Consensus of multiple second-order vehicles with a time-varying reference signal under directed topology

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**Abstract**

This paper investigates two kinds of different consensus strategies for multi-vehicle systems with a time-varying reference velocity under directed communication topology, where the systems are modeled by double-integrator dynamics. For the fixed communication topology case, we provide a necessary and sufficient condition for all the vehicles with reference velocity to reach consensus by the use of a new graphic methodology. We then extend this method to deal with the general case, that is, both the communication topologies and weighting factors are dynamically changing. In particular, it is shown that all the vehicles can reach consensus even though the dynamically changing interaction topology may not have a globally reachable node.

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1. Introduction

Information consensus has recently received much attention in the system and control community. A critical problem for information consensus is how to design an appropriate update law such that the information states of all the vehicles can reach a common value in the presence of limited and unreliable information and dynamically changing communication topologies. Most work on consensus focuses on algorithms taking the form of single-integrator dynamics (Cao, Morse, & Anderson, 2008; Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-saber & Murray, 2004; Ren & Beard, 2005; Vicsek, Czirok, Jacob, Cohen, & Schochet, 1995; Xiao & Wang, 2006), please refer to Olfati-saber, Fax, and Murray (2007) and references therein for details. Extensions of consensus algorithms to double-integrator dynamics are investigated in Olfati-saber and Murray (2003), Ren and Atkins (2007), Ren (2008), Tanner, Jadbabaie, and Pappas (2003a), Tanner, Jadbabaie, and Pappas (2003b) and Xie and Wang (2007), to name a few.

For most of the consensus strategies studied in the existing literature, the final consensus value to be reached is inherently a constant, which usually depends on the initial values of the vehicles. This may not be appropriate in practical applications when each vehicle's information state evolves over time. Furthermore, most of the existing consensus strategies guarantee that the information states converge to a common value but do not provide the specific value, especially when the interaction topology is dynamically changing. In practical applications, this may not be appropriate either.

With this background, the objective of this paper is to investigate consensus strategies for multiple vehicles with a time-varying group reference signal in the context of directed communication topology. A series of consensus strategies to make a group of vehicles with a reference state reach consensus are proposed in Ren (2007b), where each vehicle is modeled by a single-integrator dynamics. This paper focuses on consensus strategy for double-integrator dynamics, which is more challenging than those for single-integrator dynamics. Two different consensus strategies, the variants of which are investigated in Ren (2007a, 2008) under fixed topology, are systematically revisited in this paper. By employing totally different methods, the consensus analysis is performed from a new viewpoint, in which the conditions imposed to realize the final consensus take simpler forms than the corresponding ones in Ren (2007a, 2008). Moreover, the convergence...
analysis for the case with fixed topology is extended to deal with the general case where both the communication topology and the weighting factors are dynamically changing. In particular, we show that consensus can be reached asymptotically for the vehicles with a reference signal if the union of the collection of communication topologies across some time intervals has a globally reachable node frequently enough. It is worthwhile mentioning that when the reference velocity is zero, the consensus strategy without the measurement of the relative velocities between neighboring vehicles is also investigated in Xie and Wang (2007), however, only in the context of undirected fixed/dynamic communication topology. In Xie and Wang (2007), the consensus analysis relies heavily upon the fact that the Laplacian matrix associated with the communication topology is symmetric, which thereby cannot be extended to deal with the case with directed topology. To the best of our knowledge, this is one of the first independent attempts that address the consensus problems for second-order continuous multi-vehicle systems with both dynamically changing weighting factors and directed communication topologies which do not necessarily have a globally reachable node.

An additional contribution of our paper is that we use a totally different graphic methodology, which may be of independent interest in graph theory, in dealing with the above mentioned problem. By the use of this graphic method, we are able to extend most of the existing results for vehicles with single-integrator dynamics to the double-integrator case, which is certainly nontrivial.

In addition to the multi-agent consensus algorithms with a reference signal, various flocking and swarm tracking algorithms are also studied when there exists a virtual leader. The objective of flocking or swarm tracking with a virtual leader is that a group of agents track the leader while maintaining a desired configuration among all the agents (Cao & Ren, in press; Olfati-saber, 2006; Su, Wang, & Lin, 2009). In Olfati-saber (2006), the flocking algorithms are studied based on the assumptions that all the agents should be informed by the virtual leader and also the virtual leader travels at a constant velocity. Su et al. (2009) extends the result in Olfati-saber (2006) from two aspects, i.e., the case that the virtual leader travels at a constant velocity and the case that the virtual leader travels at a time-varying velocity. More recently, Cao and Ren (in press) systematically studied the distributed consensus and swarm tracking problems for agents under both fixed and switching network topologies based on the assumption that only a subset of the agents have access to the leader and only partial measurements of the states of the virtual leader and the followers are available. However, the study in all the literature mentioned above is confined in the context of undirected communication topology. In particular, the communication topology should be kept connected when the switching case is considered in Cao and Ren (in press). Clearly, the analysis method used in the literature mentioned above does not work for the directed case. To the best of our knowledge, there has not been any work being reported in the literature concerning the flocking or swarm tracking problems in the context of directed communication topology where the communication topology can be relaxed to only have a directed spanning tree or a globally reachable node. With no need to maintain the desired geometrical configuration among all the vehicles, this paper investigates the consensus problem with a reference signal under fixed/dynamically changing and directed communication topologies. It should be noted that the case with directed topology is much different from the undirected case.

The rest of the paper is organized as follows. Some concepts about matrix theory and graph theory are described in Section 2 while the problems to be investigated are formulated in Section 3. In Section 4, necessary and/or sufficient conditions under which all the vehicles with a time-varying reference signal can reach consensus are systematically investigated for both the cases with fixed and dynamically changing interaction topologies. Finally, some concluding remarks are drawn in Section 5.

2. Background and preliminaries

Let \( \mathbf{1}_n \in \mathbb{R}^n \) denote the column vector of all ones and \( \mathbf{l}_m \) denote the \( m \times m \) identity matrix, \( n, m \in \mathbb{Z}_+ \), where \( \mathbb{Z}_+ \) is the set of positive integers. A matrix \( \mathbf{M} \in \mathbb{R}^{n \times n} \) is nonnegative, denoted as \( \mathbf{M} \geq 0 \), if all its entries are nonnegative. Furthermore, \( \mathbf{M} \) is said to be a stochastic matrix if all its row sums are \(+1\). Let \( \mathbf{N} \in \mathbb{R}^{n \times n} \). We write \( \mathbf{M} \geq \mathbf{N} \) if \( \mathbf{M} - \mathbf{N} \geq 0 \). Throughout the paper, we let \( \mathbf{L} = \mathbf{M}_1 = \mathbf{M}_2 \mathbf{M}_1 \cdots \mathbf{M}_t \) denote the left product of matrices \( \mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_t \). A stochastic matrix \( \mathbf{M} \) is called ergodic (or indecomposable and aperiodic) if there exists a column vector \( \mathbf{f} \in \mathbb{R}^n \) such that \( \lim_{k \to \infty} \mathbf{M}^k = \mathbf{f} \mathbf{1}^t \).

A digraph (or directed graph) will be used to model communication topologies among vehicles. Let \( \mathbf{g} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) be a weighted digraph of order \( n \) with a finite nonempty set of nodes \( \mathcal{V} = \{1, 2, \ldots, n\} \), a set of edges \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \), and a weighted adjacency matrix \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n} \) with nonnegative adjacency elements \( a_{ij} \). An edge of \( \mathbf{g} \) is denoted by \((i, j)\). The adjacency elements associated with the edges are positive, i.e., \( a_{ij} > 0 \). Moreover, we assume \( a_{ii} = 0 \) for all \( i \in \mathcal{V} \). The set of neighbors of node \( i \) is denoted by \( \mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\} \).

The Laplacian matrix \( \mathbf{L} = [l_{ij}] = \mathcal{L}(\mathbf{g}) \) of weighted digraph \( \mathbf{g} \) is defined by \( l_{ij} = -a_{ij}, i \neq j \), and \( l_{ii} = \sum_{k=1,k\neq i}^{n} a_{ik} \).

For the edge \((i, j)\), \( i \) is the parent node and \( j \) is the child node. A directed path is a sequence of edges in a directed graph of the form \((i_1, i_2), (i_2, i_3), \ldots\). A digraph \( \mathbf{g} \) is called strongly connected if between any pair of distinct nodes \( i, j \), \((i, j) \) is a directed path from \( i \) to node \( j \). Note that we will denote \((i \rightarrow j)\) as a directed path from node \( i \) to node \( j \) in the sequel. A node of a digraph is globally reachable if it can be reached from any other node by a directed path. Clearly, a digraph is strongly connected if and only if every node is globally reachable. A digraph has a spanning tree if there exists at least one node, called the root node, having a directed path to all the other nodes.

Given a nonnegative matrix \( \mathbf{S} = [s_{ij}] \in \mathbb{R}^{n \times n} \), the weighted digraph of \( \mathbf{S} \), denoted by \( \Gamma(\mathbf{S}) \), is the digraph with node set \( \mathcal{V} = \{1, 2, \ldots, n\} \) such that there is an edge in \( \Gamma(\mathbf{S}) \) from \( i \) to \( j \) if and only if \( s_{ij} > 0 \), and moreover, the entries of the adjacency matrix \( \mathcal{A} = [a_{ij}] \) of this digraph satisfy \( a_{ij} = s_{ij} \). It is easy to verify that if \( S_1 \geq \eta S_2 \) for some \( \eta > 0 \) and \( \Gamma(S_1) \) has a globally reachable node, then the digraph \( \Gamma(S_t) \) also has a globally reachable node.

3. Consensus strategy

Suppose that the network system under consideration consists of \( n \) vehicles. Each vehicle is regarded as a node in a directed graph \( \mathbf{g} \). Each edge \((i, j) \in \mathcal{E}(\mathbf{g}(t))\) corresponds to an unidirectional information flow from vehicle \( j \) to vehicle \( i \) at time \( t \), which means that vehicle \( i \) can receive or obtain information from vehicle \( j \) at time \( t, i, j \in \mathcal{V} \).

Suppose that the dynamics of each vehicle is modeled by double-integrator dynamics

\[
\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \tag{1}
\]

where \( x_i \in \mathbb{R}^3 \) and \( v_i \in \mathbb{R}^3 \) are the position and velocity of vehicle \( i \), respectively, and \( u_i \in \mathbb{R}^3 \) is the control input of vehicle \( i \). In practical applications, it may be desirable to guarantee that the positions of all the vehicles reach consensus and the velocities of all the vehicles approach a reference. Toward this end, we consider the following consensus strategy as investigated in Ren (2008) for multi-vehicle systems with a time-varying reference:

\[
u_i = -2\gamma_x (v_i - v_j) + v_j + \sum_{j \in \mathcal{N}(i)} a_{ij}(t)(x_j - x_i), \tag{2}
\]
where $\gamma > 0$ denotes the velocity damping gain and $N_i(t)$ denotes the set of neighbors of vehicle $i$ at time $t$. $a_q(t) \geq 0$ is a weighting factor chosen from any finite set $\bar{a}$, and $a_q(t) > 0$ if vehicle $i$ can receive information from vehicle $j$ at time $t$ and $a_q(t) = 0$ otherwise. Here, the weighting factors $a_q(t)$ are allowed to be dynamically changing to represent possible time-varying relative confidence of each vehicle's information state or relative reliability of different information exchange links between vehicles.

Note that in the consensus strategy given above, there is no measurement of the relative velocity information between neighboring vehicles. If each vehicle can also get the velocity information from its neighboring vehicles, the following consensus strategy is considered in Ren (2007a):

$$u_i = -\gamma (v_i - v_\ell) + \dot{v}_\ell + \sum_{j \in N(i)} a_q(t) [(x_j - x_i) + \gamma_1 (v_j - v_i)], \quad (3)$$

where $\gamma > 0$ denotes the coupling strength of relative velocities between neighboring vehicles and $\gamma > 0$ denotes the velocity damping gain.

Throughout the paper, we say that the consensus strategy $u_i(t)$ asymptotically makes all the vehicles with reference signal reach consensus, if for all $x_i(0)$ and $v_i(0)$, the states of the vehicles satisfy $\lim_{t \to \infty} (x_i(t) - x_\ell(t)) = 0$ and $\lim_{t \to \infty} (v_i(t) - v_\ell(t)) = 0, \forall i, j \in \mathcal{V}$, where $v_\ell(t) \in \mathbb{R}^q$ is a time-varying reference for $v_i(t)$.

### 4. Main results

#### 4.1. The case with fixed topology and constant communication weighting factors

In this subsection, we take a look at under what conditions the consensus can be achieved for networks with fixed communication topology and constant weighting factors by applying strategies (2) and (3), respectively. To this end, we need to present some useful lemmas first.

The following lemma shows that the digraphs constructed in a certain way based on an original digraph will share the property of having a globally reachable node with the original digraph.

**Lemma 1.** Let $\mathcal{G}(\mathcal{V}, \varepsilon)$ be any given digraph; and let $\mathcal{G}'(\mathcal{V}', \varepsilon')$ be a graph with only $n$ nodes and no edges, that is, $\mathcal{V}' = \{1', 2', \ldots, n'\}$ and $\varepsilon' = \varnothing$. Assume that $\mathcal{G}'(\mathcal{V}', \varepsilon')$ is a digraph constructed from $\mathcal{G}(\mathcal{V}, \varepsilon)$ and $\mathcal{G}'(\mathcal{V}', \varepsilon')$ according to the following rules:

(A1) The node set $\mathcal{V}' = \mathcal{V} \cup \mathcal{V}' = \{1, 1', 2, 2', \ldots, n, n'\}$;

(B1) There is no edge between node $i$ and node $j$ or node $i'$ and node $j'$ for any $i, j \in \mathcal{V}$ and $i', j' \in \mathcal{V}'$, $i \neq j$, $i' \neq j'$;

(C1) Edge $(i, i') \in \varepsilon'$ if and only if edge $(i, j) \in \varepsilon$, for any $i, j \in \mathcal{V}$, $i \neq j$.

Then, digraph $\mathcal{G}$ has a globally reachable node if and only if digraph $\mathcal{G}'$ has a globally reachable node.

**Proof.** We first prove the first statement.

**Necessity:** Assume that $i$ is a globally reachable node in digraph $\mathcal{G}$. In fact, we can prove that node $i'$ is a globally reachable node in digraph $\mathcal{G}$. For this purpose, it suffices to prove there are directed paths from node $j$ to node $i'$ and from node $j'$ to node $i$, respectively for any $j \in \mathcal{V}, j \neq i$. Since $i$ is the globally reachable node in $\mathcal{G}$, there is a directed path, say $(j, j_1, j_2, \ldots, j_{m-1}, j_m, j_m, i)$, from node $j$ to node $i$. It then follows from rule (D1) that there is a directed path $(j, j_1, j_2, j_3, \ldots, j_m, j_m, i, i')$ from node $j$ to node $i'$ in digraph $\mathcal{G}$. Clearly, $(j, j_1, j_2, j_3, \ldots, j_m, j_m, i, i')$ is a directed path in digraph $\mathcal{G}$.* Similarly, if $i$ is a globally reachable node in $\mathcal{G}$, then $i'$ is also a globally reachable node in $\mathcal{G}$.

This completes the proof of the first statement.

For the second statement, here we only need to prove the sufficiency since the sufficiency part of this statement can be proved by using completely the same technique as above. It follows from rule (C2) that digraph $\mathcal{G}$ is isomorphic to the subgraph of $\mathcal{G}$ which is induced by the node set $\mathcal{V}'$. This, combined with rule (B2), immediately completes the proof for the necessity. \(\square\)

The following lemma is a summary of the work of Moreau (2005), Olaffi-saber and Murray (2004) and Ren and Beard (2005).

**Lemma 2.** Suppose that $\xi = [\xi_1^T, \ldots, \xi_n^T]^T$ with $\xi_i \in \mathbb{R}^q$ and $L \in \mathbb{R}^{m \times m}$ is the corresponding Laplacian matrix of digraph $\mathcal{G}$ of order $m$, $m, q \in \mathbb{Z}$. Then, consensus is reached asymptotically for system $\dot{\xi} = -L \otimes I_q \xi$, i.e.,

$$\lim_{t \to \infty} (\xi_i(t) - \xi_j(t)) = 0, \quad i, j \in \{1, 2, \ldots, m\}$$

if and only if digraph $\mathcal{G}$ has a globally reachable node.

**Remark 1.** If the adjacency matrix of a digraph is defined as $(i, j) \in \varepsilon \Leftrightarrow a_q > 0$, then the above lemma can be rephrased as that consensus is reached asymptotically for system $\dot{\xi} = -(L \otimes I_q)\xi$ if and only if digraph $\mathcal{G}$ has a spanning tree. Note that in this context, $(i, j) \in \varepsilon$ implies that vehicle $i$ can receive information from vehicle $j$.

Now we proceed to present our first main result. Note that one key analysis technique used in the proof of the following main result is that of converting the system with double-integrator dynamics to an augmented system with single-integrator dynamics, which paves the way for employing the existing results on consensus in single-integrator systems to complete the consensus analysis.

**Theorem 1.** Consider a directed network of vehicles with fixed communication topology and constant weighting factors. Assume that the velocity damping gain satisfies $\gamma \geq \sqrt{\max_{k=1, k \neq l} a_q(l)}$. Then, the update strategy (2) asymptotically makes all the vehicles with a time-varying reference signal $v_i(t)$ reach consensus if and only if the communication topology $\mathcal{G}$ has a globally reachable node.
Proof. Let $v_i^* = v_i - v_r$ and $x_i^* = x_i - x_r$, where $x_r = \int_0^t v_r(t) \, dt$. From (1) and (2), we have

$$\begin{align*}
\dot{x}_i &= x_i - (v_i - v_r), \\
\dot{v}_i &= -2\gamma (v_i - v_r) + \sum_{j \in N_i} a_{ij}(t) \left[ (x_j - x_i) - (v_j - v_i) \right],
\end{align*}$$

which is equivalent to

$$\begin{align*}
\dot{x}_i^* &= v_i^*, \\
\dot{v}_i^* &= -2\gamma v_i^* + \sum_{j \in N_i} a_{ij}(t) (x_j^* - x_i^*). 
\end{align*}$$

(4)

Let $\xi^* = [(x_1^*)^T, (x_2^*)^T, \ldots, (x_{2n-1}^*)^T, (x_{2n}^*)^T]^T$

$$\xi^* = \left[ (x_1^*)^T, (y_1^*)^T, \ldots, (x_i^*)^T, (y_i^*)^T \right]^T, \quad y_i^* = \frac{1}{\gamma} v_i^* + x_i^*. $$

Then, applying the update strategy (2), Eq. (4) can be rewritten in matrix form as

$$\dot{\xi}^*(t) = \left[ [I_{2n} \otimes A - L(t) \otimes B] \otimes I_4 \right] \xi^*(t),$$

where

$$A = \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. $$

When the communication topology is fixed, the network dynamics is

$$\dot{\xi}^*(t) = - \left[ (L \otimes B - I_n \otimes A) \otimes I_4 \right] \xi^*(t).$$

(5)

where $L$ is the Laplacian matrix associated with communication topology $G$ (also known as digraph $G$ without confusion).

Since $\gamma \geq \max_{1 \leq i, j \leq k} a_{ij}$, all the off-diagonal elements of matrix $I_n \otimes A - L \otimes B$ are nonnegative. Moreover, all the rows sums of matrix $I_n \otimes A - L \otimes B$ are zero, thus matrix $L \otimes B - I_n \otimes A$ can be considered as the Laplacian matrix of a digraph $G^*$ with 2n nodes, say, 1, 2, ..., n, n + 1, ..., 2n. Let matrices $A = [a_{ij}] \in \mathbb{R}^{2n \times 2n}$ and $A^* = [b_{ij}] \in \mathbb{R}^{2n \times 2n}$ denote the adjacency matrices of digraph $G$ and $G^*$, respectively. It follows from the definition of the Laplacian matrix that $A^* = I_n \otimes \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} - L \otimes B$.

Sufficiency: Firstly, we will prove that if digraph $G$ has a globally reachable node, then digraph $G^*$ also has a globally reachable node. Clearly, $b_{2i-1,2i} = \gamma > 0$ for any $i \in V$ which implies that there is an edge from node $2i - 1$ to node $2i$ for any $i \in V$ as shown in Fig. 1. Since $b_{2i-1,2i} = \gamma$, there may exist an edge from node $2i$ to node $2i - 1$, $i \in V$. Recall that $A$ is the adjacency matrix of graph $G$, then checking the following submatrix of $A^*$ we have

$$\begin{bmatrix} b_{21} & b_{21} & \ldots & b_{2,2n-1} \\ b_{41} & b_{43} & \ldots & b_{4,2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{2n-1,2n-1} & b_{2n,3} & \ldots & b_{2n,2n-1} \end{bmatrix} = \frac{1}{\gamma} A.$$

Relabel all the nodes as shown in Fig. 1, then we get a new digraph with the node set $V^* = \{1, 2, 3, \ldots, n, n\}$ which is homomorphic to digraph $G^*$. Because this operation does not change the property of having a globally reachable node, we still denote $G^*$ as the newly relabeled digraph. Then, by comparing the new labels of the nodes with the original ones, we can obtain $a_{ij} = \gamma b_{2i-1,2i} > 0$ in digraph $G$ if and only if $b_{ij} > 0$ in digraph $G^*$, and also $b_{2i-1,2i} = \gamma > 0$ implies $b_{ij} = \gamma > 0$ for any $i, j \in V$. On this basis, we find that $G^*$ is precisely the digraph constructed from digraph $G$ and $G^*$ (with node set $\{1, 2', \ldots, n\}$) according to the rules described in the first statement of Lemma 1. It then follows from Lemma 1 that $G^*$ also has a globally reachable node if $G$ has a globally reachable node.

Now consider the network of vehicles with fixed topology $G^* = (V^*, *, A^*)$, where each vehicle is modeled by single-integrator dynamics and the corresponding network dynamics is Eq. (5). According to Lemma 2, we can get $\lim_{t \to \infty} (\xi^*(t) - \xi^*(t)) = 0$, which implies $\lim_{t \to \infty} (x_i^*(t) - x_i^*(t)) = 0$, i.e.,

$$\lim_{t \to \infty} v_i^*(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} (x_i^*(t) - x_i^*(t)) = 0.$$

This in turn implies that the update strategy (2) asymptotically solves the consensus problem for the vehicles with reference implies that

$$\lim_{t \to \infty} v_i^*(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} (x_i^*(t) - x_i^*(t)) = 0.$$
where

\[
A_1 = \begin{bmatrix}
\frac{-1}{\gamma_1} & \frac{1}{\gamma_1} \\
\frac{1}{\gamma_1} & \frac{1}{\gamma_1} \\
\gamma - \frac{1}{\gamma_1} & \frac{1}{\gamma_1} - \gamma
\end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}.
\]

Note that \(A_1\) and \(B_1\) given above are very different from \(A\) and \(B\) given in the proof of Theorem 1. Since \(\gamma \geq \frac{1}{n}\), all the off-diagonal elements of matrix \(I_n \otimes A_1 - L \otimes B_1\) are nonnegative and all the diagonal elements are non-positive. Moreover, all the row sums of matrix \(I_n \otimes A_1 - L \otimes B_1\) are 0, thus matrix \(L \otimes B_1 - I_n \otimes A_1\) can also be considered as the Laplacian matrix of a digraph \(g^*\) with \(2n\) nodes, say, 1, 2, \ldots, \(n\), \(n+1\), \ldots, \(2n\). Let matrix \(A = \{a_{ij}\} \in \mathbb{R}^{n \times n}\) and \(A^* = \{b_{ij}\} \in \mathbb{R}^{2n \times 2n}\) denote the adjacency matrix of graph \(g\) and \(g^*\), respectively. By some manipulation, we can get that

\[
A^* = I_n \otimes \begin{bmatrix}
0 & b_{12} & \cdots & b_{1,2n} \\
0 & b_{2,4} & \cdots & b_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{2n,2} & b_{2n,4} & \cdots & 0
\end{bmatrix} + A \otimes B_1.
\]

**Sufficiency:** Firstly, we will prove that if digraph \(g\) has a globally reachable node, then digraph \(g^*\) also has a globally reachable node. Clearly, \(b_{2i-1,2i} = \frac{1}{\gamma_1} > 0\) and \(b_{2i,2i-1} = \gamma - \frac{1}{\gamma_1} \geq 0\) imply that there is a directed edge from node \(2i - 1\) to node \(2i\) while edge \((2i, 2i-1)\) may or may not exist in digraph \(g^*\) for any \(i \in \mathcal{V}\) as shown in Fig. 2. Recall that \(A\) is the adjacency matrix of graph \(g\), then checking the following submatrix of \(A^*\) we have

\[
\begin{bmatrix}
0 & b_{4,2} & \cdots & b_{4,2n} \\
0 & b_{2,4} & \cdots & b_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{2n,2} & b_{2n,4} & \cdots & 0
\end{bmatrix} = \gamma_1 A.
\]

Relabel all the nodes as shown in Fig. 2, then we get a new digraph with the node set \(\mathcal{V}^* = \{1', 2', 2', \ldots, n', n'\}\) which is homomorphic to digraph \(g^*\). Here, we still denote \(g^*\) as the newly relabeled digraph. Then, we can obtain \(a_{ij} = \frac{1}{\gamma_1} b_{2i,2j}\), which implies that \(a_{ij} > 0\) if and only if \(b_{2i,2j} > 0\). Thus we can get that \(a_{ij} \neq 0\) in digraph \(g\) if and only if \(b_{2i,2j} \neq 0\) in digraph \(g^*\) by comparing the new labels of the nodes with the original ones. On this basis, we find that \(g^*\) is precisely the digraph constructed from digraph \(g\) and \(g'\) (with node set \(\{1', 2', \ldots, n'\}\)) according to the rules described in the second statement of Lemma 1. It then follows from Lemma 1 that \(g^*\) also has a globally reachable node if \(g\) has a globally reachable node.

Now consider the network of vehicles with fixed topology \(G^* = (\mathcal{V}^*, \mathcal{E}^*, A^*)\) and the corresponding network dynamics (6). According to Lemma 2, we can get \(\lim_{t \to \infty} (\xi_i^*(t) - \xi_j^*(t)) = 0, i,j \in \{1, 2, 2n - 2, 2n\}\), which means \(\lim_{t \to \infty} (x_i^*(t) - y_i^*(t)) = 0, i,j \in \mathcal{V}\). Noticing that \(y_i^*(t) = x_i^* + \gamma_1 v_i^*(t)\), thus we have \(\lim_{t \to \infty} v_i^*(t) = 0\) if \(\lim_{t \to \infty} x_i^*(t) = 0\). This in turn proves the sufficiency part.

**Necessity:** Mimicking a similar proof for the necessity of Theorem 1, this part can be proved easily. \(\square\)

**Remark 2.** The consensus analysis for algorithms (2) and (3) under fixed communication topology is performed in Ren (2007a, 2008), respectively. In this paper, we revisit these strategies and perform the convergence analysis by using totally different methods from those in Ren (2007a, 2008) which rely heavily on analyzing the eigenvalues of the transformed system matrix. In particular, the methods employed in this paper present the corresponding results in simpler forms (i.e., the requirements imposed on \(\gamma\) and/or \(\gamma_1\) take simpler forms) than those in Ren (2007a, 2008). On the other hand, it is noticed that the theoretical findings in Theorems 1 and 2 do not always show an improvement in terms of conservatism reduction in the sense that there exist examples in which the results in this paper may be less conservative than the existing ones while some other examples may indicate the opposite. However, it is worth pointing out that the result in Theorem 2 is independent of the weighting factors of the information flow. This allows the absolute velocity damping gain \(\gamma\) to take almost any positive number once an appropriate \(\gamma_1\) is chosen. In contrast, in Theorem 3.2 in Ren (2007a), \(\gamma_1\) varies with \(\gamma\), the weighting factors as well as the topological structure of the information flow. In addition, under the assumption that \(\gamma \geq \frac{1}{n}\), it is shown in Theorem 2 that having a globally reachable node is a necessary and sufficient condition for all the vehicles to reach consensus. This may be viewed as an advantage of the result in Theorem 2 over that in Theorem 3.2 in Ren (2007a).

It should be mentioned here that the key point in introducing the new consensus analysis is that it presents a different viewpoint for understanding the double-integrator consensus strategies. More importantly, the method used to handle the case with fixed topology can be extended to deal with the dynamic case by means of product properties of nonnegative matrices and the notion of dwell time. This is also one of the main contributions of the paper. Indeed, as will be seen from the next subsection, the convergence analysis for the dynamic case takes full advantage of the proof techniques employed in treating the fixed case.

For illustration, consider a team of \(n = 4\) vehicles. Two different communication topologies under consideration are shown in Fig. 3, where the reference velocity is \(v_r = \sin t\) and the velocity damping gain is taken as \(\gamma = 2\) for each case. When the communication topology is \(G_2\), which has a globally reachable node, using the update strategy (2), the position and velocity trajectories of all vehicles reach agreement as shown in Fig. 4. This is consistent with the sufficiency of Theorem 1. When the communication topology is \(G_1\) which does not have a globally reachable node, as then shown in Fig. 5, the position trajectories of all vehicles cannot reach agreement. This is consistent with the necessity of Theorem 1.

For the update strategy (3), we take \(\gamma_1 = 1\) and \(\gamma = 1\). Obviously, \(\gamma\) satisfies the condition described in Theorem 2, that is, \(\gamma_1 > \frac{1}{n}\) and \(\gamma_1\) is irrelevant to the out-degree of the communication topology. It can be seen from Fig. 6, which depicts the position and velocity trajectories of the vehicles with reference velocity under communication topology \(G_2\), that all the vehicles reach agreement.
Remark 3. Ren (2007b) investigates the consensus problem for multiple single-integrator vehicles with a time-varying virtual leader (see algorithms (7) and (8) therein). With the reference signal being treated as a virtual vehicle with index $n+1$, it is proved in Theorem 3.3 in Ren (2007b) that the consensus can be achieved if and only if the digraph associated with the $n+1$ vehicles, say $g_{n+1}$, has a spanning tree (corresponding to a globally reachable node in the context of this paper, please see Remark 1 for more details). Clearly, in this case, node $n+1$ must be the root node (i.e., the globally reachable node in this paper) of graph $g_{n+1}$. The intuitive idea for that result is that the state information can be transmitted to all the vehicles from node $n+1$ (i.e., the virtual leader). Motivated by the work in Ren (2007b), one may ask whether the algorithms investigated in this paper can still enable the final consensus even if the state information of the reference signal is available to a subset of the vehicles (agents). By choosing a group of different values for $\gamma, \gamma_1$ in the simulations (see below for a simulation result for the case $\gamma = 2$ and $\gamma_1 = 1$ for example), it is found that consensus cannot be reached even if the globally reachable node 2 of the communication topology can receive information from the reference signal $v_r(t)$. Although we have not found some proper values for $\gamma, \gamma_1$ that can guarantee the consensus, this does not imply that such values do not exist. Considering now that it has also not been able to theoretically prove that consensus cannot be reached by choosing any values for $\gamma, \gamma_1$, a definite conclusion to the above question cannot be drawn at this stage. It is reasonable to envisage that to ensure the final consensus, some extra requirement may need to be imposed on the strategies. This is an interesting topic which deserves further investigation in our future work. The following is an example which shows that the case with single-integrator as investigated in Ren (2007b) is different from the algorithms (2) and (3) considered in this paper for some chosen $\gamma, \gamma_1$. We still consider a team of $n = 4$ vehicles. Assume that the communication topology is modeled by digraph $G_2$ in Fig. 3, and the reference velocity is $v_r(t) = \sin t$. Assume that for both strategies (2) and (3), only the globally reachable node 2 can receive the reference signal $v_r(t)$. We further choose $\gamma = 2$ and $\gamma_1 = 1$ for both strategies. Figs. 7 and 8 depict the state trajectories of the vehicles under algorithms (2) and (3), respectively, which show that consensus cannot be reached.

4.2. The case with dynamically changing communication topologies

In this section, we apply the dwell time to the continuous-time update schemes, which implies that the communication topology
and weighting factors are constrained to change only at discrete times, that is $\mathcal{S}(t) = I_n \otimes A - L(t) \otimes B$ and $\mathcal{S}_i(t) = I_n \otimes A - L_i(t) \otimes B_i$ are piecewise constant. Let $\hat{g} = \{\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m\}$ denote the finite set of all possible communication topologies. The union of a group of digraphs $\{\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_k\}$ is a digraph with the node set $V = \{1, 2, \ldots, n\}$ and the edge set given by the union of the edge sets of $\hat{g}_j$, $j = 1, \ldots, k$. Let $\bar{\tau}$ be a finite set of arbitrary positive numbers and let $\chi$ be the infinite set generated from $\bar{\tau}$ under binary operation addition and multiplication by integers, i.e., $\chi = \{pt_1 + qt_2 : p, q \in \mathbb{Z}_+, t_1, t_2 \in \bar{\tau}\}$. By choosing the set $\bar{\tau}$ properly, the dwell time can be chosen from the infinite set $\chi$, which is capable of approximating switching times arbitrarily well via appropriate choices of $\bar{\tau}$. Consider an infinite sequence of nonempty, uniformly bounded, and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, with $t_0 = 0$ and $t_{k+1} - t_k \leq T (k \geq 0)$ for some constant $T > 0$. Suppose that in each interval $[t_k, t_{k+1})$ there is a sequence of nonoverlapping subintervals

$[t_{i_0}, t_{i_1}), \ldots, [t_{i_{m_k-1}}, t_{i_{m_k}})$

with $t_{i_0} = t_k$, $t_{i_{m_k}} = t_{k+1}$ satisfying $t_{i_{m_k}} - t_{i_0} = t_k \in \chi$, $0 \leq j < m_k$ for some integer $m_k > 0$ such that during each of such subintervals, both the communication topology and the weighting factors do not change (i.e., they are piecewise constant). Note that in this subsection, we mainly perform consensus analysis for the first update strategy (2) while the second update strategy (3) can be analyzed similarly.

In order to facilitate our analysis, we need the following results which are mainly concerned with nonnegative matrices. Note that Lemma 3 is generalized from the work of Horn and Johnson (1987), Lin, Francis, and Maggiore (2005) and Ren and Beard (2005).

**Lemma 3.** If $A$ is a stochastic matrix which has an eigenvalue $\lambda = 1$ with algebraic multiplicity 1 and $v \in \mathbb{R}^n$ being one corresponding eigenvector, and all the other eigenvalues satisfy $|\lambda| < 1$, then $A$ is ergodic, that is, $\lim_{m \to \infty} A^m = 1_n v^t$, where $v^t A = v^t$ and $v^t 1_n = 1$.

In particular, if $A$ is a stochastic matrix with positive diagonal elements and the digraph associated with $A$ has a globally reachable node, then $A$ is ergodic.

**Lemma 4 (Jadbabaie et al., 2003).** Let $m \geq 2$ be a positive integer and let $P_1, P_2, \ldots, P_m$ be nonnegative $n \times n$ matrices with positive diagonal elements. Then $P_1 P_2 \cdots P_m \geq \delta (P_1 + P_2 + \cdots + P_m)$, where $\delta > 0$ can be specified from matrices $P_i$, $i = 1, 2, \ldots, m$.

**Lemma 5 (Wolfowitz, 1963).** Let $M_1, M_2, \ldots, M_n$ be a finite set of ergodic matrices with the property that for each subsequence $M_{i_1}, M_{i_2}, \ldots, M_{i_k}$ of positive length, the matrix product $M_{i_1} M_{i_2} \cdots M_{i_k}$ is ergodic. Then for each infinite sequence $M_{i_1}, M_{i_2}, \ldots$ there exists a vector $C \in \mathbb{R}^n$ such that

$$\lim_{j \to \infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = C \cdot I_n.$$

In the sequel, let $\Delta = \max \{\sum_{j=1}^{n} a_i(j) : \gamma = 0\}$. The maximum of $\Delta$ always exists and can be achieved because both $\bar{g}$ and $\bar{\alpha}$ are finite sets. Note that $\Delta$ is just the maximum out-degree of all the possible communication topologies.

**Lemma 6.** Assume that the velocity damping gain $\gamma$ satisfies $\gamma \geq \sqrt{\Delta}$. Then, each $\exp \{\mathcal{S}(t_k) r_k\}$ is a stochastic matrix with positive diagonal elements, for $k \in \mathbb{Z}_+$, $0 \leq i \leq m_k - 1$. Moreover,

$$\mathcal{S} = \{\exp \{\mathcal{S}(t_k) r_k\} : k \in \mathbb{Z}_+, 0 \leq i \leq m_k - 1\}$$

is a finite set.

**Proof.** Since $L(t) 1_n = 0$, we have

$$\mathcal{S}(t) 1_n = [I_n \otimes A - L(t) \otimes B] 1_n = 0.$$

Recall that for any square matrix $P \in \mathbb{R}^{m \times m}$, $m \in \mathbb{Z}_+$, $\exp(Pr) = I_n + \sum_{k=1}^{\infty} \frac{1}{k!} P^k t$. Then,

$$\exp \{\mathcal{S}(t_k) r_k\} 1_n = I_n 1_n + \sum_{i=1}^{\infty} \frac{1}{i!} I_n 1_n \mathcal{S}(t_k) r_k = I_n 1_n.$$

(7)

On the other hand, according to the definition of $\mathcal{S}(t_k)$, we can denote $\mathcal{S}(t_k)$ as $\mathcal{S}(t_k) = -\gamma I_n + P t_k$, where $P t_k = I_n \otimes \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix} - L(t_k) \otimes B$. It follows from $\gamma \geq \sqrt{\Delta}$ that matrix $P t_k$ is nonnegative, and thus

$$\exp \{\mathcal{S}(t_k) r_k\} = \exp \{-\gamma r_k I_n + P t_k\}$$

$$\geq \exp \{-\gamma r_k I_n + P t_k I_n + P t_k r_k\}$$

$$\geq \exp \{-\gamma r_k I_n\},$$

which, together with Eq. (7), implies that $\exp \{\mathcal{S}(t_k) r_k\}$ is a stochastic matrix with positive diagonal elements, for $k \in \mathbb{Z}_+$, $0 \leq i \leq m_k - 1$.

Since both $\bar{g}$ and $\bar{\alpha}$ are finite sets,

$$\mathcal{S}(t_k) : k \in \mathbb{Z}_+, 0 \leq i \leq m_k - 1$$

is also a finite set. Furthermore, given that $\bar{\alpha}$ is a finite set and all the time intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, are uniformly bounded by $T > 0$, it follows that the set $\{t_k : k \in \mathbb{Z}_+, 0 \leq i \leq m_k - 1\}$ is also finite. Combining these two facts we can easily get that $\mathcal{S}$ is a finite set. \(\square\)

Based on the preceding work, we can present a sufficient condition for the vehicles with reference velocity to reach consensus when both the communication topologies and the weighting factors are dynamically changing.

**Theorem 3.** Assume that the velocity damping gain $\gamma$ satisfies $\gamma \geq \sqrt{\Delta}$. Consider an infinite sequence of nonempty, uniformly bounded, and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, with $t_0 = 0$ and $t_{k+1} - t_k \leq T (k \geq 0)$ for some constant $T > 0$ such that in each interval $[t_k, t_{k+1})$ there is a sequence of nonoverlapping subintervals $[t_{i_0}, t_{i_1}), \ldots, [t_{i_{m_k-1}}, t_{i_{m_k}})$.
with \( t_k = t_0, t_{k+1} = t_{k\theta} \), satisfying \( t_{k+1} - t_k = \tau_k \in \chi, 0 \leq j < m_k - 1 \) for some integer \( m_k > 0 \). If for each \( k \geq 0 \), the union of digraphs \( \tilde{g}(t_{k\theta}), \tilde{g}(t_{k\theta+1}), \ldots, \tilde{g}(t_{k\theta+m_k-1}) \) has a globally reachable node, then the update strategy (2) asymptotically makes all the vehicles with a time-varying reference velocity \( v_i(t) \) reach consensus.

**Proof.** As discussed in the proof of Theorem 1, we only need to prove that consensus can be reached asymptotically for system

\[
\hat{\xi}^-(t) = (\mathcal{E}(t) \otimes I_k)\hat{\xi}^+(t).
\]

Let \( \Phi(t_m, t_n) = I_{2n}, m \geq 0, \) and

\[
\Phi(t_{k+1}, t_k) = \prod_{i=0}^{m_k-1} \exp \left[ \mathcal{E}(t_k) \tau_k \right],
\]

where \( \tau_k = t_{k+1} - t_k \). As described in Lemma 6, \( \mathcal{E}(t_k) = -\gamma I_{2n} + P_{t_k} \), where \( P_{t_k} \) is a nonnegative matrix. Additionally,

\[
\exp \left[ \mathcal{E}(t_k) \tau_k \right] = \exp \left[ \left( -\gamma I_{2n} \right) L(t_k) \right] \exp \{ P_{t_k} \tau_k \}
\]

\[
\geq \exp \left( -\gamma I_{2n} \right) L(t_k) \exp \{ P_{t_k} \tau_k \} \geq \rho_k \exp \{ P_{t_k} \tau_k \}
\]

for some \( \rho_k > 0, 0 \leq \gamma \leq m - 1 \). Let \( \rho_k = \min_{0 \leq \gamma \leq m - 1} \{ \rho_k \} \). Then, applying Lemma 4 we have

\[
\Phi(t_{k+1}, t_k) = \prod_{i=0}^{m_k-1} \exp \left[ \mathcal{E}(t_k) \tau_k \right] \geq \epsilon_k \rho_k \prod_{i=0}^{m_k-1} P_{t_k}
\]

(8)

for some \( \epsilon_k > 0 \). Note that

\[
P_{t_k} = I_n \otimes \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} - L(t_k) \otimes B,
\]

thus

\[
\sum_{i=0}^{m_k-1} P_{t_k} = \sum_{i=0}^{m_k-1} \left( I_n \otimes \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} - L(t_k) \otimes B \right) = [\tilde{b}] \in \mathbb{R}^{2n \times 2n}.\]

Mimicking a similar proof for Theorem 1, we consider the following submatrix of \( [\tilde{b}] \), then

\[
\begin{bmatrix}
\tilde{b}_{21} & \tilde{b}_{23} & \cdots & \tilde{b}_{2,2n-1} \\
\tilde{b}_{41} & \tilde{b}_{43} & \cdots & \tilde{b}_{4,2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{b}_{2n,1} & \tilde{b}_{2n,3} & \cdots & \tilde{b}_{2n,2n-1}
\end{bmatrix}
\]

\[
\approx \begin{bmatrix}
0 & \tilde{b}_{23} & \cdots & \tilde{b}_{2,2n-1} \\
\tilde{b}_{41} & 0 & \cdots & \tilde{b}_{4,2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{b}_{2n,1} & \tilde{b}_{2n,3} & \cdots & 0
\end{bmatrix}
= \frac{1}{\gamma} \tilde{A},
\]

where

\[
\tilde{A} = \begin{bmatrix}
0 & m_{k-1} \sum_{i=0}^{m_k-1} a_{12}(t_k) & \cdots & m_{k-1} \sum_{i=0}^{m_k-1} a_{1n}(t_k) \\
m_{k-1} \sum_{i=0}^{m_k-1} a_{12}(t_k) & 0 & \cdots & \sum_{i=0}^{m_k-1} a_{2n}(t_k) \\
\vdots & \vdots & \ddots & \vdots \\
m_{k-1} \sum_{i=0}^{m_k-1} a_{1n}(t_k) & m_{k-1} \sum_{i=0}^{m_k-1} a_{2n}(t_k) & \cdots & 0
\end{bmatrix}.
\]

Note that digraph \( \Gamma'(\tilde{A}) \) is equivalent to the union of digraphs \( \tilde{g}(t_{k\theta}), \tilde{g}(t_{k\theta+1}), \ldots, \tilde{g}(t_{k\theta+m_k-1}) \) and thus has a globally reachable node according to the given condition. Then, using similar proving techniques as in the proof for the sufficiency part of Theorem 1, we can get that digraph \( \Gamma' \left( \sum_{i=0}^{m_k-1} \left( t_i \otimes \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} - L(t_i) \otimes B \right) \right) \) has a globally reachable node, which together with inequality (8) implies that digraph \( \Gamma'(\Phi(t_{k+1}, t_k)) \) also has a globally reachable node. On the other hand, since by the related definitions stochastic matrices with positive diagonal elements are closed under matrix multiplication, it follows from Lemma 6 that each matrix \( \Phi(t_{k+1}, t_k) \) is also a stochastic matrix with positive diagonal elements. Then, according to the second statement of Lemma 3, matrix \( \Phi(t_{k+1}, t_k) \) is ergodic. Moreover, the set of possible \( \Phi(t_{k+1}, t_k), k \in \mathbb{Z}_+, \) must be finite because \( \Phi(t_{k+1}, t_k) \) is a product of at most \( \frac{m}{2} + 1 \) matrices from a finite matrix set \( \mathcal{E} \) (see Lemma 6 for the definition), where \( \frac{m}{2} + 1 \) denotes the largest integer not greater than \( \frac{m}{2} \) and \( \tau = \min \{ t : t \in \mathbb{R} \} \).

Based on the above arguments, the remaining part of the proof can be completed by mainly using the result on the product properties of nonnegative matrices in Lemma 5. By applying the analysis techniques that can be found from the existing literature concerning the convergence analysis for agents with single-integrator dynamics under dynamically changing communication topologies (see, e.g., Jadabaei et al., 2003, Lin, Francis, & Maggiore, 2008 and Ren & Beard, 2005), it can be shown that

\[
\lim_{t \to \infty} \Phi(t, 0) = I_{2n} \otimes T.
\]

This, together with the fact that \( \hat{\xi}^+(t) = \left[ \Phi(t, 0) \otimes I_k \right] \hat{\xi}^+(0) \) implies

\[
\lim_{t \to \infty} \hat{\xi}^+(t) = (I_{2n} \otimes I_k)(u^T \otimes I_k) \hat{\xi}^+(0),
\]

which in turn means that \( \lim_{t \to \infty} (\hat{\xi}^+(t) - \hat{\xi}^-(t)) = 0, i, j \in \{ 1, 2, \ldots, 2n - 1, 2n \} \), thereby completing the proof. □

Employing the consensus strategy (3), we have the following results which can be proved using similar techniques as above. This can be easily seen by observing Eq. (6) and the proof for Theorem 2.

**Theorem 4.** Assume that the velocity damping gain \( \gamma \) satisfies \( \gamma \geq \frac{1}{\gamma} \). Consider an infinite sequence of nonempty, uniformly bounded, and contiguous time intervals \( \{ t_k, t_{k+1} \}, k = 0, 1, \ldots, \) with \( t_0 = 0 \) and \( t_{k+1} - t_k \leq T \) \( (k \geq 0) \) for some constant \( T > 0 \) such that in each interval \( \{ t_k, t_{k+1} \} \) there is a sequence of nonoverlapping subintervals \( \{ t_{k0}, t_{k1}, \ldots, t_{k\theta}, t_{k\theta+1}, \ldots, t_{k\theta+m_k-1}, t_{k\theta+m_k} \} \) with \( t_k = t_{k0}, t_{k+1} = t_{k\theta} \) satisfying \( t_{k+1} - t_k = \tau_k \in \chi, 0 \leq j < m_k - 1 \) for some integer \( m_k > 0 \). If for each \( k \geq 0 \), the union of digraphs \( \tilde{g}(t_{k\theta}), \tilde{g}(t_{k\theta+1}), \ldots, \tilde{g}(t_{k\theta+m_k-1}) \) has a globally reachable node, then the update strategy (3) asymptotically makes all the vehicles with a time-varying reference velocity \( v_i(t) \) reach consensus.

**Remark 4.** By comparing all the results between the consensus strategies (2) and (3), it can be easily found that the condition on the velocity damping gain for the update strategy (3) is more relaxed than that of the strategy (2). In the case of the strategy (3), the velocity damping gain can be relaxed to be irrelevant to the weighting factors. This is mainly because in the strategy (3), each vehicle can get more information (mainly the relative velocity information) from its neighboring vehicles when compared with the strategy (2). And this also enables the vehicles to converge more quickly to the common value.
5. Conclusion

In this paper we have studied the consensus strategies for multi-vehicle systems with a time-varying reference signal under directed communication topology. A new graphic methodology, which plays an important role in transforming the network dynamics, has been proposed to solve the problems. For the case with the fixed communication topology, we have shown that the necessary and sufficient condition for a group of vehicles with a time-varying reference signal to reach consensus is that the communication topology has a globally reachable node. We have also derived a sufficient condition for all the vehicles with a reference signal to reach consensus under time-varying weighting factors and dynamically changing communication topologies which might not have a globally reachable node.

Finally, since the weighting factors are required to be chosen from a finite set $\tilde{\alpha}$ and $\chi$ (i.e., the set from which the length of each dwell time is chosen) is required to be generated from a finite set $\bar{r}$, we have been able to use Wolfowitz’s work concerning the product properties of stochastic matrices (Wolfowitz, 1963) to help perform the consensus analysis in this paper. However, such requirements imposed on the choices of the weighting factors and the length of the dwell time are somewhat restrictive. A more general and practical case may be that the set $\tilde{\alpha}$ is infinite and the lengths of all the dwell times can be arbitrarily chosen except that they are all upper bounded by $T$ and lower bounded by a positive integer (unlike the case in this paper, the set $\bar{r}$ produced by these assumptions will be infinite). A possible way to tackle this challenging problem is to appeal to the work given in Lin et al. (2008) which deals with the infinite case for single-integrator dynamics by using the generalized result of Lemma 5 in Wolfowitz (1963). This presents an interesting topic for future research.

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References


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