Using Discriminant Curves to Recover a Surface of \( \mathbb{P}^4 \) From Two Generic Linear Projections

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ABSTRACT
We study how an irreducible smooth and closed algebraic surface \( X \) embedded in \( \mathbb{P}^4 \), can be recovered using its projections from two points onto embedded projective hyperplanes. The different embeddings are unknown. The only input is the defining equation of each projected surface. We show how both the embeddings and the surface in \( \mathbb{P}^4 \) can be recovered modulo some action of the group of projective transformations of \( \mathbb{P}^4 \).

We show how in a generic situation, a characteristic matrix of the pair of embeddings can be recovered. Then we use this matrix to recover the class of the couple of maps and as a consequence to recover the surface.

For a generic situation, two projections define a surface with two irreducible components. One component has degree \( d(d-1) \) and the other has degree \( d \), being the original surface.

Categories and Subject Descriptors
I.1.2 [Computing Methodologies]: Symbolic and Algebraic Manipulation—Algebraic Algorithms

Keywords
Computational Algebraic Geometry, Discriminant Curves, Projection

1. INTRODUCTION
The study of varieties and their projections has old roots in algebraic geometry [7]. In the past three decades, an application of this subject, in computer vision, became of major importance. See for further details [8][3]. However these applications mainly concerns the case of configurations of points or lines in \( \mathbb{P}^3 \) projected over embedded projective planes (or mere camera images). More recently the case of curves received some attention [10]. Due to the similarity of our setting, we borrow from computer vision some of its classical terminology.

2. NOTATIONS AND PRELIMINARIES
We denote by \( \mathbb{P}^n \) the projective space, defined as the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by scalar multiplication. The notation \( L \mathbb{P}^n \) designates the Grassmannian variety parameterizing lines in \( \mathbb{P}^n \). If \( p, q \) are two points in \( \mathbb{P}^n \) we shall denote the line joining \( p \) and \( q \) by \( p \lor q \). If some point \( e \in \mathbb{P}^n \) is given, we denote by \( L_e \mathbb{P}^n \) the closed subset of lines in \( \mathbb{P}^n \) containing \( e \). It is isomorphic to \( \mathbb{P}^{n-1} \) in a non canonical way: once a hyperplane \( H \subset \mathbb{P}^n \) missing \( e \) has been fixed, one obtains an isomorphism \( H \sim \mathbb{L}^n_e \), by mapping a point \( p \in H \) to the line \( p \lor e \). By a linear projection we mean a rational onto map \( p : \mathbb{P}^n \rightarrow \mathbb{P}^{n-1} \) induced by a linear map \( \mathbb{P}^n \rightarrow \mathbb{C}^n \), represented by a full rank \( n \times (n+1) \) matrix once two coordinates systems have been fixed on the source and target spaces. Since \( \mathbb{P}^n \) has a one-dimensional kernel such a map \( p \) is undefined at a point \( e \in \mathbb{P}^n \) called its center.

The set \( Pr(\mathbb{P}^n, \mathbb{P}^{n-1}) \) of linear projections comes naturally with a right action of the linear group \( PGL_{n+1} \): \( p \cdot g(x) = p(g^{-1} \cdot x) \). In a similar manner \( PGL_{n+1} \) operates on pairs of projections: \( (p_1, p_2) \cdot g = (p_1 \cdot g, p_2 \cdot g) \). Given now two projections \( p_1, p_2 \in Pr(\mathbb{P}^n, \mathbb{P}^{n-1}) \), which centers are \( c_1 \) and \( c_2 \), we shall refer to the points \( e_1 := p_1(c_2) \) and \( e_2 := p_2(c_1) \) in \( \mathbb{P}^{n-1} \) as the epipoles. With the data pair \( (p_1, p_2) \) in hand one defines the following fundamental map:

\[
F : L_{e_1} \rightarrow L_{e_2}  \\
L \mapsto p_2 p_1^{-1}(L)
\]

In the previous definition we first take the inverse image \( L \) of \( L \) by \( p_1 \), which is a 2–plane in \( \mathbb{P}^n \) containing both centers, then \( L \) is projected by \( p_2 \) onto a line in \( \mathbb{P}^{n-1} \) containing \( e_2 \). It should be noted that \( F \) is actually an isomorphism: indeed its inverse map is expressed similarly as \( F^{-1}(L) = p_1 p_2^{-1}(L) \). Moreover, by the previous observations, once two hyperplanes \( H_1 \) and \( H_2 \) in \( \mathbb{P}^{n-1} \) are fixed with the constraint \( e_i \notin H_i \) for \( i = 1, 2 \), the map \( F \) induces a linear isomorphism
$\tilde{F}: H_1 \rightarrow H_2$ making the following diagram commutative:

$$
\begin{array}{ccc}
L_{r_1} & \xrightarrow{F} & L_{r_2} \\
\downarrow & & \downarrow \\
H_1 & \xrightarrow{F} & H_2
\end{array}
$$

The data $(\epsilon_1, \tilde{F}, \epsilon_2)$ determines explicitly $F$. The following proposition is nothing new (see [10]), but we present here another proof.

**PROPOSITION** – 1. The $3$-tuple $(\epsilon_1, \epsilon_2, \tilde{F})$ introduced above determines the pair of projections $(p_1, p_2) \in P_r(\mathbb{P}^3, \mathbb{P}^{n-1})^2$ up to the right action of $\mathrm{PGL}_{n+1}$. Indeed if $(\alpha_1, \ldots, \alpha_{n+1})$ is any generator set of $\mathbb{P}^n$, and $\{ u_1, \ldots, u_{n-1} \}$ is any generator set of $H_1$, the pair $(p_1, p_2)$ is $\mathrm{PGL}_{n+1}$-equivalent to the pair $(q_1, q_2)$ such that: 

$$
\alpha_j \text{ center of } q_j, \quad q_j(\alpha_{3-j}) = \epsilon_j \quad \text{for } j = 1, 2, \text{ and } q_1(\alpha_j) = u_{j-2},
$$

$q_2(\alpha_j) = \tilde{F}(u_{j-2})$ for $j = 3, \ldots, n + 1$.

**Proof:** Let $v_1, \ldots, v_{n-1}$ be points of $\mathbb{P}^n$ such that $p_1(v_i) = u_i$ for each $i$; then $\{c_1, c_2, v_1, \ldots, v_{n-1}\}$ generates $\mathbb{P}^n$. Take the transformed points $w_i = \tilde{F}(u_i) \in H_2$ for $i = 1, 2, \ldots, n - 1$, which generate $H_2$; for all $i$ we have

$$p_2(v_i) \in c_2 \vee w_i \quad i = 1, 2, \ldots n - 1 \quad (1)$$

Now pick up vectors $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}^{n+1}$ such that $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}^n$ and for $i = 1, 2, \ldots, n - 1, \tilde{c}_i \in \mathbb{C}^{n+1}$, $\tilde{v}_i \in \mathbb{C}^n$ and $\tilde{v}_i \in \mathbb{C}^{n+1}$, representing the classes of $c_1, c_2, c_1^j, c_2^j, v_1, u_i, w_i$, and let also $\tilde{p}_1, \tilde{p}_2 : \mathbb{P}^n \rightarrow \mathbb{C}^n$ be two linear maps representing $p_1$ and $p_2$. By multiplying the previous vectors by scalars, one can assume the identities:

$$
\begin{align*}
\tilde{p}_1(\tilde{c}_2) &= \tilde{c}_1, \\
\tilde{p}_2(\tilde{c}_1) &= \tilde{c}_2.
\end{align*}
$$

Hence for each $1 \leq i \leq n - 1$ one has: $\tilde{p}_1(\tilde{v}_i - \lambda_i \tilde{c}_1) = \tilde{u}_i$ and $\tilde{p}_2(\tilde{v}_i - \lambda_i \tilde{c}_1) = \tilde{w}_i$. Given a generator set $\{c_1, \ldots, c_{n+1}\}$ of $\mathbb{P}^n$, define the automorphism $g \in \mathrm{PGL}_{n+1}$ such that $g(\alpha_j) = c_j$ for $j = 1, 2$, and for $3 \leq j \leq n + 1$, $g(\alpha_j)$ equal to the class of $\tilde{v}_j - \lambda_j \tilde{c}_j$ in $\mathbb{P}^n$. We have clearly $(p_1, p_2) \cdot g = (q_1, q_2)$ where $(q_1, q_2)$ is the pair explicit in the proposition.

3. PROBLEM PRESENTATION

3.1 General statement

The problem we have in mind is the following. Let $X \subset \mathbb{P}^4$ be an irreducible smooth closed surface. Given two images $X_1$ and $X_2$ of $X$ by two different generic projections $(p_1, p_2) \in P_r(\mathbb{P}^4, \mathbb{P}^3)^2$, how could we recover these two projections and $X$ itself up to projective equivalence, and at most up to finite ambiguity? More precisely let $c_1$ and $c_2$ be the (unknown) projection centers, and the epipoles defined by $c_1 := p_1(c_2), \quad c_2 := p_2(c_1)$. Given two equations $f_1 = 0$ and $f_2 = 0$ of $X_1$ and $X_2$ respectively, by prop[1] we wish to recover the $3$-tuple $(c_1, \tilde{F}, c_2)$ up to finite ambiguity. Let $(q_1, q_2)$ be a known pair of projections in the same $\mathrm{PGL}_{n+1}$-orbit of $(p_1, p_2)$; then we have to recover $X$ as a component of the cones intersection $p_1^{-1}(X_1) \cap p_2^{-1}(X_2)$.

Several comments are in order: first $X_1$ and $X_2$ do not in general correspond through some linear transformation of $\mathbb{P}^3$. We have also to suppose that each $\epsilon_i$ does not lie on some already fixed hyperplane $H_i$ of $\mathbb{P}^3$; we suppose moreover that the restricted projections $p_i|_X : X \rightarrow X_i$ are birational. Moreover each projected surface $X_i$ has a codimension $1$ singular locus which is the image of the curve in $X$ where the projection from the center $c_i$ is not a local isomorphism; equivalently, it is the locus in $X_i$ of points where the fibre of $p_i$ contains more than one point (including multiplicities).

The reader may refer to [3] and [12] for much more detailed and general account of the topics presented now.

**DEFINITION** – 1. For any closed projective subvariety $X \subset \mathbb{P}^n$, a line $L \subset \mathbb{P}^n$ is said to be tangent to $X$ if there is a smooth point $x \in X$ such that projective closure $\overline{T_Xx}$ of the tangent space $T_Xx$ in $\mathbb{P}^n$ contains $L$; The closed subvariety $\overline{X} \subset L$ is defined as the closure of the subset of lines tangent to $X$ in the above sense.

Let $X^0 \subset X$ be the open subset of smooth points, and denote by $TX^0 \rightarrow X^0$ the tangent bundle over $X^0$; there exists a natural dominant map $\psi : \mathbb{P}(TX^0) \rightarrow \overline{X}$, showing that $\overline{X}$ is irreducible of dimension $\dim X \leq 2 \dim X - 1$. This inequality is strict as soon as the generic fibre of $\psi$ is non-finite, i.e. when $X$ is ruled.

Suppose now $X \subset \mathbb{P}^n$ is a projective surface. For every codimension $3$ subspace $L \subset \mathbb{P}^n$ avoiding $X$ we define the locally closed subset $R^0 \subset X$ of smooth points $x \in X$ such that the span $(L, x)$ is tangent to $X$ in the meaning that $(L, x) \cap \overline{T_Xx}$ contains at least a line. Denote also by $R$ the closure of $R^0$ in $X$.

**DEFINITION** – 2. Under the notations stated above, if $p_{1, 2} : X \rightarrow \mathbb{P}^3$ is the restriction of the linear projection from center $L$ (it is a finite covering as soon as $L$ does not intersect any line contained in $X$), the discriminant curve of $p_{1, 2}$ is defined as the curve $B = p_{1, 2}(R)$. The curve $R$ is referred as the ramification divisor of $p_{1, 2}$.

Definition[2] is a particular case of a more general one defining the discriminant curve of a finite covering as the image of the locus where the differential of the covering is not into. If $G_3(\mathbb{P}^n)$ is the Grassmanian of codimension $3$ subspaces, there is a Zariski open subset $U \subset G_3(\mathbb{P}^n)$ such that for every $L \subset U$ (avoiding $X$) the projection $p_L$ is a generic morphism (see [11]); in other words $p_L$ is finite, its discriminant curve $B \subset \mathbb{P}^3$ is irreducible with ordinary cusps and nodes as only singularities, and $p_{L, 2}(B)$ is $2R + C$, where $C$ reduced and $p_{L, 2} : R \rightarrow B$ is the normalization of $B$. Moreover for all $L \subset U$ the curve $B$ has fixed (even) degree, genus, and numbers of cusps and nodes.

Going back now to our surface $X$ embedded in $\mathbb{P}^4$ consider the projection $p_L : X \rightarrow \mathbb{P}^2$ from the line $l = c_1 \vee c_2$ joining $c_1$ and $c_2$. Then there exists a Zariski open subset $U \subset \mathbb{P}^4 \times \mathbb{P}^4$ such that if $(c_1, c_2) \in U$ then $p_L$ is a generic projection with discriminant curve $B$ of fixed degree $2\delta$, genus $g$, and numbers $c$ and $n$ of cusps and nodes. From now on we shall suppose that $c_1$ and $c_2$ lie in such subset $U$. For $i = 1, 2$ let $\pi_i \in P_r(\mathbb{P}^4, \mathbb{P}^2)$ be the projection from the epipole $c_i$, hence $p_{L, i} = \pi_i p_L = \pi_2 p_{L, 2}$. Let $S_1 \subset X$ be the closed subset of points lying on a secant to $X$ passing through $c_i$; its image by $p_i$ is the curve $D_i \subset X_i$ of singular points. We shall always assume that $S_1 \cup S_2$ does intersect properly the ramification divisor of $p_L$; this ensures that $\pi_{L, i}(B)$ and $D_i$ are distinct curves, and the ramification divisor $B$ of $\pi_i$ is non-empty.
The two discriminant curves of \( \pi_1 \) and \( \pi_2 \) are isomorphic to \( B \), and we have \( \pi_i(R_i) \simeq B \).

We end with few words about the numerics: let \( d \) be the degree of \( X \), and \( d_i \) be the degree of the curve of singular points of \( X \). Next lemma, proved in [11] (lemma 1), gives a relation between \( 2\delta \) and \( d_i \).

**LEMMA** - 1. We have

\[
d_i \leq \frac{(d - 1)(d - 2)}{2}, \quad 2\delta = d(d - 1) - 2d_i.
\]

Somewhat harder relations between the two curves, involving not only their degrees but also the genera and their numbers of singularities, can be derived from Riemann-Roch and Noether formulas (see [3]).

### 3.2 Unicity result: Kulikov’s theorem

The following main result of [11] solves the famous Chisini’s conjecture (see [2]) in the particular case of coverings which are restrictions of linear projections.

**THEOREM** - 1. Let \( X \subset \mathbb{P}^n \) be a smooth projective surface and let \( f = p_{1X} \) be the restriction to \( X \) of a generic linear projection \( p \in \text{Proj} \mathbb{P}^n, \mathbb{P}^2 \) in the meaning above. Denote by \( B \subset \mathbb{P}^2 \) the discriminant curve of \( f \). Then provided \( X \) is not the image of the Veronese embedding \( \mathbb{P}^2 \to \mathbb{P}^3 \), the covering \( (X, f) \) is uniquely determined up to isomorphism by its discriminant curve \( B \subset \mathbb{P}^2 \) in the following meaning: if \( g : X \to \mathbb{P}^2 \) is another generic covering which discriminant curve \( B' \) is projectively isomorphic to \( B \) by \( \pi : \mathbb{P}^2 \to \mathbb{P}^2 \), there exists an isomorphism \( \sigma : X \to X' \) making the following diagram commutative:

\[
\begin{array}{ccc}
   X & \xrightarrow{f} & X' \\
   \downarrow \sigma & & \downarrow \pi \\
   \mathbb{P}^2 & \xrightarrow{\tau} & \mathbb{P}^2
\end{array}
\]

Note that in Th. 1 the morphism \( f \) is unique as a covering, not only as the restriction of a projection, though we shall just need this weaker consequence. We shall now specialize to the only case of interest for us, namely when \( X \) is embedded in \( \mathbb{P}^4 \). The following result follows readily from Th. 1 above. When \( x \) is a point of \( \mathbb{P}^3 \) we denote by \( \pi_x \) the projection onto \( \mathbb{P}^2 \) from center \( x \).

**COROLLARY** - 1. Suppose the surface \( X \subset \mathbb{P}^4 \) is embedded by its complete linear series \( |O_X(1)| \), and has moreover no non-trivial automorphism induced by some linear automorphism of \( \mathbb{P}^4 \); there are two non-empty Zariski open subsets \( U_1 \subset \mathbb{P}^3 \) and \( U_2 \subset \mathbb{P}^3 \) containing respectively \( \epsilon_1 \) and \( \epsilon_2 \) such that if \( (\epsilon_1, \epsilon_2) \in U_1 \times U_2 \) is a pair of points such that if \( \pi_{\epsilon_i}(R_i) \) for \( i = 1, 2 \) are projectively equivalent curves, where \( R_i \subset X \) is the ramification divisor of \( \pi_{\epsilon_i} \), then \( (\epsilon_1, \epsilon_2) = (\epsilon_2, \epsilon_1) \).

**Proof:** Indeed one can find some neighbourhood \( \Omega_1 \) of \( \epsilon_1 \), keeping \( \epsilon_2 \) otherwise fixed, such that the open subset of lines joining \( \epsilon_2 \) to some point of \( \Omega_1 \) do define generic projections in the meaning explained above, because this is verified already for \( L = \epsilon_1 \vee \epsilon_2 \) (in other words, this is an “open” property). Similarly one can find some neighbourhood \( \Omega_2 \) of \( \epsilon_2 \) verifying the same property with the roles of \( \epsilon_1 \) and \( \epsilon_2 \) swapped. Therefore there exist two open subsets \( U_1 \) and \( U_2 \in \mathbb{P}^3 \) containing respectively \( \epsilon_1 \) and \( \epsilon_2 \), such that for any \( \epsilon_1 \in U_1 \) the line \( L = \pi_{\epsilon_1}^{-1}(\epsilon_1) \) induces a generic projection \( p_{\epsilon_1} : X \to \mathbb{P}^2 \). If such a pair \( (\epsilon_1, \epsilon_2) \in U_1 \times U_2 \) is distinct from \( (\epsilon_1, \epsilon_2) \), the two lines \( L_i = \pi_{\epsilon_i}^{-1}(\epsilon_i) \) for \( i = 1, 2 \) in \( \mathbb{P}^3 \) are also distinct; if the discriminant curves \( \pi_{\epsilon_i}(R_i) \) for \( i = 1, 2 \) are projectively equivalent, the previous theorem asserts the existence of an automorphism \( \sigma \) of \( X \) compatible with the two coverings, in other words for some automorphism \( \tau \) of \( \mathbb{P}^2 \) we have the commutative diagram

\[
\begin{array}{ccc}
   X & \xrightarrow{\tau} & X \\
   \downarrow p_{\epsilon_1} & & \downarrow \pi_{\epsilon_2} \\
   \mathbb{P}^2 & \xrightarrow{\sigma} & \mathbb{P}^2
\end{array}
\]

We would have necessarily

\[
\sigma^*(O_X(1)) \simeq \sigma^*(p_{\epsilon_1}^*O_{\mathbb{P}^2}(1)) \\
\simeq p_{\epsilon_2}^*\tau^*O_{\mathbb{P}^2}(1) \simeq O_X(1)
\]

Since \( X \) is supposed to be embedded in \( \mathbb{P}^2 \) by the complete linear series \( |O_X(1)| \), or in other words that \( H^0(O_X(1)) \simeq H^0(O_{\mathbb{P}^2}(1)) \) so the isomorphism \( \sigma : X \to X \) would be necessarily induced by some linear automorphism in \( \text{PG}_{\mathbb{P}^2} \). By assumption this last one could be only the identity, hence the linear projections \( \tau p_{\epsilon_1} \) and \( \pi_{\epsilon_2} \) would coincide on \( X \) and therefore would have equal centers, since \( X \) is not contained in any hyperplane. The equalities \( e_1 = l_2 \) and hence \( e_1, e_2 \) follow at once.

In the next section we seek for the pair of epipoles \( (\epsilon_1, \epsilon_2) \). We shall use the previous corollary to assert that this pair is an isolated point in a family of potential candidates.

### 4. EPIPOLAR RECOVERY BY DISCRIMINANT CURVES

#### 4.1 The basic principle

We keep the notations and assumptions introduced in sections 2 and 3. For any point \( \epsilon_i \in \mathbb{P}^3 \), not in \( X \), let \( \hat{X}_{\epsilon_i} \subset \mathbb{P}^3 \) be the closure of the locally closed subset of lines in \( \mathbb{P}^3 \) containing \( \epsilon_i \) and tangent to \( X \). As such \( \hat{X}_{\epsilon_i} \), is isomorphic to the discriminant curve of \( \pi_{\epsilon_i} \); suppose a hyperplane \( H_i \subset \mathbb{P}^3 \) was chosen with \( \epsilon_i \not\in H_i \), we can consider \( \hat{X}_{\epsilon_i} \), as a curve in \( H_i \) (every point is simply mapped to the line joining it to \( \epsilon_i \)). Again it is isomorphic to the discriminant curve of the projection \( p_{\epsilon_i}^{-1}(\epsilon_i) : X \to \mathbb{P}^2 \) from the line \( p_{\epsilon_i}^{-1}(\epsilon_i) \), which contains the center \( c_{\epsilon_i} \).

**PROPOSITION** - 2. For generic projections centers \( \epsilon_1 \) and \( \epsilon_2 \) in \( \mathbb{P}^4 \), the map \( \hat{F} : H_1 \to H_2 \) induces an isomorphic \( \hat{X}_{\epsilon_1} \simeq \hat{X}_{\epsilon_2} \). Finally the point \((\epsilon_1, \epsilon_2) \) is isolated in the subset \( Z \) of all points \((\epsilon_1, \epsilon_2) \in (\mathbb{P}^3 \times H_2) \) such that \( \hat{X}_{\epsilon_i} \), for \( i = 1, 2 \) are projectively isomorphic plane curves.

**Proof:** Let again \( S = S_1 \cup S_2 \) be the curve of \( X \) of points where \( p_1 \) or \( p_2 \) is not a local isomorphism. Let \( i \) be 1 or 2, and define \( j = 3 - i \). Any point \( x \in p_1(X \setminus S) \subset X_i \), necessarily smooth, lies on the divisor \( R_i \), as soon as the line \( \epsilon_i \vee x \) is tangent to \( X_i \) at \( x \). Necessarily the \( 2 \)-plane \( p_{\epsilon_i}^{-1}(\epsilon_i \vee x) \) in \( \mathbb{P}^3 \) is tangent to \( X_i \), hence the projected line \( p_j(p_{\epsilon_i}^{-1}(\epsilon_i \vee x)) \) is tangent to \( X_j \) at some smooth point \( y \). Since the line \( \epsilon_j \vee y \) is the
image of $e_i \vee x$ by $F$, we obtain two rational maps between the two curves $\tilde{X}_i, e_i$ for $i = 1, 2$ which are inverse to each other. Since $F$ is linear, hence is globally defined these birational maps can be extended to isomorphisms on the whole curves (this point is crucial since a birational map between possibly singular curves may not be extended to a global isomorphism).

Last assertion follows readily from corollary [1] inside an open subset $U_1 \times U_2$ containing $(e_1, e_2)$ the pair $(e_1, e_2) = (\epsilon_1, e_2)$ is the sole point such that the curves $\tilde{X}_i, \epsilon_i$ are projectively equivalent.

In the next paragraph we shall explain the algorithm.

4.2 The algorithm

We keep the notations of the previous sections. The idea is to look for the subset of all 3-tuples $(e_1, F, e_2)$ where $e_1$ and $e_2$ are two points in $\mathbb{P}^3 \setminus H_1$ and $\mathbb{P}^3 \setminus H_2$ respectively, and $F \in PGL(H_1, H_2)$ induces an isomorphism $\tilde{X}_1, e_1 \simeq \tilde{X}_2, e_2$. Let us introduce homogeneous coordinates on $\mathbb{P}^3$: the point $e_i = [e_i^0 : \cdots : e_i^4]$ is the unknown, for $i = 1, 2$. We shall suppose that each $H_i$ is the hyperplane at infinity defined by the vanishing of the last coordinate. So we assume $e_i^4 \neq 0$, and for such $e_i$ we shall make use of the isomorphism between $\mathbb{P}^3 \setminus H_i$ and $H_i$, mapping each point $p = [x_0 : x_1 : x_2 : 0]$ to $H_i$. The line $L_p = p \vee e_i$. For each such $p$ the intersection $L_p \cap \tilde{X}_i$ is given by (potentially multiple) roots of the polynomial

$$f_i(s ; e_i^0 + t x_0 ; \cdots ; s e_i^2 + t x_2 ; s e_i^3) = \sum_{j=0}^d A_j(e_i ; x_0, x_1, x_2) s^{d-j} t^j$$

We shall denote the polynomial in [3] by $g_{e_i, p}(s, t)$: its coefficients $A_j$ are bihomogeneous forms of degree $d - j$ in the coordinates of $e_i$, and of degree $j$ in $x_0, x_1, x_2$. If $L_p$ is tangent to $\tilde{X}_i$ at some smooth point, or if $L_p$ meets the double points curve $D_i$, then $g_{e_i, p}(s, t)$ has at least one multiple root. This property is characterized by the vanishing of the following discriminant (see appendix [4]), taking entries of $e_i$ and $p$ as parameters:

$$G_i(e_i, x_0, x_1, x_2) = \text{Disc}_{[x_0, x_1, x_2]}(g_{e_i, p}(s, t))$$

The projective curve $G_i(e_i, x_0, x_1, x_2) = 0$ is of degree $d(d-1)$ in the coordinates $[x_0, x_1, x_2]$. It is the union of the generators of the cone of vertex $e_i$ over $D_i$ and $\tilde{X}_i, e_i$, hence contains at least two irreducible components. We have the following factorization in the ring $\mathbb{C}(e_i^0 ; \cdots ; e_i^4)[x_0, x_1, x_2]$

$$G_i(e_i, x_0, x_1, x_2) = \Delta_i(e_i, x_0, x_1, x_2) \cdot \pi_i(e_i, x_0, x_1, x_2)$$

In [5], the polynomial $\Delta_i(e_i, x_0, x_1, x_2)$ generates the ideal of $\pi_i(e_i, R_i)$, and $\pi_i(e_i, x_0, x_1, x_2)$ generates the ideal of $\pi_i(e_i, R_i)$. If $e_i$ is generic the polynomial $\pi_i(e_i, x_0, x_1, x_2)$ is irreducible, because $p_{R_i}^{-1}(e_i) : X \rightarrow \mathbb{P}^3$ is a generic covering and therefore $R_i$ is the image by $p_i$ of the normalization of an irreducible curve, hence the set $\pi_i(e_i, x_0, x_1, x_2) = 0$ must be also irreducible. Beside, $\Delta_i(e_i, x_0, x_1, x_2)$ might be reducible.

We are looking for all pairs $(e_1, e_2) \in (\mathbb{P}^3 \setminus H_1) \times (\mathbb{P}^3 \setminus H_2)$ together with the class of a linear automorphism, which can be explicit as:

$$\tilde{F}([x_0 : x_1 : x_2] = [l_0(x_0, x_1, x_2) : l_1(x_0, x_1, x_2) : l_2(x_0, x_1, x_2)]$$

verifying:

$$\pi_1(e_1, x_0, x_1, x_2) = \pi_1(e_1, x_0, x_1, x_2)$$

We need to identify the factors $\pi_i(e_i, x_0, x_1, x_2)$ for $i = 1, 2$, which we do by using elimination: consider the closed subset of points $[e_i^0 : \cdots : e_i^4 ; [x_0 : x_1 : x_2] ; s : t]$ satisfying:

$$\nabla f_i(s e_i^0 + t x_0 ; s e_i^1 + t x_1 ; s e_i^2 + t x_2 ; s e_i^3) = 0$$

(7)

This gives a system of 4 equations, from which we eliminate the variables $s, t$, in other words we obtain the ideal of the image of the closed subset $[\pi_i]$ by the projection on the variables of $e_i$ and $p$. It is generated by a polynomial in $x_0, x_1, x_2$ of degree $\Delta_i$ with coefficients depending on $e_i$ ones. This is the factor $\pi_i(e_i, x_0, x_1, x_2)$. We can summarize up our algorithm now.

**ALGORITHM:**

1. Given two equations $f_i = 0$ of the two projected surfaces $X_i$ in $\mathbb{P}^3$:

   - For $i = 1, 2$ eliminate the variables $s, t$ in $[\pi_i]$ to get $\pi_i(e_i, x_0, x_1, x_2)$;
   - compute $G_i(e_i, x_0, x_1, x_2)$ by [3]; form $\pi_i(e_i, x_0, x_1, x_2) = G_i(e_i, x_0, x_1, x_2)$;
   - introduce the unknown plane transformation $\tilde{F}[x_0 : x_1 : x_2] = [l_0(x_0, x_1, x_2) : l_1(x_0, x_1, x_2) : l_2(x_0, x_1, x_2)]$ and write down the ideal generated by equality $[\tilde{F}]$, which amounts to equalizing $e_i^4 + 1$ coefficients of two degree $2\Delta_i$ homogeneous polynomials in 3 variables;
   - then fix $e_i^4 = 1$ for $i = 1, 2$ and find zero-dimensional components of the image of this ideal by the projection $e_i \mapsto (e_1, e_2)$

After the last step of the previous algorithm is completed, we have recovered the pair of epipoles up to a finite ambiguity. In the next section we work out an example.

4.3 An example

We worked out an example of the algorithm. We considered a surface of $\mathbb{P}^3$, being a complete intersection of two quadrics:

$$\begin{align*}
  f_1 &= X_1^2 - X_2^2 + 2X_2X_4 + X_3^2 - X_1X_3 + X_2^2 + X_3X_0 \\
  f_2 &= X_2X_3 - X_0^2 + X_2^2 + X_1X_3 + X_3X_3
\end{align*}$$

The computation was performed using Maple 13, with includes the FastCGB library [4], known as being one of the best available Groebner bases implementations. The surface is irreducible and smooth.

The projection maps were defined as follows:

$$M_1 = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1 & -1 & 0 & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1
\end{bmatrix}$$

In this setting the centers of projection in $\mathbb{P}^4$ are quite simple:

$$M_1 \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix}, \quad M_2 \begin{bmatrix}
  0 \\
  0 \\
  -1 \\
  0
\end{bmatrix} \in \mathbb{P}^4.$$
\(\epsilon_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \epsilon_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{P}^3,\) while the fundamental matrix \(\mathbf{f}\) is the unit matrix \(I_3\).

The equation \(\mathcal{R}_1\) obtained from the first projection for the set of lines passing through a symbolic epipole \(\epsilon_1 = [\epsilon_1^1, \epsilon_1^2, \epsilon_1^3, 1]^T\) not lying on the plane at infinity and tangent to the surface at smooth points is far too long to be incorporated here. However the realization for the actual epipole \(\epsilon_1 = [0, -1, 0, 1]^T\) is given by:

\[
\begin{align*}
-\frac{9}{4} x_2^6 &+ \frac{6}{3} x_1 x_2^5 - \frac{4}{2} x_1^2 x_2^4 + \frac{1}{1} x_1^3 x_2^3 - \frac{6}{5} x_0 x_2^3 x_1 + \frac{9}{8} x_1^4 x_2^2 - \frac{3}{7} x_0^2 x_2^2 x_1^2 + \frac{4}{9} x_0^3 x_2 x_1^3 \nonumber \\
&\frac{1}{1} x_0^4 x_1^4 + \frac{1}{1} x_0^5 x_2 + \frac{1}{1} x_0^6 x_1 x_2^2 + \frac{1}{1} x_0^7 x_2^2 x_1^2 + \frac{1}{1} x_0^8 x_2^3 x_1^3 + \frac{1}{1} x_0^9 x_2^4 x_1^4 + (3456)^1 x_0 x_1 x_2 \text{ and ordinary double points. More precisely, a Maple computation shows that} \\
&G_{\epsilon_1}(x_0, x_1, x_2). \text{ A similar step is performed after the elimination of } s \text{ and } t \text{ from } G_{\epsilon_1} \text{ we discard the multiplicities, which outputs the polynomial:} \\
&G_{\epsilon_1}(x_0, x_1, x_2) \text{ is straightforward.} \\
&\text{As expected the singularities of the resulting curve are cusps and ordinary double points. More precisely, a Maple computation provides the following singular points:}
\end{align*}
\]

\[
\begin{bmatrix}
1, 0, 0, 2, 1, 2, 1, 0, 0, 2, 1, 2, 1, & 2, 1, 2 \\
[\{x_0, x_1, x_2\}, 1, 0, 2, 1, 2] \\
\text{RootOf}(3456Z^6 - 1728Z^5 + 3204Z^4 - 376Z^3 + 48Z^2 + 987Z - 947), \\
\text{RootOf}(334501Z^2 - 2709968+ 2615728\text{RootOf}(3456Z^6 - 1728Z^5 + 376Z^4 + 48Z^2 + 987Z - 947)) \\
1, 2, 1, 1 \\
\text{RootOf}(3456Z^6 - 1728Z^5 + 3204Z^4 - 483Z^3 + 987Z - 947), \\
\text{RootOf}(3456Z^6 - 1728Z^5 + 3204Z^4 - 483Z^3 + 987Z - 947) \\
1, 1 \\
\end{bmatrix}
\]

Each singular point is given by its projective coordinate, followed by a list of three invariants: its multiplicity, its delta invariant and the number of local branches passing through the point. Thus the three first singularities are ordinary double points, while the last one is a cusp.

Once the reduced corresponding discriminant curves \(\mathcal{R}_1\) and \(\mathcal{R}_2\) have been computed, the remaining steps of the algorithm can be performed as follows: (i) from equation \(\mathcal{R}_1\), we get a system of equation in the epipoles and the fundamental matrix, say \(S(\epsilon_1, \epsilon_2, \tilde{F})\). In our example, this systems contained 45 equations. One can perform a projection to get a system over the epipoles only. Let \(S(\epsilon_1, s)\) be this final system. The extraction of the zero dimensional component can be performed by first computing the radical of the ideal defined by \(S(\epsilon_1, s)\) (we denote \(K(\epsilon_1, \epsilon_2)\) this radical ideal), and then by looking for

5. SURFACE RECOVERY

Once the epipolar geometry is computed, one can recover the pair of projections as shown in section 2. Then the surface itself lies in the intersection of the cones defined by the projected surfaces and the centers of projections. Let us write \(\Delta_1 = p_1^{-1}(X_1)\) and \(\Delta_2 = p_2^{-1}(X_2)\) for these two cones. In the case the surface is irreducible, the intersection of the cones contains for generic projections two irreducible components. More precisely we have the following theorem.

**THEOREM** 2. Let \(X\) be an irreducible and smooth closed surface embedded in \(\mathbb{P}^3\) of degree \(d\). For a generic position of the centers of projection, namely when no epipolar hyperplane (hyperplane containing the two centers) is tangent twice to the surface \(X\), the surface defined by \((\Delta_1 = 0, \Delta_2 = 0)\) has two irreducible components. One has degree \(d\) and is the actual solution of the reconstruction. The other one has degree \(d(d-1)\).

PROOF. The proof relies on a similar theorem previously introduced for curves in \(\mathbb{P}^3\) projected over planes. See [10] for more details. Thus we consider a section by a generic epipolar hyperplane. Since the center of projections are generic, by Bertini’s theorem [9], the curve in this section is smooth and irreducible. This reduces the configuration to the case of curves, for which the theorem has already been proven. \(\boxdot\)

6. CONCLUSION

We have presented a solution for recovering a smooth surface embedded in \(\mathbb{P}^3\) from two generic linear projections. Our algorithm is based on the discriminant curves and relies on Kulikov’s theorem on the Chisini conjecture. We have also showed that when the surface is irreducible, it can be finally recovered as the single component of the right degree in the intersections of the two cones defined by the projections. Future work will consider other methods and algorithms for the recovery of the surface, as well as generalizations to general codimension 2 varieties.

7. APPENDIX: DISCRIMINANT OF A HOMOGENEOUS POLYNOMIAL

Suppose \(P(s, t) = a_0 s^n + a_1 s^{n-1} t + \cdots + a_{n-1} s t^{n-1} + a_n t^n\) is a homogeneous polynomial of degree \(n\). We define the discriminant of \(P\) respectively to the variables \([s, t]\) by the following expression:

\[
\text{Disc}_{(s,t)}(P) = \left( -\frac{\sigma_{n-1}}{a_n} \right) \text{Res}_s(P(s, 1), \frac{\partial P}{\partial s}(s, 1)),
\]

where \(\text{Res}_X(F, G)\) denotes the resultant between two polynomials \(F, G \in R[X]\) with \(R\) being some commutative ring.
It is proved (see for instance [1]) that \( \text{Disc}_{s,t}(P) = 0 \) if and only if \( P \) contains a factor of the form \( (t_0s - s_0t)^2 \), i.e. \( P \) vanishes at order \( \geq 2 \) along a direction in \( [s,t] \).

8. REFERENCES