Existence of weak solutions for a $p$-Laplacian problem involving Dirichlet boundary condition

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**Abstract**

In this work, by virtue of topological degree theory and critical point theory, we are mainly concerned with the existence of weak solutions for a Dirichlet boundary value problem with the $p$-Laplacian operator.

**1. Introduction**

In this paper, we investigate the existence of weak solutions for the $p$-Laplacian problem with Dirichlet boundary condition

$$
\begin{align*}
-\Delta_p u &= f(x, u), & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$, $p \in (1, \infty)$ and $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian. The nonlinearity $f$ is of continuous functions on $\Omega \times \mathbb{R}$ and has subcritical growth, i.e.,

1. There exist $c_1 > 0$ and $r \in (1, p^*)$ such that

$$
|f(x, t)| \leq c_1 (|t|^{p^*} + 1), \quad \forall (x, t) \in \Omega \times \mathbb{R},
$$

(1.2)

where $p^* := \begin{cases} np/(n - p), & p < n, \\
\infty, & p > n \end{cases}$ is the critical Sobolev exponent.

Nonlinear boundary value problems with $p$-Laplacian operator $-\Delta_p$ occur in a variety of physical phenomena, for instance, non-Newtonian fluids, reaction–diffusion problems, petroleum extraction, flow through porous media, etc. From Perera’s elegant work [1], in which a new sequence of eigenvalues of $-\Delta_p$ was constructed using the Yang index, we can obtain enough sufficient information about such problems. After Perera’s breakthrough paper [1], a significant research area for the $p$-Laplacian problems has been established. Therefore, more and more mathematicians pay their attention to the study of such problems, see [2–12] and the references therein.

In [2], Yuan and Ou studied the problem (1.1) by virtue of some critical point theorems involving the homotopical linking, for more details about this theory, please refer the recent book [3], which summarizes some excellent results on the $p$-Laplacian problems.

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0096-3003/© 2014 Published by Elsevier Inc.
In [4], A. Candela and G. Palmieri investigated the existence of one or more critical points of a family of functionals which generalizes the model problem
\[
J(u) = \int_{\Omega} A(x, u) |\nabla u|^p dx - \int_{\Omega} G(x, u) dx
\]

in the Banach space \(W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\), being \(\Omega\) a bounded domain in \(\mathbb{R}^N\). It is worth noting that \(W^{1,p}_0(\Omega)\) is decomposed according to a “good” sequence of finite dimensional subspaces.

Let \(W\) be the Sobolev space \(W^{1,p}_0(\Omega)\) with the usual norm
\[
\|u\| = \left(\int_{\Omega} |\nabla u|^p dx\right)^{\frac{1}{p}}, \text{ for } u \in W^{1,p}_0(\Omega).
\]

By the Sobolev embedding theorem, for \(r < p\), the embedding \(W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega)\) is compact. Hence we find there is a constant \(C_{\text{emb}} > 0\) such that
\[
\|u\|_{L^r(\Omega)} \leq C_{\text{emb}} \|u\|_{W^{1,p}_0(\Omega)}, \quad \text{for all } u \in W^{1,p}_0(\Omega),
\]

(1.3)

where \(\|u\|_{L^r(\Omega)} := \left(\int_{\Omega} |u|^r \right)^{1/r}\) denotes the norm of \(L^r(\Omega)\).

Define the energy functional \(J : W^{1,p}_0(\Omega) \to \mathbb{R}\)
\[
J(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(x, u) dx,
\]

(1.4)

where \(F(x, t) = \int_0^t f(x, s) ds\). Since \(f(x, t)\) satisfies (H1), by the standard argument, \(J\) is of the class \(C^1\). From the variational point of view, we easily know that weak solutions of (1.1) coincide with the critical points of the \(C^1\)-functional \(J\). Furthermore, the derivative of \(J\) is
\[
\langle J'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \int_{\Omega} f(x, u) v dx, \quad \forall u, v \in W^{1,p}_0(\Omega).
\]

(1.5)

It is well known that the \(p\)-homogeneous boundary value problem
\[
\begin{align*}
-\Delta_p u &= \lambda |u|^{p-2} u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.6)

has an increasing eigenvalue sequence \(0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \cdots \to +\infty\), and the corresponding eigenfunctions \(\varphi_k \in W^{1,p}_0(\Omega)\) associated with \(\lambda_k\) for all \(k\). Moreover, the first eigenvalue \(\lambda_1\) is simple and isolated in \(\sigma(-\Delta_p)\), the spectrum of \(-\Delta_p\), and \(\varphi_1\) is positive in \(\Omega\). In this case, \(\lambda_1\) can be expressed explicitly by
\[
\lambda_1 = \inf_{u \in W^{1,p}_0(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx},
\]

and hence, the following Poincaré inequality
\[
\int_{\Omega} |\nabla u|^p dx \geq \lambda_1 \int_{\Omega} |u|^p dx, \quad \forall u \in W^{1,p}_0(\Omega)
\]

(1.7)

holds. Note that we can normalize \(\varphi_k\) such that
\[
\int_{\Omega} |\varphi_k|^p dx = 1, \quad \text{for all } k.
\]

In what follows, we shall introduce the finite dimensional decomposition of \(W^{1,p}_0(\Omega)\) by [4]. Fixed any \(k \geq 1\) and defined \(Y_k = \text{span}\{\varphi_1, \ldots, \varphi_k\}\) and

\[
Z_k = \bigcap_{j=1}^k \ker(L_j) = \{u \in W^{1,p}_0(\Omega) : L_1(u) = \cdots = L_k(u) = 0\},
\]

where
\[
L_j u = \int_{\Omega} |\varphi_j|^{p-2} \varphi_j u dx.
\]

The conclusions are
\[
W^{1,p}_0(\Omega) = Y_k \oplus Z_k, \quad \dim Y_k = k,
\]

and
\[
\lambda_{k+1} \int_{\Omega} |u|^p dx \leq \int_{\Omega} |\nabla u|^p dx, \quad \text{for all } u \in Z_k \quad \text{and } k \geq 1.
\]

(1.8)
Note the definitions of $\lambda_k$ and $\varphi_k$, by (1.6), we have
\[ -\Delta_p \varphi_k = \lambda_k |\varphi_k|^{p-2} \varphi_k, \quad \text{in } \Omega \] (1.9)
For each $v \in W^1_{0,p}(\Omega)$, multiply by $v$ on both sides of (1.9) to obtain
\[ \int_\Omega |\nabla \varphi_k|^{p-2} \nabla \varphi_k \nabla v \, dx = \lambda_k \int_\Omega |\varphi_k|^{p-2} \varphi_k v \, dx. \] (1.10)
In particular, choosing $v = \varphi_k$, we see $\int_\Omega |\nabla \varphi_k|^p \, dx = \lambda_k$ for all $k$.

2. Main results

In [11], X. Shang investigated the nonlinear elliptic boundary value problem
\[
\begin{align*}
-\Delta_p &= \lambda f(u), \quad x \in \Omega, \\
 u &= 0, \quad x \in \partial \Omega,
\end{align*}
\] (2.1)
where $f : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous function satisfying the condition $f(0) < 0$. Moreover, the another two hypotheses on $f$ are as follows:

(Shang1) $\lim_{s \to +\infty} \frac{f(s)}{s^{\frac{p}{p-1}}} = 0$,
(Shang2) $F(\beta) > 0$ for some $\beta > 0$, where $F(s) = \int_0^s f(t) \, dt$.

By using degree theoretic arguments based on the degree map for operators of type (S)$_+$, the author proved the existence of at least two nontrivial solutions for (2.1).

Inspired by this paper, we shall consider the existence of weak solutions for (1.1) by the degree map for operators of type (S)$_+$. In the following, we introduce the fundamental definitions and theorems for topological degree theory needed in the ensuing results.

**Definition 2.1** (see [11, Definition 2.1]). Let $X$ be a reflexive real Banach space and $X^*$ its dual. The operator $T : X \rightarrow X^*$ is said to satisfy the (S)$_+$ condition if the assumptions $u_n \rightharpoonup u_0$ weakly in $X$ and $\limsup_{n \to \infty} \langle T(u_n) - T(u_0), u_n - u_0 \rangle \leq 0$ imply $u_n \rightarrow u_0$ strongly in $X$.

**Definition 2.2** (see [13]). The operator $T : X \rightarrow X^*$ is said to be demicontinuous if $T$ maps strongly convergent sequences in $X$ to weakly convergent sequences in $X^*$.

**Lemma 2.1** (see [11, Remark 2.2]). Let $T : X \rightarrow X^*$ satisfy the (S)$_+$ condition and let $K : X \rightarrow X^*$ be a compact operator. Then the sum $T + K : X \rightarrow X^*$ satisfies the (S)$_+$ condition.

**Lemma 2.2** (see [11,14,15]). Let $T : X \rightarrow X^*$ be a bounded and demicontinuous operator satisfying the (S)$_+$ condition. Let $D \subset X$ be an open, bounded and nonempty set with the boundary $\partial D$ such that $T(u) \neq 0$ for $u \in \partial D$. Then there exists an integer $\deg(T, D, 0)$ such that

1. $\deg(T, D, 0) \neq 0$ implies that there exists an element $u_0 \in D$ such that $T(u_0) = 0$.
2. If $D$ is symmetric with respect to the origin and $T$ satisfies $T(u) = -T(-u)$ for any $u \in \partial D$, then $\deg(T, D, 0)$ is an odd number.
3. Let $T_\lambda$ be a family of bounded and demicontinuous mappings which satisfy the (S)$_+$ condition and which depend continuously on a real parameter $\lambda \in [0, 1]$, and let $T_\lambda(u) \neq 0$ for any $u \in \partial D$ and $\lambda \in [0, 1]$. Then $\deg(T_\lambda, D, 0)$ is constant with respect to $\lambda \in [0, 1]$.

Now, we list our assumptions on $f.$

(H2) $\lim_{s \to +\infty} \frac{f(s)}{s^p} = a$ uniformly for $x \in \Omega$ and $a$ is a finite number.
(H3) $\lim_{s \to 0} \frac{f(s)}{s^p} = b$ uniformly for $x \in \Omega$ and $b$ is a finite number, $b \neq \lambda_k \sigma(-\Delta_p)$.
(H4) $f(x, t) + f(x, -t) = 0$ for all $x \in \Omega.$
Lemma 2.3 (see [16, Lemma 2.2]). Let (H2) and (H3) hold. If there is a sequence \( \{u_n\} \subset W^{1,p}_0 \) such that \( \|u_n\| \to \infty \), then
\[
\int_{\Omega} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx \to \int_{\Omega} b |w|^{p-2} w \, dx,
\]
where \( w_n = \frac{u_n}{|u_n|} \) and \( w_n \rightharpoonup w \) weakly in \( W^{1,p}_0 \).

**Proof.** Clearly, \( \{w_n\} \) is bounded in \( W^{1,p}_0(\Omega) \) and we may suppose that there is a \( w \in W^{1,p}_0(\Omega) \) such that
\[
w_n \rightharpoonup w \ \text{weakly in} \ W^{1,p}_0(\Omega), \quad w_n \to w \ \text{strongly in} \ L^p(\Omega), \quad w_n \to w \ \text{a.e.} \ x \in \Omega.
\]
Moreover, by [20, Remark 1.2.18], there exists \( w_0 \in L^p(\Omega) \) such that \( |w_n| \leq w_0 \) a.e. in \( \Omega \) for all \( n \).

By (H2) and (H3), we see that there exists \( M_1 > 0 \) such that \( \frac{|f(x,u_n)|}{|u_n|^{p-2}u_n} \leq M_1, \forall x \in \Omega \) and \( t \in \mathbb{R} \). Let us define \( \Omega_0 := \{ x \in \Omega : w(x) = 0 \} \), \( \Omega_1 := \{ x \in \Omega : w(x) \neq 0 \} \). For \( \Omega_0 \), by Hölder inequality, we have
\[
\int_{\Omega_0} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx \leq \int_{\Omega_0} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-1} |x| \, dx \leq M_1 \int_{\Omega_0} |w_n|^{p-1} |x| \, dx \leq M_1 \left( \int_{\Omega_0} |w_n|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega_0} |x|^p \, dx \right)^{\frac{1}{p}}.
\]
For the reason that \( w_n \to w \) strongly in \( L^p(\Omega_0) \) and \( \Omega_0 \subset \Omega \), we see \( w_n \to w \) strongly in \( L^p(\Omega_0) \). So
\[
\int_{\Omega_0} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx \to 0 = \int_{\Omega_0} b |w|^{p-2} w \, dx.
\]
For \( \Omega_1 \), we first note that
\[
\left| \int_{\Omega} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx \right| \leq M_1 |w_0|^{p-1}.
\]
Therefore, we can utilize the dominated convergence theorem to obtain
\[
\lim_{n \to \infty} \int_{\Omega} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx = \int_{\Omega} \lim_{n \to \infty} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx.
\]
Since \( \|u_n\| \to \infty \), we have \( |u_n| = |w_n||u_n| \to \infty \) for \( x \in \Omega_1 \). So
\[
\int_{\Omega_1} \lim_{n \to \infty} \frac{f(x,u_n)}{|u_n|^{p-2}u_n} |w_n|^{p-2} w_n \, dx = \int_{\Omega_1} b |w|^{p-2} w \, dx.
\]
This completes the proof. \( \square \)

We shall define three operators as follows
\[
(A_p(u), \nu) := \int_{\Omega} \nabla u \nabla u \nabla \nu \, dx, \quad (B_p(u), \nu) := \int_{\Omega} |u|^{p-2} u \nu \, dx, \quad (S(u), \nu) := \int_{\Omega} f(x,u) \nu \, dx.
\]
We list the properties of the operators\( A_p, B_p \) and \( S \), for more details about their proof, please see [12].

(i) \( A_p^{-1} \) exists, \( A_p, A_p^{-1}, B_p \) and \( S \) are bounded and continuous operators;
(ii) \( B_p \) and \( S \) are compact operators;
(iii) \( A_p \) satisfies the \((S)_+\) condition.

**Theorem 2.1.** Let (H1)–(H3) hold. Then (1.1) has a weak solution.

**Proof.** By the definition of \( A_p, S \) and (1.5), we obtain \( u \in W^{1,p}_0(\Omega) \) is a weak solution of (1.1) if \( (A_p(u), \nu) = (S(u), \nu), \forall \nu \in W^{1,p}_0(\Omega) \). Thus, to find a weak solution of (1.1) is equivalent to finding a \( u \in W^{1,p}_0(\Omega) \) which satisfies the operator equation \( A_p(u) = S(u) = 0 \).

In order to apply Lemma 2.2, we define a homotopy
\[
T_\tau(u) = A_p(u) - (1 - \tau)S(u) - \tau b B_p(u), \quad \text{for} \quad \tau \in [0,1], \ u \in W^{1,p}_0(\Omega),
\]
where \( b \) is determined by (H3). From Lemma 2.1, we know that \( T_\tau \) satisfies the \((S)_+\) condition. We shall prove that there exists a large enough \( R > 0 \) such that this homotopy is admissible with respect to the ball \( B(0,R) \subset W^{1,p}_0(\Omega) \). If the claim is false, for any \( n \in \mathbb{N} \), there exist \( \tau_n \in [0,1] \) and \( u_n \in W^{1,p}_0(\Omega) \), \( \|u_n\| \geq n \) such that \( T_{\tau_n}(u_n) = 0 \), i.e., \( A_p(u_n) - (1 - \tau_n)S(u_n) - \tau_n b B_p(u_n) = 0 \). It is equivalent to the integral equation
\[
\int_{\Omega} \nabla u_n \nabla \nu \, dx - (1 - \tau_n) \int_{\Omega} f(x,u_n) \nu \, dx - \tau_n b \int_{\Omega} |u_n|^{p-2} u_n \nu \, dx = 0, \quad \forall \nu \in W^{1,p}_0(\Omega).
\]

Note that \( \omega_n = \frac{n}{|\Delta n|} \) and divided (2.5) by \( \|u_n\|^{p-1} \) to get
\[
\int_{\Omega} |\nabla \omega_n|^{p-2} \nabla \omega_n \nabla \psi dx - (1 - \tau_n) \int_{\Omega} |\nabla u_n|^{p-2} |\nabla u_n| \cdot |\nabla \psi| dx - \tau_n b \int_{\Omega} |\omega_n|^{p-2} \omega_n \psi dx = 0.
\] (2.6)

Due to the reflexivity of \( W^{1,p}_0(\Omega) \) and the compactness of the interval \([0,1]\), we may assume that \( \omega_n \rightharpoonup \omega \) weakly in \( W^{1,p}_0(\Omega) \), \( w_n \rightharpoonup w \) strongly in \( L^p(\Omega) \), and \( \tau_n \to \tau \in [0,1] \). The continuity of \( A_p \) and \( B_p \), combining with Lemma 2.3, enables us to obtain
\[
\int_{\Omega} |\nabla \omega|^{p-2} \nabla \omega \nabla \psi dx - (1 - \tau) \int_{\Omega} |\nabla v|^{p-2} |\nabla v| \cdot |\nabla \psi| dx - \tau b \int_{\Omega} |\omega|^{p-2} \omega \psi dx = 0, \quad \forall \psi \in W^{1,p}_0(\Omega),
\] (2.7)
i.e., \( (A_p, \omega, v) = b(B_p, \omega, v) \) for \( \|\omega\| = 1 \) and for each \( v \in W^{1,p}_0(\Omega) \), which contradicts \( b \) is not the eigenvalue of \( -\Delta_p \),

This prove that the homotopy \( T \) is admissible with respect to the ball \( B(0,R) \) if \( R \) is large enough. Hence, Lemma 2.2 (3) yields that
\[
\deg(A_p - S, B(0,R), 0) = \deg(A_p - bB_p, B(0,R), 0),
\] (2.8)

Note that the value of the degree on the right-hand side of (2.8) is an odd number by Lemma 2.2 (2) and \( b \) is not the eigenvalue of \( -\Delta_p \). Hence \( \deg(A_p - S, B(0,R), 0) \neq 0 \), and Lemma 2.2 (1) indicate that (1.1) has a weak solution. This completes the proof. \( \Box \)

As we have mentioned, we will utilize critical point theory to discuss the problem (1.1). Let us collect some definitions and lemmas that will be used below. One can refer to [17–21] for more details.

**Definition 2.3.** Let \( X \) be a real Banach space, \( J \in C^1(X, \mathbb{R}) \). We say that \( J \) satisfies the \( (Ce)_c \) condition at the level \( c \in \mathbb{R} \) if any sequence \( \{u_m\} \subset X \) with \( J(u_m) \to c, \ (1 + \|u_m\|) J'(u_m) \to 0 \), possesses a convergent subsequence in \( X \).

**Definition 2.4.** Let \( X \) be a real Banach space, \( J \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). We say that \( J \) satisfies the \( (PS)_c \) condition if the existence of a sequence \( \{u_m\} \subset X \) such that \( J(u_m) \to c \) and \( J'(u_m) \to 0 \) as \( m \to \infty \) lead to \( c \) is a critical value of \( J \).

**Lemma 2.4 (Mountain Pass Theorem).** Suppose that \( J \in C^1(X, \mathbb{R}) \) satisfies
\[
\max\{J(0), J(e)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} J(u)
\]
for some \( \alpha < \beta, \rho > 0 \) and \( e \in X \) with \( \|e\| > \rho \).

Let
\[
\Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e \} \quad \text{and} \quad c = \text{inf}_{\gamma \in \Gamma} J(\gamma(1)).
\]

Then \( c \geq \beta > 0 \) and there exists a sequence \( \{u_n\} \subset X \) such that
\[
J(u_n) \to c, \quad (1 + \|u_n\|) J'(u_n) \to 0.
\]
Moreover, if \( J \) satisfies the \( (Ce)_c \) condition, then \( c \) is a critical point of \( J \).

**Lemma 2.5 (Fountain Theorem).** Let \( W^{1,p}_0(\Omega), Y_k \) and \( Z_k \) be defined in Section 1. Suppose that

(A1) \( J \in C^1(W^{1,p}_0(\Omega), \mathbb{R}) \) is an even functional.

If for every \( k \in \mathbb{N} \), there exist \( \rho_k > r_k > 0 \) such that

(A2) \( a_k := \max_{|\Delta u| \leq r_k} J(u) < 0 \).

(A3) \( b_k := \inf_{|\Delta u| \geq r_k} J(u) \to -\infty \) as \( k \to \infty \).

(A4) \( J \) satisfies the \( (PS)_c \) condition for all \( c > 0 \).

Then \( J \) has an unbounded sequence of critical values.

**Theorem 2.2.** Let (H1)–(H3) hold. If \( a < \lambda_1, \ b > \lambda_1 \) and \( r > p \). Then (1.1) has a nontrivial weak solution.

**Proof.** By (H1) and (H2), there exist \( \varepsilon_1 \in (0, \lambda_1) \) and \( c_2 > 0 \) such that
\[
F(x,t) \leq \frac{1}{p} (\lambda_1 - \varepsilon_1) |t|^p + c_2 |t|^r, \quad \text{for any } t \in \mathbb{R} \text{ and } x \in \Omega.
\] (2.9)
Therefore, (1.3) and (1.7) enable us to obtain
\[
J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x, u) \, dx \geq \frac{1}{p} ||u||^p - \int_\Omega \left( \frac{1}{p} (\lambda_1 + \varepsilon_1) ||u||^p + c_2 |u|^r \right) \, dx \\
\geq \frac{1}{p} ||u||^p - \frac{1}{p} \frac{\lambda_1}{\lambda_1} ||u||^p - c_2 C_{\text{emb}} ||u||^r = \frac{\lambda_1}{p} \frac{\lambda_1}{\lambda_1} ||u||^p - c_2 C_{\text{emb}} ||u||^r.
\]

(2.10)

Hence, we can choose \( \rho = \frac{\alpha}{2p \lambda_1} \) such that
\[
J(u) \geq \frac{\lambda_1}{2p \lambda_1} \rho^p, \quad \text{when } ||u|| = \rho.
\]

On the other hand, we assume that \( \lim_{|x| \to \infty} \frac{f(x, t)}{|t|^{p-2} t} \geq \lambda_1 \) uniformly for \( x \in \Omega \). By this, there are \( \varepsilon_2 > 0 \) and \( M_2 > 0 \) such that
\[
F(x, t) \geq \frac{1}{p} (\lambda_1 + \varepsilon_2) |t|^p, \quad \text{for all } x \in \Omega \quad \text{and} \quad |t| \geq M_2.
\]

By the continuity of \( F \), there exists \( c_3 > 0 \) such that
\[
F(x, t) \geq \frac{1}{p} (\lambda_1 + \varepsilon_2) |t|^p - c_3, \quad \text{for all } x \in \Omega \quad \text{and} \quad t \in \mathbb{R}.
\]

(2.11)

Consequently, for \( t > 0 \), we have
\[
J(t \varphi_1) = \frac{t^p}{p} \int_\Omega |\nabla \varphi_1|^p \, dx - \int_\Omega F(x, t \varphi_1) \, dx \leq \frac{t^p \lambda_1}{p} \int_\Omega \left( \frac{1}{p} (\lambda_1 + \varepsilon_2) |t \varphi_1|^p - c_3 \right) \, dx = \frac{t^p \lambda_1}{p} \frac{1}{p} (\lambda_1 + \varepsilon_2) + c_3 |\Omega| \]
\[
= \frac{\varepsilon_2 t^p}{p} + c_3 |\Omega|.
\]

(2.12)

So, for \( t_0 > 0 \) large enough and choosing \( ||e|| = t_0 > \rho \) such that \( J(e) < 0 \).

Assume that \( \{u_n\} \subset W^{1,p}_0(\Omega) \) is a \((C)\)-sequence for \( c > 0 \),
\[
J(u_n) \to c, \quad (1 + ||u_n||)J'(u_n) \to 0 \quad \text{when } n \to \infty.
\]

Note that
\[
(f(u_n), u_n) = ||u_n||^p - \int_\Omega f(x, u_n) u_n \, dx \to 0
\]

(2.13)

and
\[
\int_\Omega |\nabla u_n|^p |\nabla u_n| v \, dx - \int_\Omega f(x, u_n) v \, dx \to 0
\]

(2.14)

for any \( v \in W^{1,p}_0(\Omega) \).

We first claim that \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \). In fact, if not, we may assume that there is a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)) with \( ||u_n|| \to +\infty \), and \( w_n = \frac{u_n}{||u_n||} \). Clearly, \( \{w_n\} \) is bounded in \( W^{1,p}_0(\Omega) \) and we may suppose that there is a \( w \in W^{1,p}_0(\Omega) \) such that (2.2) holds true. Then we claim that \( w \neq 0 \). Otherwise, if \( w = 0 \), we know that \( w_n \to 0 \) strongly in \( L^p(\Omega) \). Dividing (1.4) by \( ||u_n||^p \), we have
\[
\frac{J(u_n)}{||u_n||^p} = \frac{1}{p} - \frac{1}{p} \int_\Omega F(x, u_n) \, dx = \frac{1}{p} - \frac{1}{p} \int_\Omega F(x, u_n) \frac{|u_n|^p}{||u_n||^p} \, dx = \frac{1}{p} \int_\Omega \frac{F(x, u_n)}{||u_n||^p} |w_n|^p \, dx = o(1).
\]

Note that, by (H2) and (H3), there exists \( M_3 > 0 \) such that \( \frac{|f(x, u)|}{|u|^{p-1}} \leq M_3 \), \( \forall x \in \Omega \) and \( t \in \mathbb{R} \). Consequently, we see
\[
\frac{1}{p} = \int_\Omega \frac{f(x, u)}{|u|^p} |w_n|^p + o(1) \leq M_3 \int_\Omega |w_n|^p + o(1) \to 0
\]

which is impossible, so \( w \neq 0 \).

As a result, dividing (2.14) by \( ||u_n||^{p-1} \), we get
\[
\int_\Omega \frac{|\nabla u_n|^{p-2} |\nabla u_n|}{||u_n||^{p-2}} |\nabla v| \, dx \to 0.
\]

That is
\[
\int_\Omega |\nabla w_n|^{p-2} |\nabla w_n| |v| \, dx = \int_\Omega \frac{|f(x, u_n)|}{||u_n||^{p-2}} |w_n|^{p-2} w_n |v| \, dx \to 0.
\]

Lemma 2.3 enables us to find
\[
\int_\Omega |\nabla w|^{p-2} \nabla w \, v \, dx = b \int_\Omega |w|^{p-2} w \, v \, dx.
\]
It is contrary to the fact that \( b \neq \lambda_b \). Hence \( \{u_n\} \) is bounded, as required. Thus, there is a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)), and \( u \in W_0^1(\Omega) \) such that \( u_n \rightharpoonup u \) weakly in \( W_0^1(\Omega) \) and \( u_n \to u \) strongly in \( L^q(\Omega) \). Consequently,
\[
\lim_{n \to \infty} (f(u_n), u_n - u) = 0. \tag{2.15}
\]

From (H1) and the boundedness of \( \{u_n\} \), Hölder inequality enables us to see
\[
\left| \int_{\Omega} f(x, u_n)(u_n - u) \, dx \right| \leq c_1 \int_{\Omega} \left( 1 + |u_n|^\gamma \right) |u_n - u| \, dx \leq c_1 (|\Omega|^{\gamma^{-1}} + |u_n|^{\gamma^{-1}}) |u_n - u|_r \leq c_1 (|\Omega|^{\gamma^{-1}} + C_{\text{emb}} |u_n|^{\gamma^{-1}}) |u_n - u|_r \to 0, \quad \text{as } n \to \infty. \tag{2.16}
\]

Combining (2.15) and (2.16), we obtain that \( \int_{\Omega} |\nabla u|^p \, dx \to 0 \), as \( n \to \infty \). Similarly, we also arrive at \( \int_{\Omega} |\nabla u|^2 \, dx \to 0 \), as \( n \to \infty \), and hence, \( \int_{\Omega} |\nabla u|^2 \, dx \to 0 \), as \( n \to \infty \). Note that
\[
\int_{\Omega} (|\nabla u|^p - |\nabla u|^2 - |\nabla u|^2) (|\nabla u|^p - |\nabla u|^2) \, dx = \|u\|_p^p + \|u\|_p^2 - \left( \int_{\Omega} |\nabla u|^p - |\nabla u|^2 - |\nabla u|^2 \, dx \right) \geq \|u\|_p^p + \|u\|_p^2 - \|u\|_p^{p-1} \|u\| - \|u\|^{p-1} \|u\|_p.
\]
We arrive at immediately \( \lim_{n \to \infty} \int_{\Omega} |\nabla u - \nabla P| \, dx = 0 \), i.e., \( u_n \to u \) strongly in \( W_0^1(\Omega) \). Up to now, all conditions of Lemma 2.4 hold true, and then (1.1) has a nontrivial weak solution. This completes the proof. \( \square \)

In Section 1, we know that for any fixed \( k \), \( \dim Y_k = k < \infty \). For any \( u \in Y_k \), it is easy to verify that \( \| \cdot \|_p \) is a norm of \( Y_k \). Since all the norms of a finite dimensional normed space are equivalent, so there exist two positive constants \( \mu_1, \mu_2 \) such that
\[
\mu_1 \|u\|_p^p \leq \|u\|_p^p \leq \mu_2 \|u\|_p^p, \quad \text{for } u \in Y_k. \tag{2.17}
\]

On the other hand, there exists \( k \) such that \( \lambda_{k+1} > \mu_1^{-1} \) for the fact that \( \lambda_k \to +\infty \). So we have

**Theorem 2.3.** Let (H1)–(H4) hold. Assume that there exists \( k \) such that \( \lambda_{k+1} > \mu_1^{-1} \) and \( \bar{b} \in (\mu_1^{-1}, \lambda_{k+1}) \). Then (1.1) has infinitely many weak solutions.

**Proof.** By the similar method in Theorem 2.2, we see \( I \) satisfies the (PS)_c condition.

By \( \lim_{|t| \to \infty} \frac{f(t)}{|t|^p} \geq \mu_1^{-1} \), there exist \( M_3 > 0 \) and \( \varepsilon_3 > 0 \) such that \( f(x, t) \geq (\mu_1^{-1} + \varepsilon_3) |t|^{p-2} t \), for all \( |t| \geq M_3 \) and \( x \in \Omega \). We know \( f(x, t) = (\mu_1^{-1} + \varepsilon_3) |t|^{p-2} t \) is continuous and bounded on \( x \in \Omega \) and \( |t| \leq M_3 \), and thus there exists \( c_4 > 0 \) such that \( -c_4 < f(x, t) - (\mu_1^{-1} + \varepsilon_3) |t|^{p-2} t \leq c_4 \). Therefore, \( f(x, t) \geq (\mu_1^{-1} + \varepsilon_3) |t|^{p-2} t - c_4 \), \( \forall (x, t) \in \Omega \times \mathbb{R} \). Thus, we have
\[
F(x, t) = \int_0^t f(x, s) \, ds \geq \frac{\mu_1^{-1} + \varepsilon_3}{p} |t|^p - c_4 |t|, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \tag{2.18}
\]

For any \( u \in Y_k \), and in view of (2.17), by Hölder inequality, we obtain
\[
J(u) = \frac{1}{p} \|u\|_p^p - \int_{\Omega} F(x, u) \, dx \leq \frac{1}{p} \|u\|_p^p - \int_{\Omega} \frac{\mu_1^{-1} + \varepsilon_3}{p} |u|^{p-1} \, dx + c_4 |\Omega|^{\frac{1}{p-1}} \|u\|_p \leq \frac{1}{p} \|u\|_p^p \left( 1 - (\mu_1^{-1} + \varepsilon_3) \mu_1 \right) + c_4 \sqrt[p]{\mu_2} |\Omega|^{\frac{1}{p-1}} \|u\|. \tag{2.19}
\]

Since \( 1 - (\mu_1^{-1} + \varepsilon_3) \mu_1 < 0 \), note that \( p > 1 \), then there exists positive constants \( d_4 \) such that
\[
J(u) \leq 0, \quad \text{for each } u \in Y_k \quad \text{and } \|u\| \geq d_4. \tag{2.20}
\]

On the other hand, by \( \lim_{|t| \to \infty} \frac{f(t)}{|t|^p} \leq \lambda_{k+1}^{-1} \), there exist \( M_5 > 0 \) and \( \varepsilon_4 \in (0, \lambda_{k+1}) \) such that \( f(x, t) \leq (\lambda_{k+1}^{-1} - \varepsilon_4) |t|^{p-2} t \), \( \forall |t| \geq M_5 \) and \( x \in \Omega \). For the reason that \( f(x, t) - (\lambda_{k+1}^{-1} - \varepsilon_4) |t|^{p-2} t \) is continuous and bounded on \( |t| \leq M_5 \) and \( x \in \Omega \), then there is a \( c_5 > 0 \) such that \( f(x, t) - (\lambda_{k+1}^{-1} - \varepsilon_4) |t|^{p-2} t \leq c_5, \quad \forall |t| \leq M_5 \) and \( x \in \Omega \). Consequently, \( f(x, t) \leq (\lambda_{k+1}^{-1} - \varepsilon_4) |t|^{p-2} t + c_5, \quad \forall (x, t) \in \Omega \times \mathbb{R} \). Furthermore,
\[
F(x, t) \leq \frac{\lambda_{k+1}^{-1} - \varepsilon_4}{p} |t|^p + c_5 |t|, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \tag{2.21}
\]

For any \( u \in Z_k \), let \( \beta_k := \sup_{(x, t) \in \Omega \times \mathbb{R}} \|u\|_p \). Since \( W_0^1(\Omega) \) is compactly embedded into \( L^q(\Omega) \), there holds, \( \beta_k \to 0, \text{as } k \to \infty \), see Lemma 3.8 of [18]. By (2.21), (1.8) and Hölder inequality, we arrive at
\begin{equation}
J(u) = \frac{1}{p} \left\| u \right\|^p - \frac{1}{p} \int_{\Omega} F(x, u) \, dx \geq \frac{1}{p} \left\| u \right\|^p - \frac{1}{p} \int_{\Omega} \frac{j_{k+1} - \phi_4}{p} |u|^p \, dx - \int_{\Omega} c_5 |u| \, dx
\end{equation}

Choosing \( r_k := 1/\mu \), we easily \( r_k \to \infty \) as \( k \to \infty \), then

\begin{equation}
J(u) \geq \frac{1}{p} \left( 1 - \frac{j_{k+1} - \phi_4}{\lambda_{k+1}} \right) r_k \left( 1 - \frac{j_{k+1} - \phi_4}{\lambda_{k+1}} \right) - c_5 |\Omega|^{\frac{p-1}{2}} \to \infty, \quad \text{as} \quad k \to \infty.
\end{equation}

Hence, \( b_k := \inf_{u \in V_k \setminus \{0\}} J(u) \to -\infty \) as \( k \to \infty \). Combining this and (2.20), we can take \( \rho_k := \max\{d_k, r_k + 1\} \), and thus

\( a_k := \max_{u \in Y_k \setminus \{0\}} \rho_k J(u) \leq 0 \).

(H1) and (H4) enable us to obtain that (A1) and (A4) in Lemma 2.5 are satisfied. Up until now, we have proved the functional \( J \) satisfies all the conditions of Lemma 2.5, then \( J \) has an unbounded sequence of critical values. Equivalently, (1.1) has infinitely many weak solutions. This completes the proof. \( \square \)

References


