdot; \phi) = \phi \in C_\beta$. By similar arguments as in section 5, it follows that $Q_t$ satisfies (A4) if $t > \tau$ and (A4') if $t \in (0, \tau]$.

Let $Q_\tau$ be the restriction of $Q_t$ on $X_\beta$. Denote the derivative $DQ_t[0]$ of $Q_t$ by $\bar{M}_{0,t}$, then $\bar{M}_{0,t}$ is the solution map of the following functional equation:

\[
\frac{du}{dt} = \bar{L}(0)u_t = a(0)u(t) + L_1(0)u_t.
\] (6.33)

By assumptions (E2) and (E4), it follows that system (6.33) admits a principle eigenvalue $s_0$ with an associated eigenfunctions $v_0 := e^{s_0t}$ (see [34, Theorem 5.5.1]). More precisely, $\bar{M}_{0,t}[v_0] = e^{s_0 t} v_0$. Furthermore, [34, Corollary 5.5.2] implies that $s_0 < 0$ since $f'(0) < 0$. Therefore, there exists $\delta_0(t) > 0$ such that

\[
\begin{aligned}
Q_t[\delta v_0] &= \bar{Q}_t[0] + D\bar{Q}_t[0][\delta v_0] + o(\delta^2) \\
&= \delta \bar{M}_{0,t}[v_0] + o(\delta^2) \\
&= \delta e^{s_0 t} v_0 + o(\delta^2) \\
&= \delta v_0 + \delta[e^{s_0 t} - 1]v_0 + o(\delta^2) \ll \delta v_0, \quad \forall \delta \in (0, \delta_0(t)].
\end{aligned}
\]

43
Similarly, there exists $\delta_\alpha(t), v_\alpha$ and $\delta_\beta(t), v_\beta$ such that

$$
\bar{Q}_t[\beta - \delta v_\beta] \gg \beta - \delta v_\beta, \quad \forall \delta \in (0, \delta_\beta(t)]
$$

and

$$
\bar{Q}_t[\alpha + \delta v_\alpha] \gg \alpha + \delta v_\alpha, \quad \bar{Q}_t[\alpha - \delta v_\alpha] \ll \alpha - \delta v_\alpha, \quad \forall \delta \in (0, \delta_\alpha(t)].
$$

Till now, it remains to show (A6) is also true. Indeed, we see from [25, Theorem 2.17] that the solution semiflows $\{Q_t\}_{t \geq 0}$ restricted on $[0, \alpha]_C$ and $[\alpha, \beta]_C$ admit a spreading speed $c^*(0, \alpha)$ and $c^*(\alpha, \beta)$, respectively. Let $M_t^l$ be the solution maps of the linear system

$$
\begin{cases}
  u(t, \cdot) = T(t)\phi(0, \cdot) + \int_0^t T(t - r)L_\epsilon(u_r(\cdot, \cdot))dr, & t > 0 \\
  u(\theta, \cdot) = \phi(\theta, \cdot), & \theta \in [-\tau, 0].
\end{cases}
$$

Then assumption (E5) guarantees that $Q_t[\phi] \geq M_t^l[\phi]$ when $\phi \in C_3$, where $\delta = \delta(\epsilon)$ is defined in (E5). Therefore, we see from [25, Theorem 3.10] that $c^*(0, \alpha) \geq \bar{c}$ and $c^*(\alpha, \beta) \geq \bar{c}$, where $\bar{c}$ is positive number determined by the linearized system of (6.29) at $u \equiv \alpha$, and hence, (A6) holds. Consequently, Theorem 5.1 completes the proof. \(\square\)

**Remark 6.2.** At this moment we are unable to present a general result on the uniqueness and global attractivity of bistable waves under the current abstract setting. However, one may use the convergence theorem for monotone semiflows (see [47, Theorem 2.4]) and the similar arguments as in the proof of [47, Theorem 10.2.1] and [45, Theorem 3.1] to obtain the global attractivity (and hence, uniqueness) of bistable waves for four examples in this section.

7 Appendix

In this appendix, we present certain properties of Banach lattices and countable subsets in $\mathbb{R}$, and some convergence results for sequences of monotone functions, including an abstract variant of Helly’s theorem.

**Proposition 7.1.** A Banach lattice $X$ has the following properties:

1. For any $u, v \in X$ with $v \in X^+$, if $-v \leq u \leq v$, then $\|u\|_X \leq \|v\|_X$.
2. If $u_k \to u$ and $v_k \to v$ in $X$ with $u_k \geq v_k$, then $u \geq v$.

**Proposition 7.2.** The space $C$ has the following properties:

1. Let $\phi$ be a monotone function in $C$. If $x_k \in H$ nondecreasingly tends to $x \in H \cup \{+\infty\}$ and $\lim_{k \to \infty} \phi(x_k) = u \in X$, then $\lim_{y \to x} \phi(y) = u$. The similar result holds if $x_k$ nonincreasingly tends to $x \in H \cup \{-\infty\}$.

44
(2) Assume that \( h, h_k : \mathcal{H} \to \mathcal{H} \) are continuous and \( \phi_k \to \phi \) in \( \mathcal{C} \). If \( h_k(x) \to h(x) \) uniformly for \( x \) in any bounded subset of \( \mathcal{H} \), then \( \phi_k \circ h_k \to \phi \circ h \) in \( \mathcal{C} \).

Propositions 7.1 and 7.2 can easily be proved. Here we omit the proofs.

**Proposition 7.3.** Assume that \( D \) is a countable subset of \( \mathbb{R} \). Then for any \( c \in \mathbb{R} \), there exists another countable subset \( A \) of \( \mathbb{R} \) such that \( (\mathbb{R} \setminus A) + cm \subset \mathbb{R} \setminus D, \forall m \in \mathbb{Z}^+ \).

**Proof.** It suffices to show the set \( A := \{ x \in \mathbb{R} : \text{there exists } m \text{ such that } x + cm \in D \} \) is countable. Indeed, we have \( A = \bigcup_{m=1}^{\infty} (D - cm) \). This implies \( A \) is countable. □

**Proposition 7.4.** For any countable subset \( \Gamma_1 \) of \( \mathbb{R} \), there exists another countable \( \Gamma_2 \) such that it is dense in \( \mathbb{R} \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \).

**Proof.** Since \( \Gamma_1 \) is countable and \( \bigcup_{\alpha \in \mathbb{R}} (\alpha + \mathbb{Q}) = \mathbb{R} \), there must exist a sequence \( \alpha_n \) such that \( \Gamma_1 \subset \bigcup_{n=1}^{\infty} (\alpha_n + \mathbb{Q}) \). Note that \( \bigcup_{n=1}^{\infty} (\alpha_n + \mathbb{Q}) \) is countable. Then we see that there exists \( \alpha \in \mathbb{R} \) such that \( \alpha \notin \bigcup_{n=1}^{\infty} (\alpha_n + \mathbb{Q}) \). This means that \( \alpha - \alpha_n \notin \mathbb{Q}, \forall n \geq 1 \), and hence, \( (\alpha + \mathbb{Q}) \cap (\alpha_n + \mathbb{Q}) = \emptyset, \forall n \geq 1 \). Define \( \Gamma_2 := \alpha + \mathbb{Q} \). We then see that \( \Gamma_2 \) is countable and dense in \( \mathbb{R} \), and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). □

**Proposition 7.5.** Assume that \( f, f_n : \mathbb{R} \to \mathcal{X} \) are nondecreasing and the set \( D \) is dense in \( \mathbb{R} \). If \( s_n \to 0 \), \( f(s) \) is continuous on \( D \) and \( f_n(s) \to f(s) \) for every \( s \in D \), then \( f_n(s + s_n) \to f(s) \) for every \( s \in D \).

**Proof.** Let \( s \in D \) be fixed. For any \( \delta > 0 \), since \( D - s \) is dense in \( \mathbb{R} \), we can choose \( \delta_+ \in (D - s) \cap (0, \delta) \) and \( \delta_- \in (D - s) \cap (-\delta, 0) \). Clearly, \( s + \delta_+ \in D \), \( s + \delta_- \in D \). Thus, there exists an integer \( N_\delta \) such that \( s + s_n \in (s + \delta_-, s + \delta_+), \forall n \geq N_\delta \). Since

\[
|f_n(s + s_n) - f_n(s)|_\mathcal{X} \leq |f_n(s + \delta_+) - f_n(s + \delta_-)|_\mathcal{X}, \forall n \geq N_\delta,
\]

we have

\[
\|f_n(s + s_n) - f_n(s)\|_\mathcal{X} \leq \|f_n(s + \delta_+) - f_n(s + \delta_-)\|_\mathcal{X}, \forall n \geq N_\delta.
\]

It then follows that

\[
\|f_n(s + s_n) - f(s)\|_\mathcal{X} \leq \|f_n(s + s_n) - f_n(s)\|_\mathcal{X} + \|f_n(s) - f(s)\|_\mathcal{X} \\
\leq \|f_n(s + \delta_+) - f_n(s + \delta_-)\|_\mathcal{X} + \|f_n(s) - f(s)\|_\mathcal{X} \\
\leq \|f(s + \delta_+) - f_n(s + \delta_-)\|_\mathcal{X} + \|f(s + \delta_-) - f(s + \delta_+)\|_\mathcal{X} \\
+ \|f_n(s + \delta_-) - f(s + \delta_-)\|_\mathcal{X} + \|f_n(s) - f(s)\|_\mathcal{X}
\]

for all \( n \geq N_\delta \). Now the pointwise convergence of \( f_n \) in \( D \) and the continuity of \( f \) on \( D \) complete the proof. □

To end this section, we prove a convergence theorem for sequences of monotone functions from \( \mathbb{R} \) to the special Banach lattice \( C(M, \mathbb{R}^d) \) defined in section 2, which is a variant of Helly’s theorem \([17, P.165]\) for sequences of monotone functions from \( \mathbb{R} \) to \( \mathbb{R} \).
Theorem 7.1. Let $D$ be a dense subset of $\mathbb{R}$ and $f_n, n \geq 1$ be a sequence of nondecreasing functions from $\mathbb{R}$ to the Banach lattice $\mathcal{X} := C(M, \mathbb{R}^d)$. Assume that

(i) for any $s \in D$, $f_n(s)$ is convergent in $\mathcal{X}$.

(ii) there exists a countable set $D_1 \subset \mathbb{R}$ such that for any $s \in \mathbb{R} \setminus D_1$, the limits

$$\lim_{m \to \infty} \lim_{n \to \infty} f_n(s_{\pm,m})$$

exist in $\mathcal{X}$, where $s_{-m} \uparrow s$ and $s_{+m} \downarrow s$ with $s_{\pm,m} \in D$.

Then $f_n(s)$ is convergent in $\mathcal{X}$ almost for all $s \in \mathbb{R}$.

Proof. Due to assumption (ii), we can define $f : \mathbb{R} \to \mathcal{X}$ by

$$f(s) := \begin{cases} \lim_{x \uparrow s} \lim_{n \to \infty} f_n(x), & s \in \mathbb{R} \setminus D_1 \\ \text{any value}, & s \in D_1. \end{cases} \quad (7.1)$$

We first show that the discontinuous points of $f$ are at most countable. Define the sets

$$A := \{s \in \mathbb{R} \setminus D_1 : f(s^-), f(s^+) \text{ both exits}\}$$

and

$$B := \{s \in A : f(s^-) < f(s^+)\}.$$

For any $s \in B$, there exists $x_1 \in M$ and $1 \leq i \leq d$ such that $(f(s^-)(x_1))_i < (f(s^+)(x_1))_i$. Recall that $M$ is compact, so there is a countable dense subset $M_1$. It then follows that there must be $x_2 \in M_1$ such that $(f(s^-)(x_2))_i < (f(s^+)(x_2))_i$. Therefore,

$$B = \bigcup_{i=1}^m \bigcup_{x \in M_1} \{s \in A : (f(s^-)(x))_i < (f(s^+)(x))_i\}.$$

Since for each fixed $i$ and $x$, $(f(s)(x))_i$ is a nondecreasing function from $\mathbb{R} \setminus D_1$ to $\mathbb{R}$, we know that $\{s \in A : (f(s^-)(x))_i < (f(s^+)(x))_i\}$ is at most countable, and hence so is the set $B$.

Now we can prove the conclusion. Assume that $s \in \mathbb{R}$ is a continuous point of $f$. For any $\delta > 0$, choose $\delta_- \in D \cap (s-\delta, s)$ and $\delta_+ \in D \cap (s, s+\delta)$. Then we have

$$f_n(\delta_-) - f_n(\delta_+) \leq f_n(s) - f_n(\delta_-) \leq f_n(\delta_+) - f_n(\delta_-), \quad \forall n \geq 1,$$

which, together with Proposition 7.1(2), implies that

$$\|f_n(s) - f_n(\delta_-)\|_X \leq \|f_n(\delta_+) - f_n(\delta_-)\|_X, \quad \forall n \geq 1. \quad (7.2)$$

In the other hand, by (7.2) and the triangular inequality we have

$$\|f_n(s) - f(s)\|_X \leq \|f_n(s) - f_n(\delta_-)\|_X + \|f_n(\delta_-) - f(s)\|_X \leq \|f_n(\delta_+) - f_n(\delta_-)\|_X + \|f_n(\delta_-) - f(s)\|_X \leq \|f_n(\delta_+) - f(\delta_-)\|_X + \|f(\delta_-) - f(s)\|_X \leq \|f(\delta_-) - f(s)\|_X, \quad \forall n \geq 1.$$

Now the pointwise convergence of $f_n$ in $D$ and the continuity of $f$ at $s$ complete the proof. □
Acknowledgment. J. Fang’s research is supported in part by the NSF of China (grant 10771045) and the Collaborative Research Groups Program at HIT. X.-Q. Zhao’s research is supported in part by the NSERC of Canada and the MITACS of Canada.

References


