Functionally-Weighted Lagrange Interpolation of Band-limited Signals from Non-uniform Samples

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Abstract—A modification of the conventional Lagrange interpolator is proposed in this paper, that allows one to approximate a band-limited signal from its own non-uniform samples with high accuracy. The modification consists in applying the Lagrange method to the signal, but pre-multiplied by a fixed function, and then solving for the desired signal value. Its efficiency lies in the fact that the fixed function is independent of the sampling instants. It is shown in this paper that the function can be selected so that the interpolation error decreases exponentially with the number of samples, for the case in which the sampling instants have a maximum deviation from a uniform grid. The paper includes a low-complexity recursive implementation of the method. Its accuracy is validated in the numerical examples by comparison with several interpolators in the literature, and by deriving upper and lower bounds for its maximum error.

I. INTRODUCTION

In a variety of applications a band-limited signal must be reconstructed from its own non-equally spaced samples. These include the implementation of fast A/D converters [1]–[4], the demodulation of FM signals from its zero crossings, [5, chapter 16], and the computation of the inverse non-uniform DFT [6]. Reference [5] is a general survey on this kind of interpolation and its applications. In contrast with the interpolation from equally-spaced samples, for which there is a wide range of efficient techniques [7], [8], the non-uniform case involves a high computational burden, because the processing is usually performed block-wise and by means of iterative methods, [5, chapter 5], [9], [10]. Recently, the design of fast interpolators for this problem has been addressed by several authors. In [3], [4] a fast procedure based on a bank of FIR filters is proposed, assuming that the sampling is non-uniform but periodic. In [11], the approach consists in fitting a low-order polynomial to the set of known samples. Finally, the efficient implementation of the Lagrange method has been studied in [12].

The purpose of this paper is to present a new variant of the conventional Lagrange interpolator that provides a significant accuracy improvement. Basically, if \( s(t) \) is the signal to interpolate, the proposed method consists in computing the Lagrange interpolant for the product \( s(t)\gamma(t) \), where \( \gamma(t) \) is a fixed function that is independent of the sampling instants, and then solving for \( s(t) \), that is, dividing by \( \gamma(t) \). It is shown that, under specific restrictions on the spreading of the sampling instants, there are functions \( \gamma(t) \) for which this procedure is very accurate. More precisely, if the sampling instants are the result of perturbing a uniform sampling grid by less than half its repetition period, a specific \( \gamma(t) \) can be constructed for which the accuracy increases exponentially with the number of samples. If the sampling instants are equally spaced, this interpolant coincides with that presented in [13].

The implementation of the proposed method involves the efficient computation of the function \( \gamma(t) \) and of the Lagrange interpolant. As will be shown, \( \gamma(t) \) can be approximated using polynomials in a simple way. Regarding the Lagrange interpolant, the algorithms currently available are able to compute it in \( O(N^2) \) operations, where \( N \) is the number of samples, [14, chapter 3], [12]. In this paper, an algorithm will be presented that reduces this complexity to \( O(N) \), if the implementation is recursive.

The paper has been organized as follows. In the next section, the problem of interpolating a band-limited signal from non-equally spaced samples is first analyzed. This is done by viewing the possible interpolation formulas as the result of truncating an infinite sampling series. Then, the procedure proposed in this paper, called functionally-weighted (FW) interpolator, is introduced at the end of the section. Its approximation error is then analyzed in Sec. [II] assuming a maximum deviation from a uniform sampling grid. For this case it is shown that the error decreases exponentially with the number of samples. Afterward, the recursive implementation of the FW interpolator is studied in Sec. [IV] where a method is presented that makes it possible to compute it with complexity proportional to the number of samples. Sec. [V] contains several numerical examples. First, the function \( \gamma(t) \), the error bound and the complexity of the recursive implementation are respectively analyzed in sections [V-A], [V-B], and [V-C]. Then, in Sec. [V-D] the interpolator’s performance in the frequency-domain is compared with that of the conventional Lagrange interpolator and with the upper bound derived in Sec. [II]. In Sec. [V-E] the interpolator’s performance is assessed relative to a lower bound in the time-domain \( L^\infty \) norm. Finally, in Sec. [VI] the FW interpolator is compared with other methods in the literature.

II. LA GRANGE-TYPE SERIES AND THE FUNCTIONALLY-WEIGHTED INTERPOLATOR

Consider the general problem in which a signal \( s(t) \) with spectrum lying in \([-B/2, B/2]\) must be interpolated from its own non-equally spaced samples. More precisely, if \( s(t) \) is known at the set of distinct instants \( t_1, t_2, \ldots, t_N \), the problem consists in finding a set of functions \( g_n(t) \) for which the formula

\[
s(t) \approx \sum_{n=1}^{N} s(t_n) g_n(t)
\]  

(1)
is accurate. From the theoretical point of view, a convenient way to determine the functions \( g_n(t) \) is to consider that (1) is the truncated version of a sampling series
\[
s(t) = \sum_{n=-\infty}^{\infty} s(t_n)g_n(t).
\]
This approach is convenient because there is available a batch of results in the Sampling Theory and Complex Analysis fields that provide series of this type, [15]–[18]. The basic result in these fields is that \( s(t) \) can be perfectly reconstructed from its own non-equally spaced samples by means of a generalization of the Lagrange formula, under specific assumptions on the infinite set \( \{t_n\} \).

In order to introduce this generalization, recall that the Lagrange procedure for the finite set \( t_1, t_2, \ldots, t_N \) starts with the construction of a polynomial that is zero at each of these instants. The polynomial of smallest degree with this property, up to a scale factor, is the Lagrange kernel
\[
L(t) = \prod_{n=1}^{N} \frac{t-t_n}{t(t-t_n)}.
\]
Then, the signal \( s(t) \) can be interpolated by canceling each of these zeros and forming a sum using as weights the samples \( s(t_n) \),
\[
s(t) \approx \sum_{n=1}^{N} s(t_n) \frac{L(t)}{L(t_n)(t-t_n)}.
\]
If the set \( \{t_n\} \) is infinite, the Lagrange procedure is similar. First, it is necessary to form a kernel \( \phi(t) \) whose zeros are the instants \( \{t_n\} \). This can be done by forming the infinite product
\[
\phi(t) = A \prod_{n=-\infty}^{\infty} (1-t/t_n)e^{t/t_n}.
\]
were the prime (') means that the factor should be replaced with \( t \) if \( t_n = 0 \), and \( A \) is any constant, \( A \neq 0 \). The factors \( e^{t/t_n} \) have been introduced in order to guarantee the convergence. (This characterization of a band-limited signal is a direct application of Hadamard’s theorem, [18, page 27].)

The next step consists in forming the Lagrange formula with kernel \( \phi(t) \),
\[
s(t) = \sum_{n=-\infty}^{\infty} s(t_n) \frac{\phi(t)}{\phi'(t_n)(t-t_n)}.
\]
From [18, Lecture 22], this equality is true if \( \phi(t) \) is a so-called sine-type function. Specifically, \( \phi(t) \) must fulfill the following conditions:

1) It is a band-limited signal with two-sided bandwidth \( 1/T \) whose set of zeros is \( \{t_n\} \).
2) The set \( \{t_n\} \) has a minimal separation \( \delta_o > 0 \), i.e., \( |t_n-t_k| \geq \delta_o \) for any pair of indices \( n \neq k \).
3) \( \phi(t) \) is bounded above and below on a straight line which is parallel to the real axis, i.e., there are three real constants \( h, c \) and \( C \) such that
\[
0 < c < |\phi(t+jh)| < C \tag{6}
\]
for any real \( t \).
4) All zeros \( t_n \) lie in a strip centered on the real axis of width \( 2H \), i.e., \( |\text{Im}\{t_n\}| \leq H \).
5) It is \( BT \leq 1 \). This is the Nyquist condition but including the case \( BT = 1 \).

Additionally, \( s(t) \) must belong to the Lebesgue space \( L^p \) with \( 1 < p < \infty \) when restricted to the real axis.

The most widely-known application of this result is the sinc sampling series itself. If the instants \( \{t_n\} \) are equally-spaced, \( t_n = nT \), then the kernel in (1) with \( A = \pi/T \) is
\[
\phi(t) = (\pi t/T) \prod_{n=-\infty}^{\infty} (1-t/(nT))e^{t/(nT)}.
\]
This kernel is actually the sine function \( \sin(\pi t/T) \), and the associated Lagrange formula in (3) is the sinc series,
\[
s(t) = \sum_{n=-\infty}^{\infty} s(nT)\text{sinc}(t/T - n).
\]
[The fact that (7) is the sinc function \( \sin(\pi t/T) \) is a consequence of Hadamard’s theorem, [18, page 27].]

This theoretical result suggests the following method to obtain the functions \( g_n(t) \) in (1). Given the set of sampling instants \( t_1, t_2, \ldots, t_N \), select a \( T \) following \( BT \leq 1 \), and form an infinite set by adding \( t_n \) for \( n < 1 \) and \( n > N \), so that the conditions above on \( \phi(t) \) hold. Then, truncate (5) at indices 1 and \( N \), i.e.,
\[
s(t) \approx \sum_{n=1}^{N} s(t_n) \frac{\phi(t)}{\phi'(t_n)(t-t_n)}\tag{8}
\]
The resulting functions \( g_n(t) \) in (1) are, obviously,
\[
g_n(t) = \frac{\phi(t)}{\phi'(t_n)(t-t_n)}\tag{9}
\]
and the interpolation error is the difference between (5) and (8)
\[
\epsilon(t) = \sum_{n<1,n>N} s(t_n) \frac{\phi(t)}{\phi'(t_n)(t-t_n)}.\tag{9}
\]
In order to use this method in practice, it is necessary to address two questions. First, the error \( \epsilon(t) \) must be small independently of the signal being interpolated, and second, the computational burden related with the computation of the kernel \( \phi(t) \) must be as small as possible.

Regarding the reduction of \( \epsilon(t) \), it seems convenient to use kernels with only real zeros, given that the amplitude of \( s(t) \) may increase exponentially with \( |\text{Im}\{t\}| \). (It may increase as \( e^{\pi BT\text{Im}\{t\}} \) for signals \( s(t) \) which are bounded on the real axis.) Besides, \( s(t) \) is hardly known in practice for non-real \( t \). Also, from (9), the error \( \epsilon(t) \) will be small if the absolute values of the samples \( s(t_n) \) are reduced in some way for \( n < 1 \) and \( n > N \). This reduction can be achieved by means of the windowing method in [13]. For this, assume that \( T \) has been selected so that \( BT < 1 \), and that the interpolator must be accurate for \( t \) lying in a range \( [-\eta T, \eta T] \), \( \eta > 0 \). Also, assume that the instants \( t_1, t_2, \ldots, t_N \) lie in another range \( [-T_o, T_o] \) with \( T_o > \eta T \), but lies outside \( [-T_o, T_o] \) if \( s(t_n) \) is unknown, i.e., for \( n < 1 \) or \( n > N \). Next, let \( w(t) \) denote a window function, which is a band-limited signal with spectrum lying
in \([-B_w/2, B_w/2]\) with \(0 < B_w \leq 1/T - B\). This window is selected so that it is close to one in \([-\eta T, \eta T]\), but close to zero outside \([-T_0, T_0]\), in a sense that will become apparent in the sequel. The condition \(B_w \leq 1/T - B\) implies that the series in (5) is valid for \(s(t)w(t)\). So, if the interpolator in (6) is applied to \(s(t)w(t)\) and \(s(t)\) is solved for, the result is

\[
s(t) \approx \frac{1}{w(t)} \sum_{n=1}^{N} s(t_n)w(t_n) \frac{\phi(t)}{\phi'(t)(t-t_n)},
\]

with interpolation error

\[
\epsilon_w(t) = \sum_{n<1,n>N} s(t_n)w(t_n) \frac{\phi(t)}{\phi'(t)(t-t_n)}.
\]

Now, comparing this last expression with (9), it turns out that each of the summation terms has been damped by the corresponding factor \(w(t_n)/w(t)\); \(w(t_n)\) is close to zero, \(w(t)\) is close to one. Thus, the criterion for the selection of \(w(t)\) is to minimize the magnitude of (11) in order to improve the accuracy of (10).

If the instants \(t_1, t_2, \ldots, t_N\) are equally spaced, there are windows for which \(|\epsilon_w(t)|\) decreases exponentially with \(N\). In [13], [19]. For \(BT\) products close to one, an efficient choice is the approximate prolate (AP) window in (20),

\[
w_{ap}(t) = \frac{\sin(c_B \sqrt{T^2 - T_w^2})}{\sin(c_B T_w)},
\]

with \(B_w \equiv 1/T - B\) and \(T_w \approx T_0\). The favorable truncation properties of the AP window have already been validated for the uniform case in [13], [20]–[22], though other choices are also efficient like the self-truncating window in [19], [23]. In this paper, \(w_{ap}(t)\) will be the window employed and, in Section III it will be shown that the interpolation error decreases exponentially with \(P\) for this window, if the deviation from a uniform grid is smaller than \(T/2\).

It must be noted that the introduction of \(w(t)\) makes the interpolation procedure in (11) applicable to signals \(s(t)\) that belong to \(L^\infty\) on the real axis, provided that \(w(t)\) belongs to \(L^p\) with \(1 < p < \infty\). Also, the fact that the interpolation is performed for \(t\) lying in an interval of the form \([-\eta T, \eta T]\) and for \(t_1, t_2, \ldots, t_N\) in \([-T_0, T_0]\) is not a problem in practice, given that it is always feasible to interpolate a time-shifted version of \(s(t), s(t+\tau)\), for a suitable choice of \(\tau\).

Regarding the computation of \(\phi(t)\), the main drawback is that it depends on the set of instants \(t_1, t_2, \ldots, t_N\), and these may vary from evaluation to evaluation of (10). However, this drawback can be overcome as follows. Consider a specific set \(t_1', t_2', \ldots, t_N'\) with associated kernel \(\phi_0(t)\), for which the interpolator in (10) is applicable. Next, assume that the set of instants must be substituted by a new set \(t_1, t_2, \ldots, t_N\) that also lies in \([-T_0, T_0]\). Then, one may construct a new kernel \(\phi(t)\) from \(\phi_0(t)\) by removing the zeros \(t_1', t_2', \ldots, t_N'\) and adding the zeros \(t_1, t_2, \ldots, t_N\).

\[
\phi(t) = \frac{\phi_0(t) \prod_{n=1}^{N} (t - t_n)}{\prod_{p=1}^{P} (t - t_p')},
\]

This new kernel is also a sine-type function. The straight line in (6) can be the same for both \(\phi_0(t)\) and \(\phi(t)\). The expression in (13) is the product of the Lagrange kernel \(L(t)\) with the function

\[
g(t) = \frac{\phi(t)}{\prod_{p=1}^{P} (t - t_p')},
\]

that is,

\[
\phi(t) = g(t)L(t),
\]

and \(g(t)\) is independent of either \(t_1', t_2', \ldots, t_N'\) or \(t_1, t_2, \ldots, t_N\). Now, observing that \(\phi_0(t) = g(t)g(t)\) for \(n = 1, 2, \ldots, N\), the interpolation formula in (10) can be written as

\[
s(t) \approx \frac{g(t)}{w(t)} \sum_{n=1}^{N} s(t_n)w(t_n) \frac{L(t)}{L(t_n)(t-t_n)}.\]

Finally, define the function

\[
\gamma(t) \equiv \frac{w(t)}{g(t)}
\]

in order to write (14) as

\[
s(t) \approx \frac{1}{\gamma(t)} \sum_{n=1}^{N} s(t_n)\gamma(t_n) \frac{L(t)}{L(t_n)(t-t_n)}.\]

This is the proposed interpolation method, which in the sequel will be called the functionally-weighted (FW) interpolator. The expression in (15) means that, in order to interpolate \(s(t)\) from \(s(t_1), s(t_2), \ldots, s(t_N)\), it is only necessary to evaluate the fixed function \(\gamma(t)\) at \(t_1, t_2, \ldots, t_N\), then compute the Lagrange interpolator using as data

\[
s(t_1)\gamma(t_1), s(t_2)\gamma(t_2), \ldots, s(t_N)\gamma(t_N),
\]

and finally divide the result by \(\gamma(t)\).

### III. Bound On the Error of the FW Interpolator Assuming a Maximum Deviation from a Regular Grid

In the previous section, the accuracy has been improved by means of a window \(w(t)\) which damps the terms of the error signal \(\epsilon_w(t)\) in (11). However, this error also depends on the reference kernel \(\phi_0(t)\) and on the instants \(t_1, t_2, \ldots, t_N\), and the latter may vary from evaluation to evaluation of the FW interpolator. So, it would be convenient to have a bound that only depends on a simple constraint on the set of instants, for fixed reference kernel \(\phi_0(t)\) and window \(w(t)\). The simplest and, probably, the most useful constraint consists in imposing a maximum deviation from a uniform grid with spacing \(T'\).

So, let us repeat the procedure in the previous section, but assuming this constraint on the sampling instants, in order to derive an expression for \(\gamma(t)\). Then, the bound will be deduced by analyzing the error \(\epsilon_w(t)\) for this specific \(\gamma(t)\). In the derivation that follows, it will be assumed that the period associated to the kernel \(\phi(t)\) is equal to \(T'\), i.e., \(T = T'\). This constraint is assumed for simplicity, and is motivated by the fact that this interpolator is very accurate if the instants are regularly spaced, [13].

Assume that \(s(t)\) must be interpolated for \(t\) in a range \([\eta T, \eta T]\), and that it is known at a set of instants around \(t = 0\) of the form \((p + \delta_p)T\) with \(-P \leq p \leq P\), where the
deviations are bounded by a δ, |δ| ≤ δ < 1/2. This set can be viewed as the sub-grid with indices from 1 to N = 2P + 1 of the infinite grid
t_n = \begin{cases} 
(P + n - 1 + \delta, p_n) T, & \text{if } 1 \leq n \leq 2P + 1, \\
0, & \text{otherwise.}
\end{cases}
(16)

If the deviations are zero, δ_p = 0, then \{t_n\} is the set of integer multiples of T, and its corresponding kernel is \sin(\pi t / T). This function can be taken as reference, i.e.,
\phi_o(t) = \sin(\pi t / T).

Now, the actual kernel for each specific set of deviations δ_p can be obtained by removing from \phi_o(t) the zeros of the form pT and adding the zeros \((p + \delta_p)T\) for \(-P \leq p \leq P\). So, at each evaluation the kernel is
\phi(t) \equiv \frac{\sin(\pi t / T)}{L_o(t)},
where \text{L}(t) is the polynomial in (2), with \text{N} = 2P + 1 and \text{t}_n given by (16), and
L_o(t) \equiv \prod_{p=-P}^P t - pT.

The factor that is independent of the deviations δ_p is
g(t) \equiv \frac{\sin(\pi t / T)}{L_o(t)},
and the factorization \phi(t) = g(t)\text{L}(t) holds. Finally, for a given window \text{w}(t) the function \gamma(t)
is
\gamma(t) = \frac{w(t)L_o(t)}{\sin(\pi t / T)}.
(17)

Let us proceed with the analysis of the error \epsilon_w(t) in (11) for this \gamma(t). Since \phi(t) depends on δ but not on \text{B}, and the converse happens with \text{w}(t), it is convenient to bound \epsilon_w(t) by separating these two functions. First, if \text{A}_s is a bound on the samples \text{s}(pT), \{|s(pT)| \leq \text{A}_s\}, then it is from (11):
\[|\epsilon_w(t)| \leq |s(pT)| \phi(t) \frac{w(pT)}{\phi'(pT) w(t)(t - pT)} \leq \text{A}_s \sum_{|p| > P} \left| \frac{\phi(t)}{\phi'(pT)} \frac{w(pT)}{w(t)(t - pT)} \right|.
(18)

Now, if \text{A}_g(\delta) is a bound on \phi(t) / \phi'(pT) for \{|p| > P\},
\left| \frac{\phi(t)}{\phi'(pT)} \right| \leq \text{A}_g(\delta),
the inequality in (18) can be further manipulated to yield
\[|\epsilon_w(t)| \leq \text{A}_s \text{A}_g(\delta) \sum_{|p| > P} \left| \frac{w(pT)}{w(t)(t - pT)} \right|.
(19)

Next, let \text{A}_w(\text{B}) denote a bound on the summation on the right,
\sum_{|p| > P} \left| \frac{w(pT)}{w(t)(t - pT)} \right| \leq \text{A}_w(\text{B}).
(20)

Substituting this inequality in (19) yields
\[|\epsilon_w(t)| \leq \text{A}_s \text{A}_g(\delta) \text{A}_w(\text{B}).
(21)

Finally, the expression for \text{A}_g(\delta),
\text{A}_g(\delta) = \frac{\text{E}_g(1 - \delta)(2P + 1)!}{T^{2P}(P!)^{2} \Gamma(2P + 2 - \delta)},
(22)
has been derived in Appendix I-A, where \Gamma(\cdot) is the Gamma function and \text{E}_\delta is
\text{E}_\delta \equiv \max_{t \in [-T, T]} \prod_{P+1} \left| t - pT + \delta T \right|.
(23)

And the expression for \text{A}_w(\text{B}),
\frac{\text{E}_w(1/2)}{\sinh(\pi T \sqrt{(P + 1)^2 - \eta^2})},
(24)
has been derived in Appendix I-B for the AP window in (12) with \text{T}_w = (P + 1)T. The inequality in (22), together with (23) and (24), will be employed to compute the bounds appearing in the numerical examples in this paper.

Observe that the factors \text{A}_g(\delta) and \text{A}_w(\text{B}) model the contributions of δ and B to the error bound separately. One would expect that both were increasing functions, and \text{A}_w(\text{B}) should become very large as \text{BT} converges to one. This is actually so as is shown in the numerical example in Sec. V-B.

IV. RECURSIVE LAGRANGE-TYPE INTERPOLATION FROM NON-UNIFORM SAMPLES

Up to this point, it was assumed that the instants \text{t}_1, \text{t}_2, \ldots, \text{t}_N lie in a range \([-\text{T}_o, \text{T}_o]\), in order to simplify the derivation of the interpolator. In practice, however, a usual scenario consists in a signal \text{s}(t) which is sampled at the non-uniform stream \{\text{t}_m\}, \text{t}_m < \text{t}_{m+1} for any \text{m}, and \text{s}(t) must be interpolated from its samples at the last N instants \text{t}_{m-N+1}, \text{t}_{m-N+2}, \ldots, \text{t}_m. As in the previous section, it is assumed that the instants \text{t}_m deviate at most \delta T from the uniform grid, i.e., \text{t}_m = \text{mT} + \delta \text{mT}, |\delta| \leq 1/2, and that for each \text{m} the interpolation instant \text{t} lies around \text{mT} - (\text{N} - 1)T/2 in the range
\[(\text{N} - (\text{N} - 1)/2 - \eta T), \text{t}_m - (\text{N} - (\text{N} - 1)/2 + \eta T)].
(25)

The interpolator in the previous section can be easily adapted to this scenario. For this, it is only necessary to delay the function \gamma(t) by an amount
\text{t}_{\text{m}} \equiv (\text{m} - (\text{N} - 1)/2)T.

This delay is equivalent to a time shift on \text{s}(t) that moves the instants \text{t}_{m-N+1}, \text{t}_{m-N+2}, \ldots, \text{t}_m to the range \[-\text{T}_o, \text{T}_o\] and maps the interval in (25) onto \[-\eta T, \eta T\]. The formula in (15), with the function \gamma(t) replaced with \gamma(t - \text{t}_{\text{m}}), is
\text{s}(t) \approx \frac{\text{L}_m(t)}{\gamma(t - \text{t}_{\text{m}})} \sum_{n=m-N+1}^{m} \frac{\text{s}(\text{t}_n)\gamma(t - \text{t}_n)}{\text{L}_m'(\text{t}_n)(t - \text{t}_n)}.
(26)

Here, \text{L}_m(t) is the Lagrange kernel relative to the last N samples,
\text{L}_m(t) \equiv \prod_{k=m-N+1}^{m} (t - \text{t}_k).
(27)
and $L_m(t_n)$ has the well-known expression

$$L'_m(t_n) = \prod_{k=m-N+1}^{m} t_n - t_k. \quad (28)$$

In discrete-time systems, what is usually available is not the stream $\{t_n\}$ but the stream of deviations $\{\delta_m\}$, and $s(t)$ must be interpolated at $mT + u_mT$, where $u_m$ lies in $\left[ -(N-1)/2 - \eta, -(N-1)/2 + \eta \right]$, and is given by another input stream $\{u_m\}$. The derivation for this case would be the same as the present one, though the notation would be more cumbersome. The equivalent to (26) using the streams $\{\delta_m\}$ and $\{u_m\}$ can be found in Appendix II.

The evaluation of (26) involves several evaluations of the function $\gamma(t)$ and the computation of the conventional Lagrange interpolator. The evaluation of $\gamma(t)$ can be performed by dividing its support into sub-intervals, and then pre-computing for each of them a polynomial approximation using standard procedures, like Chebyshev interpolation [14, section 5.8]. (See the numerical example in [V-A]) The total cost of these evaluations is $O(N)$.

Regarding the computation of the Lagrange interpolant, it can be seen in expression (26) that it involves various operations with complexity $O(N)$, and one operation with complexity $O(N^2)$, namely, the computation of $L'_m(t_{m-q})$ for $q = 1, 2, \ldots, N - 1$. This last operation is what gives the $O(N^2)$ complexity in standard algorithms for this interpolant like Aitken’s or Neville’s, [14, chapter 3]. The reduction of this complexity has been intensively addressed in the signal processing literature assuming uniform sampling, [24–26]. However, it seems not to be realized that it is possible to recursively compute $L'_m(t_{m-q})$ with complexity $O(N)$, and then the total complexity is also $O(N)$. For the non-uniform sampling case this possibility was only envisaged in [12]. The recursive evaluation of $L'_m(t_{m-q})$ with complexity $O(N)$ can be done as follows.

The interpolator in (26) makes use of the samples at $N$ consecutive instants. If $t_n$ is any of them, then it may be the one with the largest sub-index appearing in (26). In this case, the instant with the smallest sub-index is $t_{n-N+1}$. In any other case the smallest sub-index is larger than $n - N + 1$. All this implies that, as a function of $n$, the smallest $k$ that may appear in (28) is $n - N + 1$. Next, the first limit for $k$ in (28) can be extended down to this lower limit $n - N + 1$, if the added factors are canceled, i.e.,

$$L'_m(t_n) = \prod_{k=n-N+1}^{m} t_n - t_k \left/ \prod_{k=n-N+1}^{m-N} t_n - t_k \right.. \quad (29)$$

Here, the second product must be taken equal to one if $m = n$. In (26) and in this last equation, the difference $m - n$ ranges from 0 to $N - 1$. Thus, it is possible to replace $n$ with $m - q$, where $q$ is a new index that varies from 0 to $N - 1$. In terms of $q$, Eqs. (26) and (29) can be written as

$$s(t) \approx \frac{L_m(t)}{\gamma(t - \tau_m)} \sum_{q=0}^{N-1} s(t_{m-q}) \gamma(t_{m-q} - \tau_m) \frac{L'_m(t_{m-q})}{L'_m(t_{m-q})} (t - t_{m-q}) \quad (30)$$

and

$$L'_m(t_{m-q}) = \prod_{k=m-q-N+1}^{m} t_{m-q} - t_k \left/ \prod_{k=m-q-N+1}^{m-N} t_{m-q} - t_k \right..$$

Finally, let $\alpha_{m,q}$ and $\beta_{m,q}$ respectively denote the products in the numerator and in the denominator of this last expression,

$$L'_m(t_{m-q}) = \alpha_{m,q} / \beta_{m,q}. \quad (31)$$

It can be easily checked that $\alpha_{m,q}$ and $\beta_{m,q}$ can be recursively updated by means of the formulas

$$\alpha_{m,q} = \begin{cases} \prod_{k=m-q+1}^{m-1} t_{m-q} - t_k & \text{if } q = 0, \\ \alpha_{m-1,q-1}(t_{m-q} - t_m) & \text{if } 1 \leq q \leq N - 1, \end{cases} \quad (31)$$

and

$$\beta_{m,q} = \begin{cases} 1 & \text{if } q = 0, \\ \beta_{m-1,q-1}(t_{m-q} - t_{m-N}) & \text{if } 1 \leq q \leq N - 1. \end{cases} \quad (32)$$

Since the total cost of evaluating the last two formulas for $q = 0, 1, \ldots, N - 1$ is $O(N)$, it follows that all the required values of $L'_m(t_{m-q})$ can be computed in $O(N)$ operations.

An interesting property of the update formulas in (31) and (32) is that they restart themselves once every $N$ samples. Therefore, any errors in the values of $t_n$ cannot accumulate as $n$ increases.

V. NUMERICAL EVALUATION OF THE FW INTERPOLATOR

This section contains a numerical example whose purpose is to give specific answers on the performance of the FW interpolator. The examples also include the performance of the conventional Lagrange interpolator, in order to stress the improvements provided by the function $\gamma(t)$. These two interpolators have been termed conventional Lagrange (CL) and functionally weighted (FW).

Unless otherwise stated, the parameters have been fixed to the following values:

- Sampling period for FW interpolator: $T = 1$.
- Signal two-sided bandwidth: $B = 0.7$.
- Interpolator length: $N = 17$.
- Maximum deviation from the uniform grid: $\delta = 0.25$.
- Interpolation range semi-length: $\eta = 0.5$.
- Signal amplitude: $A_s = 1$.

The numerical evaluation of the function $\gamma(t)$ and its influence on the numerical stability of the Lagrange interpolator are studied in Sec. III. Then, the exponential dependence of the upper bound in Sec. III is checked in V-B numerically. Sec. V-C contains an analysis of the computational burden of the recursive implementation in Sec. IV. Sec. V-C compares the error performance of the FW interpolator in the frequency domain with that of the CL interpolator and with the upper bound in Sec. III. The numerical results presented in this subsection serve to assess the significant improvement in accuracy provided by the function $\gamma(t)$. Finally, in Sec. V-E, a lower
bound on the maximum interpolation error is numerically computed. The purpose of this last sub-section is to compare the performance of the FW interpolator with the optimal one in $L^\infty$ norm.

A. The $\gamma(t)$ function

Fig. 1 shows the function $\gamma(t)$ (continuous curve). It increases with |t|. This implies that the magnitude of the samples far off the range $[-\eta T, \eta T]$ is enlarged by $\gamma(t)$. This feature is beneficial, given that it reduces the sensitivity of the interpolator to any perturbation of the sample values. For instance, for the uniform sampling grid ($\delta_n = 0$), Fig. 1 contains the values of $L'(t_n)$, (upper crosses). Note that the range of variation of $L'(t_n)$ is about 82.19 dB, but the range of variation of $L'(t_n)/\gamma(t_n)$ is only 33.15 dB. So, the function $\gamma(t)$ provides a significant reduction of the variation range, and this compensates the ill-conditioning of the Lagrange interpolator for large $N$, [27]. Finally, it must be noted that $\gamma(t)$ can be multiplied by any non-zero constant without affecting the interpolator. This can be readily checked in Eq. (15). Thus, the fact that $\gamma(t)$ takes on large values in Fig. 1 is only incidental, given that $\gamma(t)$ can be replaced with for example $\gamma(t)/\gamma(0)$.

Regarding the efficient computation of $\gamma(t)$, it can be well approximated by low order polynomials in the ranges $[pT - \delta T, pT + \delta T]$,

$$\gamma(pT + x) \approx \sum_{q=0}^{Q_s-1} a_{p,q} x^q. \quad (33)$$

For example, if six digits of precision are required, then it is enough to select $Q_0 = 5$ and $Q_s = 8$, using Chebyshev interpolation, [14, section 5.8]. The resulting average number of coefficients in (33) is 5.7. The computation of $\gamma(t)$ in the range $[-\eta T, \eta T]$ would be done in a similar way.

B. Bound on the interpolation error

Fig. 2 shows the factor $A_g(\delta)$, which is independent of $B$, as a function of the maximum deviation $\delta$, for several semi-lengths $P$. Note the strong dependence of this factor on $\delta$. It must be noted that the bound in (21) is an $L^\infty$ bound, i.e., it bounds the maximum error for the worst distributions of sampling instants, like that in (35).

Fig. 3 presents the factor $A_w(B)$, which is independent of $\delta$, as a function of $BT$ for several semi-lengths $P$. It is clear form this figure how the bound increases as $BT$ approaches one. Figs. 2 and 3 allow one to compute a bound on the interpolation error for specific values of the parameters involved. For instance, if $\delta = 0.3$ and $P = 8$ it is $A_g(\delta) = 18.55$ dB (Fig. 2). If besides $BT = 0.6$, then it is $A_w(B) = -76.13$ dB (Fig. 3). Therefore, for a normalized signal, $A_s = 0$ dB, the error bound is $A_s A_g(\delta) A_w(B) = -57.58$ dB, [Eq. (21)].

C. Computational burden of the recursive implementation

A special feature of the FW interpolator is that it is usable if $t$ lies in a range $[-\eta T, \eta T]$, whose length $2\eta T$ may be larger than $T$. However, in the usual case, $t$ may take any value in consecutive $T$-length intervals $[mT - T/2, mT + T/2]$. So, $\eta = 1/2$ would be enough to carry out the interpolation. However, most of the computational cost of the recursive implementation in Sec. [V] is given by the evaluation of functions like $\gamma(t_m - q - \tau_m)$ or $L'(t_m - q)$, which do not depend on the interpolation instant $t$. This implies that there would be
significant computational savings if $\eta > 1/2$. The parameters $\eta$ and $N$ are related by the error bound: if $\eta$ is increased or $N$ is decreased, the error bound increases (less accuracy) and vice versa. Besides, $N$ and $\eta$ determine the computational burden of the interpolator: if $N$ is increased, then each evaluation of the interpolator involves a summation with more summands in (30), but if $\eta$ is increased then the values of $\gamma(t_m-q-t_m)$ and $\mathcal{L}((t_m-q)$ can be used more than once. The complexity of the recursive FW interpolator, measured using the number of multiplications/divisions per interpolated value, is

$$N_\eta - 1 + 6N + \frac{N}{2\eta}(1 + N_\eta),$$

(34)

where $N_\eta$ is the number of coefficients of the polynomial approximation to $\gamma(t)$. In this expression, it is assumed that there is one interpolated value per input sample. For $N_\eta = 5.7$, Table I contains the number of operations for several pairs $(N, \eta)$, for which the error bound is below 0.012. Observe that the less complex option is $N = 19, \eta = 2$, which corresponds to reusing the FW interpolator for four consecutive values of $t$.

### D. Frequency-domain performance

If $\hat{\phi}(f, t)$ is the value delivered by a given interpolator when its input is $\phi(f, t) \equiv e^{2\pi ft}$, the performance is evaluated in the sequel using the following measures:

- Assuming that $t$ and the deviations $\delta_n$ are uniformly distributed in $[-\eta T, \eta T]$ and $[-\delta T, \delta T]$, respectively, the root-mean-square (RMS) error is

$$\chi_{rms}(f) \equiv \sqrt{\mathbf{E}\{|\phi(f, t) - \hat{\phi}(f, t)|^2\}},$$

where $\mathbf{E}\{\cdot\}$ is the expectation operator.

- A numerical approximation to the maximum error is

$$\chi_{mc}(f) \equiv \max_{N, \eta, t \in [-\eta T, \eta T]} \max_{|\phi(f, t) - \hat{\phi}(f, t)|}.$$

Here, the notation $\max_{N, \eta, t \in [-\eta T, \eta T]}$ means that the maximum was obtained from $N, \eta$ Monte Carlo trials in which the deviations $\delta_n$ were realizations of a uniform distribution with support $[-\delta, \delta]$. In this numerical example, it is $N = 400$.

- In Appendix I, around Eq. (46), it is shown that a specially unfavorable instant distribution is

$$t_n = \begin{cases} 
 n - 1 - (N - 1)/2 - \delta & \text{if } n \leq (N + 1)/2 \\
 n - 1 - (N - 1)/2 + \delta & \text{if } n > (N + 1)/2,
\end{cases}$$

(35)

$\chi_{rms}(f)$ shows the error spectrum of both interpolators, together with the error bound in Eq. (21). The dashed-dotted curves correspond to $\chi_{rms}(f)$. Observe that the performance of the CL interpolator is excellent at low frequencies but poor at high frequencies. This feature is well known in the literature, [8]. In contrast, the FW interpolator shows an equiripple behavior in the whole band [0, $B/2$]. The continuous curves correspond to $\chi_{mc}(f)$. It is clear that this error measure is much more demanding than $\chi_{rms}(f)$. The error bound for the FW interpolator is 22.5 dB above the maximum error. However, it cannot be inferred that the error bound is un-tight, given that $\chi_{mc}(f)$ is the result of a Monte Carlo simulation, and there are too many variables involved (set of deviations $\delta_n$). In order to show that the maximum error can be larger, this figure also contains $\chi_{uc}(f)$, (dashed curves). The maximum of $\chi_{uc}(f)$ is only 11 dB below the error bound.

It is interesting to compare these results with those obtained in the uniform case, i.e., for $\delta = 0$. Fig. 5 shows $\chi_{uc}(f)$ and $\chi_{rms}(f)$ for both interpolators in this case. Note that $\chi_{mc}(f) = \chi_{uc}(f)$ since $\delta = 0$. As could be expected, the performance is better that in the non-uniform case. Actually, the bound in Fig. 5 is 23.7 dB below the bound in Fig. 4.

### E. Time-domain performance

The analysis of the interpolation error in the time domain is relevant for those applications in which large interpolation errors at specific instants cannot be tolerated. However, it is difficult to assess how large the error at any specific instant can be from the performance of the interpolator on a given random signal. To illustrate this point, consider the linear combination of sinc pulses

$$\sum_{p=-20}^{N_t+20} a_p \text{sinc}(Bt - p).$$

(36)
with \( N_s = 1000 \). The coefficients \( a_p \) have been obtained from a uniform distribution in \([-1, 1]\), but they have been re-scaled so that the maximum amplitude of \((36)\) is equal to one, \((A_s = 1)\). Then, the recursive implementation of the FW interpolator in Sec. V has been applied to the samples of \( s(t) \) taken at instants \((k + \delta_k)T\) with \( 0 \leq k < 1000 \) and \( \delta_k \) uniformly distributed in \([-\delta, \delta]\). Fig. 6 shows the interpolation error. Observe that the magnitude of the maximum error in below \( 4 \cdot 10^{-4} \) \((-70.46\, \text{dB})\). Thus, the error bound 0.011 \((-39.17\, \text{dB})\) seems very pessimistic. However, this conclusion may be wrong. For example, it was shown in Sec. V-D that the error can be only 11 dB below the error bound for unfavorable distributions of the sampling instants, [curve for \( \chi_{\text{uc}}(f) \)] in Fig. 4. In this situation, it would be convenient to have a lower bound on the error of any interpolation method, in order to assess how far from optimal the performance of the FW interpolator is. In order to obtain such a bound, it is necessary to resort to the results in [21].

In this reference, it was demonstrated that under weak constraints on the set of instants \( t_1, t_2, \ldots, t_N \), the optimal interpolator is of the form in \((8)\). Its corresponding optimal kernel \( \phi_g(t) \) is
\[
\phi_g(t) = b(t) L(t)/M(t)
\]
where \( M(t) \) is a polynomial with \( N - 1 \) real zeros \( m_k \),
\[
M(t) = \prod_{k=1}^{N-1} (t - m_k),
\]
\( Z(t) \) is a polynomial with \( N - 1 \) pairs of conjugate zeros, \((z_k, z_k^*)\),
\[
Z(t) = \prod_{k=1}^{N-1} (t - z_k)(t - z_k^*)
\]
and \( b(t) \) is the signal
\[
b(t) = \sin \left( \pi BT \int_{t_1}^{t} \frac{M(x)}{\sqrt{Z(x)}} \, dx \right). \tag{37}
\]
The coefficients \( m_k, z_k \) can be obtained from the conditions that \( b(t) \) must satisfy. First, it must be \( b(t_n) = 0 \) for \( n = 1, 2, \ldots, N \), i.e.,
\[
\int_{t_1}^{t_n} \frac{M(x)}{\sqrt{Z(x)}} \, dx = 0. \tag{38}
\]
And second, \( b(t) \) must have complex derivative at the zeros of \( Z(t) \). This condition is the same as
\[
\cos \left( \pi BT \int_{t_1}^{t_k} \frac{M(x)}{\sqrt{Z(x)}} \, dx \right) = 0 \tag{39}
\]
for \( k = 1, \ldots, N - 1 \). Eqs. \((38)\) and \((39)\) determine \( b(t) \) uniquely.

The signal \( b(t) \) can be used to numerically derive the desired lower bound. Fig. 7 shows \( b(t) \) for the unfavorable set of instants in \((35)\). Observe that, for large \( t \), \( b(t) \) is a sinusoidal of frequency \( B/2 \) and amplitude \( A_s = 1 \), and is close to zero in a range covering \( t_1 \leq t \leq t_N \). Fig. 8 depicts again \( b(t) \) but enlarging the range \([t_1, t_N]\). Note that \( b(t_n) = 0 \) exactly at each \( t_n \) in \((35)\). The fact that the sampling instants of any interpolator of the form in \((1)\) coincide with some of the zeros of \( b(t) \) implies that the output of any interpolator \((1)\) will be
Table II

<table>
<thead>
<tr>
<th>Error / bound (dB)</th>
<th>Upper bound on FW interp. error (Fig. 4)</th>
<th>-39.17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FW interp. error with unfavorable instant set (Fig. 4)</td>
<td>-50.46</td>
</tr>
<tr>
<td></td>
<td>Lower bound for any interpolator (Fig. 8)</td>
<td>-54.07</td>
</tr>
<tr>
<td></td>
<td>FW interp. error with random signal (Fig. 6)</td>
<td>-70.46</td>
</tr>
</tbody>
</table>

TABLE II

ERROR FIGURES AND BOUNDS FOR THE EXAMPLE IN Sec. V-E, ASSUMING INPUT SIGNALS WITH MAGNITUDE SMALLER OR EQUAL TO ONE. 

zero if its input is b(t). So, the magnitude |b(t)| is a lower bound for the maximum interpolation error of any interpolator \([1] \) for any t. From Fig. 8 the maximum of b(t) in \([-\eta T, \eta T]\) is \(1.98 \times 10^{-3}\), \((-54.07\, \text{dB})\). So, this figure is the desired lower bound on the error of any interpolation method.

Table II summarizes the error bounds and figures in this numerical example, assuming that the magnitude of the input signal is smaller or equal to one. The error of the FW interpolator when its input is the random signal in (36) is \(-70.46\, \text{dB}\). However, this figure is deceiving, because for undamped exponentials the error can be much larger \((-50.46\, \text{dB})\). Besides, there is no interpolator whose error is below \(-54.47\, \text{dB}\) for all input signals.

F. Variation of interpolation error with signal bandwidth B

The results in the previous sub-sections have shown that the performance of the FW interpolator is almost equi-ripple in the band \([-B/2, B/2]\). However, if B is increased one may expect a deterioration of the interpolator’s performance. This supposition is confirmed by Fig. 9 in which the maximum time-domain error for the signal in (36) is plotted versus the bandwidth B. For this figure a different random signal was generated for each value of B.

VI. COMPARISON WITH OTHER INTERPOLATORS IN THE LITERATURE

In the literature there are various approaches for the interpolation from non-uniform samples. In the sequel, four of them are briefly commented on and compared with the FW interpolator.

The first approach is based on Yen’s third theorem in [15], and consists in forming a sampling series, assuming that the set of instants \(t_1, t_2, \ldots, t_N\) is repeated periodically. Actually, Yen’s sampling series is of Lagrange-type, and most of the results in this paper can be applied to its truncated version. Let us see how this can be done.

The sampling instants \(t_1, t_2, \ldots, t_N\) can be regarded as a finite sub-grid of the infinite grid \(t_n + r/B\) with \(1 \leq n \leq N\) and integer r. Therefore, its corresponding kernel of bandwidth B is

\[
\phi_y(t) \equiv \prod_{n=1}^{N} \sin(\pi B(t - t_n)/N). \tag{40}
\]

Since \(\phi_y(t)\) fulfills the conditions in Sec. II, \(s(t)\) can be reconstructed by means of the Lagrange-type series

\[
s(t) = \sum_{n=-\infty}^{\infty} s(t_n) \frac{\phi_y(t)}{\phi_y(t_n)(t - t_n)}, \tag{41}
\]

where now \(t_n\) for \(n < 1\) or \(n > N\) is defined by repeating \(t_n\) with \(1 \leq n \leq N\) periodically, i.e.,

\[
t_n = t_{\text{mod}(n-1,N)} + \lfloor(n - 1)/N\rfloor/B.
\]

It can be easily checked from (40) that (41) is, actually, Yen’s sampling series in [15, Theorem 3, Eq. (9)]. Finally, if (41) is truncated at indices 1 and N, the result is Yen’s interpolator,

\[
s(t) \approx \sum_{n=1}^{N} s(t_n) \frac{\phi_y(t)}{\phi_y(t_n)(t - t_n)}. \tag{42}
\]

The function \(g(t)\) corresponding to \(\phi_y(t)\) is obtained by removing from \(\phi_y(t)\) the zeros at \(t = t_1, t_2, \ldots, t_N\). Doing this on (40) yields

\[
g_y(t, \{t_n\}) \equiv \prod_{n=1}^{N} \sin(B(t - t_n)/N)
\]

Since the samples \(s(t_n)\) are not weighted in (42) in any way, the corresponding window function is \(w_y(t) \equiv 1\). So, the \(\gamma(t)\)
function corresponding to $\phi_y(t)$ is

$$
\gamma_y(t, \{t_n\}) \equiv \left( \prod_{n=1}^{N} \text{sinc}(B(t - t_n)/N) \right)^{-1}.
$$

The recursive method in Sec. [IV] is therefore applicable to Yen’s interpolator. The drawback is that $\gamma_y(t, \{t_n\})$ depends on the sampling instants and, as a consequence, the complexity of evaluating $\gamma_y(t, \{t_n\})$ is $O(N)$, and the total complexity is $O(N^2)$. In the second approach, the signal is modeled using trigonometric polynomials, or is assumed to be approximately time-limited, [9], [28], [29]. For example, in [28], the model

$$
\sum_{p=-P}^{P} c_p e^{j2\pi p ft} (43)
$$

with $P\Delta f = B/2$ and $t_N - t_1 < 1/\Delta f$ is fitted to the known signal samples, using a fast implementation of the weighted least squares method. The result is the set of coefficients $c_p$, and the signal is then interpolated using (43). In this approach ([9], [28], [29]), the computation of the desired samples involves the inversion of a symmetric matrix of size $N \times N$, and the interpolation is usually performed block-wise.

The third approach is that of H. Johansson et al [30], in which $s(t)$ is interpolated by means of the formula

$$
s(t) \approx \sum_{n=1}^{N} s(t_n) a_n,
$$

where the coefficients $a_n$ depend on the instants $\{t_n\}$, and are the ones that minimize the RMS error in the band $[-B/2, B/2]$. This condition is equivalent to

$$
\{a_n\} = \arg \min_{a_p} \int_{-B/2}^{B/2} W(f) \left| 1 - \sum_{p=1}^{N} e^{-j2\pi f t_p a_p} \right|^2 df,
$$

where $W(f)$ is a positive weighting function. If $W(f) = 1$ this interpolator is the optimal one for stationary processes with flat spectrum in $[-B/2, B/2]$. The computation of the coefficients $a_p$ involves the inversion of an $N \times N$ symmetric matrix for each interpolated value.

Finally, the fourth approach is that of S. Tertinek et al. [31]. Here, $s(nT)$ is interpolated using a truncated Taylor series,

$$
s(nT) \approx s(t_n) - \sum_{k=1}^{K} s^{(k)}(nT)(t_n - nT)^k,
$$

where $t_n$ is the grid instant lying closest to $nT$, and the derivatives $s^{(k)}(nT)$ are approximated by the output of one or more FIR differentiators.

In order to compare these four approaches with that proposed in this paper, several signals of the form in (36) with $N_g = 10^4$ and $B = 0.7T$ were sampled at instants $(k+\delta_k)T$, with $0 \leq \delta_k \leq N_g$ and $N_g = 10^4$, where the deviations $\delta_k$ were uniformly distributed in $[-\delta, \delta]$ with $\delta = 0.25$. The signals were then interpolated in the uniform grid $kT$ from the $2P+1$ samples lying closest to each of the grid’s instants. This was done for the following interpolators:

**Y**: Yen’s interpolator in third theorem of [15], Eq. (9).

**ST**: Trigonometric interpolator in Strohmer’s paper [28].

**TU**: Tuncer’s interpolator in [9], Method 2, for approximately time- and band-limited signals. The sampling instants $t_n$ were moved to the closest integer multiple of $T/L$ with $L = 20$.

**MG**: Margolis’s interpolator in [29, Sec. IV] based on frames.

**JH**: Johansson’s interpolator in [30, Sec. 4] based on minimizing the RMS error with weighting function $W(f) = 1$.

**TE**: Tertinek’s differentiator-multiplier cascade in [31, Sec. 5]. For this interpolator it was $\delta = 0.1$ and it was evaluated in a separate simulation.

**FW**: Interpolator proposed in this paper using the window $w_{ap}(t)$ in (12) with $T_w = (P + 1)T$.

**Fig. 10.** RMS error versus semi-length $P$ for all interpolators except TE. Crosses (+) represent the values of interpolator TU, and the curve corresponding to interpolator ST is dashed. (δ = 0.25.)

The complexities of the interpolators FW and Y are $O(P)$ and $O(P^2)$, respectively. A more accurate operation count for the FW interpolator is that in (34). Regarding the interpolators ST, TU, MG and JH, they involve the solution of a linear system or a linear least-squares problem, and their actual complexity varies with the method used to solve it. As a guiding value, it can be said that to solve
The differentiators in Tertinek’s interpolator TE had P\delta interpolators versus the SNR for case, (Fig. 10).

The output SNR is roughly the same until a given error floor (1 with applications, given that the accuracy increases exponentially even though every of its output samples depends exclusively on 2P + 1 input samples while, for example, every output sample of the TE3 interpolator depends on 12P + 1 = 121 input samples. However, it must be mentioned that there is an alternative implementation of TE3 that reduces this number to 6P + 1 = 61. See [31, Fig. 9]. The design complexity of the TE interpolator is negligible, given that it only involves the computation of a single differentiation filter. This is also the case for the FW interpolator, because \gamma(t) is given in closed form by (17) and (12), and because the polynomial approximations in (33) can be computed by means of Chebyshev interpolation, [14, section 5.8].

VII. Conclusions

A simple modification of the conventional Lagrange interpolator, named functionally-weighted (FW) interpolator, has been presented, that permits one to interpolate a given band-limited signal from its non-uniformly spaced samples with high accuracy. Basically, the FW interpolator consists in applying the Lagrange interpolant to the signal to interpolate but multiplied by a fixed function, and then solving for the desired signal value. It has been shown that, if the sampling instants have a maximum deviation from a uniformly-spaced grid, its accuracy increases exponentially with the number samples. A low-complexity recursive implementation for this interpolator has been presented, whose complexity is proportional to the number of samples involved in the interpolation. It has been shown numerically that its error spectrum is almost equiripple, and its performance has been evaluated relative to a lower bound which is valid for any interpolator. Finally, its RMS error has been compared with that of several interpolation methods in the literature.

Appendix I

Derivation of Bound Factors

A. Bound on \(|\phi(t)/\phi'(pT)|\) in Sec. III

Let us first bound \(|\phi(t)|\) and then \(|1/\phi'(pT)|\). Assume that \(t \geq 0\) and \(\eta < P\). The kernel can be factorized as \(\phi(t) = g(t)L(t)\), where

\[
g(t) = \frac{\sin(\pi t/T)}{L_o(t)}
\]  

(44)

and

\[
L(t) = \prod_{p=-P}^{P} t - pT - \delta_T.
\]  

(45)

The absolute value of \(g(t)\) in terms of the gamma function \(\Gamma(\cdot)\) is

\[
|g(t)| = \frac{\pi T^{-(2P+1)}}{\Gamma(t/T + P + 1)\Gamma(-t/T + P + 1)}.
\]

Since this function decreases with |t|, its maximum value is located at \(t = 0\) and, recalling the property \(\Gamma(P + 1) = P!\), it follows that

\[
\left| \frac{\sin(\pi t/T)}{L_o(t)} \right| \leq \frac{\pi}{T^{2P+1}(P!)^2}.
\]  

(46)
Regarding $L(t)$, it follows from (45) that it admits the bound

$$|L(t)| \leq \prod_{p=-P}^{P} (|t - pT| + \delta T).$$

Let $E_\delta(t)$ denote the term on the right in this inequality. It can be easily checked that

$$
\frac{E_\delta(t + T)}{E_\delta(t)} = 1 + \frac{2t + T}{P'T + \delta T - t} \geq 1.
$$

This implies that the maximum of $E_\delta(t)$ for $0 \leq t \leq \eta T$ is located in the range $[(\eta - 1)T, \eta T]$. If $E_\delta$ denotes the value of this maximum,

$$E_\delta \equiv \max_{t \in [(\eta - 1)T, \eta T]} E_\delta(t),$$

then it is

$$L(t) \leq E_\delta.$$  

The bound on $\phi(t)$ is the result of combining this inequality with (46),

$$|\phi(t)| \leq \frac{\pi E_\delta}{T^{2P+1}(P!)^2}. \quad (47)$$

The derivation for $t < 0$ would be similar.

Let us proceed with the bound on $1/|\phi'(pT)|$. First, it is convenient to factor out the zero of $\phi(t)$ at $t = pT$. Since, it is

$$\sin(\pi t/T) = \text{sinc}(t/T - p)(-1)^p \pi(t/T - p),$$

the kernel in (44) can be written as

$$\phi(t) = (-1)^p \pi \frac{L(t)}{\text{Lo}(t)} \text{sinc}(t/T - p)(t/T - p).$$

Now, if this expression is differentiated, the following formula for $\phi'(pT)$ is obtained:

$$\phi'(pT) = \frac{(-1)^p \pi}{T} \frac{L(pT)}{\text{Lo}(pT)}.\quad (48)$$

For $p > P$, the quotient $|L(pT)/\text{Lo}(pT)|$ can be bounded as follows:

$$
\frac{|L(pT)|}{|\text{Lo}(pT)|} = \left| \prod_{r=-P}^{P} \frac{p - r - \delta_r}{p - r} \right| \geq \prod_{r=-P}^{P} \frac{p - r - \delta}{p - r} = \prod_{r=-P}^{P} \frac{P + 1 - r - \delta}{P + 1 - r} = \frac{\Gamma(2P + 2 - \delta)}{\Gamma(1 - \delta)(2P + 1)!}.
$$

So, it is

$$|\phi'(pT)| \geq \frac{\pi}{T} \frac{\Gamma(2P + 2 - \delta)}{\Gamma(1 - \delta)(2P + 1)!}.$$  

By symmetry this result is also valid for $p < -P$.

The bound $A_\delta(\delta)$ is the result of combining this inequality with (47),

$$\left| \frac{\phi(t)}{\phi'(pT)} \right| \leq \frac{E_\delta \Gamma(1 - \delta)(2P + 1)!}{T^{2P}(P!)^2 \Gamma(2P + 2 - \delta)} \equiv A_\delta(\delta).$$

**B. Bound on window factor**

Let $F_w(t)$ denote the window factor in (20),

$$F_w(t) \equiv \sum_{|p| > P} \left| \frac{w(pT)}{w(t)(t - pT)} \right|.$$  

If $w(t)$ is a symmetric function that decreases in $[0, \eta T]$, $F_w(t)$ can be bounded by replacing $w(t)$ by $w(\eta T)$ and $t$ by $\eta T$,

$$F_w(t) \leq \sum_{|p| > P} \left| \frac{w(pT)}{w(\eta T)(\eta - p)} \right| \leq \frac{2}{T} \sum_{p > P} \left| \frac{w(pT)}{w(\eta T)(\eta - p)} \right|.$$  

The Approximate Prolate (AP) window is

$$w(t) \equiv \frac{\text{sinc}(B_w \sqrt{T^2 - T_w^2})}{\text{sinc}(\sqrt{B_w T_w})}, \quad (49)$$

where $2T_w$ is approximately the width of its main time lobe, and $B_w \equiv 1/T - B$. Next, the quotient $w(pT)/w(\eta T)$ can be written as a quotient of sinc functions,

$$\left| \frac{w(pT)}{w(\eta T)} \right| = \left| \frac{\text{sinc}(B_w T \sqrt{T^2 - (T_w/T)^2})}{\text{sinc}(B_w T \sqrt{\eta^2 - (T_w/T)^2})} \right|. \quad (50)$$

But, since $p > P > \eta$, it is a simple task to derive a bound for the numerator in this equation and to write the denominator in terms of the hyperbolic sine function:

$$|\text{sinc}(B_w T \sqrt{T^2 - (T_w/T)^2})| \leq \frac{1}{\pi B_w T \sqrt{p^2 - (T_w/T)^2}}. \quad (51)$$

and

$$|\text{sinc}(B_w T \sqrt{\eta^2 - (T_w/T)^2})| = \frac{\sinh(\pi B_w T \sqrt{(T_w/T)^2 - \eta^2})}{\pi B_w T \sqrt{(T_w/T)^2 - \eta^2}}. \quad (52)$$

If $p = P + 1$, the numerator in (48) is smaller than or equal to one, given that $P + 1 \geq T_w/T$. Thus,

$$\left| \frac{w((P + 1)/T)}{w(\eta T)} \right| \leq \frac{\pi B_w T \sqrt{(T_w/T)^2 - \eta^2}}{\sinh(\pi B_w T \sqrt{(T_w/T)^2 - \eta^2})}. \quad (53)$$

If $p > P + 1$, the quotient in (49) can be bounded using (50) and (51),

$$\left| \frac{w(pT)}{w(\eta T)} \right| \leq \frac{\sqrt{(T_w/T)^2 - \eta^2}}{\sinh(\pi B_w T \sqrt{(T_w/T)^2 - \eta^2})} \sqrt{p^2 - (T_w/T)^2}. \quad (54)$$

Next, insert the last two bounds into (20):

$$F_w(t) \leq \frac{\pi B_w T}{P + 1 - \eta} \sum_{p>P+2} \frac{1}{(p - \eta)\sqrt{p^2 - (T_w/T)^2}}. \quad (55)$$
The summation can be bounded by comparing it with an integral:
\[
\sum_{p=P+2}^{\infty} \frac{1}{(p-\eta)\sqrt{p^2 - (P_w/T)^2}} \\
\leq \sum_{p=P+2}^{\infty} \frac{1}{(p-\eta)\sqrt{p^2 - (P+1)^2}} \\
\leq \int_{P+1}^{\infty} \frac{dx}{(x-\eta)\sqrt{x^2 - (P+1)^2}} \\
= \frac{\pi - \arccos(\eta/(P+1))}{\sqrt{(P+1)^2 - \eta^2}}.
\]

The quotient in the first line of (52) decreases with \(T_w\). Therefore, its smallest value is obtained for \(T_w = (P+1)T\),

\[
|F_w(t)| \leq \frac{2}{T} \frac{\sqrt{(P+1)^2 - \eta^2}}{\sinh(\pi B_w T \sqrt{(P+1)^2 - \eta^2})} \left(\frac{\pi B_w T}{P+1 - \eta} + \frac{\pi - \arccos(\eta/(P+1))}{\sqrt{(P+1)^2 - \eta^2}}\right).
\]

This is the final bound \(A_w(B)\). For large \(P/\eta\), \(A_w(B)\) is approximately

\[
\frac{\pi (2B_w + 1/T)}{\sinh(\pi B_w T \sqrt{(P+1)^2 - \eta^2})}.
\]

APPENDIX II

RECURSIVE FORMULA BASED ON STREAMS

Consider the following three streams:

- \(\{\delta_m\}\): Stream of deviations. The signal \(s(t)\) is sampled at \(t = (m + \delta_m)T\).
- \(\{u_m\}\): Stream of relative interpolation instants. The signal is interpolated at the instants \((m + u_m)T\), where \(u_m\) lies in the range \([- (N - 1)/2 - \eta, -(N - 1)/2 + \eta]\).
- \(\{s_m\}\): Stream of samples, \(s_m \equiv s((m + \delta_m)T)\).

Let the arrow \(\rightarrow\) denote a replacement in a given expression. The interpolation formula in (26) can be written in terms of the streams above, if the following replacements are carried out:

- \(t \rightarrow (m + u_m)T\)
- \(\tau_m \rightarrow (m - (N - 1)/2)T\)
- \(t_n \rightarrow (n + \delta_n)T\)
- \(s(t_n) \rightarrow s_n\).

If, besides, the index \(n\) in (26) is replaced with an index \(q\) ranging from 0 to \(N - 1\), the result is the formula

\[
s((m + u_m)T) \approx \frac{L_m}{\tilde{\gamma}(u_m)} \sum_{q=0}^{N-1} \tilde{\gamma}(\delta_m - q) I_{m,q}(q + u_m - \delta_m - q),
\]

where

\[
L_m = \frac{N-1}{k=0} k + u_m - \delta_m - k,
\]

\[
L_m' = \frac{N-1}{k=0 \neq q} k - q + \delta_m - q - \delta_m - k.
\]

REFERENCES


