Block Diagonalization in the MIMO Broadcast Channel with Delayed CSIT


Wireless Networking and Communications Group
Department of Electrical and Computer Engineering
The University of Texas at Austin, Austin, TX 78712-0240
Email: {jzhang2, jandrews, rheath}@ece.utexas.edu

Abstract—This paper investigates the impact of delayed channel state information at the transmitter (CSIT) on the MIMO broadcast channel with block diagonalization (BD) precoding. First, an upper bound for the achievable throughput is provided, which shows that BD is more robust to imperfect CSIT than zero-forcing precoding as it has fewer inter-user interfering streams. Due to residual inter-user interference, the throughput of BD still saturates at high SNR, which motivates switching between single-user and multi-user precoding. An accurate closed-form approximation is derived for the achievable throughput of the BD system, which provides guidance on the preferred transmission technique for a given scenario.

I. INTRODUCTION

Although not capacity achieving, linear precoding is a practical transmission technique for the MIMO broadcast channel. Zero-forcing (ZF) and block diagonalization (BD) are two popular low-complexity linear precoding techniques [1]–[3]. Channel state information at the transmitter (CSIT) is required to achieve the capacity of the MIMO broadcast channel. With perfect CSIT, ZF and BD can achieve a significant part of the ergodic sum capacity of the MIMO broadcast channel [4], [5].

There will, however, inevitably be errors in the available CSIT, such as feedback delay and quantization error due to limited feedback, which results residual inter-user interference due to imperfect spatial channel separation for different users. Recently, the ZF precoding system with imperfect CSIT has been analyzed, which shows that with fixed channel accuracy, i.e. a fixed quantization codebook size or a fixed delay, the system throughput is limited by residual interference and reaches a rate ceiling at high SNR [6]–[9]. Switching between single-user (SU) and multi-user (MU) MIMO transmission was proposed in [9] to improve the spectral efficiency.

Both ZF and BD precoding are designed to completely precancel inter-user interference at the transmitter and provide each user an interference-free channel. The difference is that ZF has a diagonal effective channel whereas BD has a block diagonal effective channel for each user. Previous research on imperfect CSIT has focused on the single-antenna receiver. Although ZF precoding can also be applied in systems with multiple receive antennas, the performance is inferior to BD precoding, as BD allows joint processing across different antennas at each receiver and has a looser constraint on precoder design.

There has been little research on BD precoding with imperfect CSIT. In [10], BD with limited feedback was investigated. Delay is another major source of CSIT imperfection, due to such effects as feedback delay, processing time, and protocol delay. In this paper, we investigate the impact of CSI delay on the BD system.

Contributions: This paper investigates the MIMO broadcast channel with multiple receive antennas at each user and with delay in the available CSIT. A rate loss analysis shows that BD precoding is more robust to imperfect CSIT than ZF precoding as it suffers from fewer interfering streams. With a fixed delay, the throughput of the BD system saturates at high SNR due to the residual inter-user interference. This motivates switching between SU and MU MIMO transmission to improve the spectral efficiency. An accurate closed-form approximation is derived from the random matrix theory for the achievable throughput of the BD system, which can be used to select the preferred transmission technique in a given scenario. In addition, the derived result is shown to be useful for the coordinated beamforming system [2], [11], [12], which provides better performance than BD precoding at low to medium SNRs.

Organization: The rest of the paper is organized as follows: the received signal and channel model are presented in Section II. Section III describes the transmission technique. The impact of delayed CSIT is investigated in Section IV. Numerical results and conclusions are in Section V and Section VI, respectively.

II. SYSTEM MODEL

Consider a MIMO broadcast channel, where the transmitter (base station) has $N_t$ antennas and each mobile has $N_r > 1$ antennas, and there are $U = N_t/N_r$ users. Denote $T_u[n]$ as the precoding matrix, then the received signal at the $u$-th user at time $n$ is given as

$$y_u[n] = H_u[n] \sum_{u'=1}^{U} T_{u'}[n] x_{u'}[n] + z_u[n],$$

where $H_u[n]$ is the $N_r \times N_t$ channel matrix from the transmitter to the $u$-th user, and $z_u[n]$ is the normalized complex Gaussian noise vector with entries distributed according to...
The capacity is given by $H$ and its delayed version $\tau V$ accurate even for small random matrix theory [14], which have been shown to be asymptotic results from systems assuming perfect CSIT, which serve as the basis for analytical results are presented for both the channel matrix. Analytical results are presented for both.

For ease of calculation, we will apply asymptotic results from

as noticed in [9].

For the numerical analysis, the classical Clarke’s isotropic point-to-point MIMO link [16]. The SVD of the channel matrix is $\text{SVD}(H)$, and it is well modeled as a spatially white Gaussian channel, with $N_t$ SNR. To assist the analysis, we assume that the channel $H[n]$ is Gaussian with zero mean and covariance $\Phi = \frac{\gamma}{N_t}I_{N_r}$. [16].

The asymptotic results for $H_{\text{iso}}(\beta,\gamma)$ is

$$
C_{\text{iso}} = \mathbb{E} \left[ \log \det \left( I_{N_r} + \frac{\gamma}{N_t} H[n] H^*[n] \right) \right].
$$

The following lemma gives the asymptotic result for this capacity as $N_t, N_r \to \infty$ with $\frac{N_t}{N_r} \to \beta$.

**Lemma 1 ([19]):** For a point-to-point MIMO link with CDIT and full CSIR, the asymptotic capacity per receive antenna is

$$
C_{\text{iso}}(\beta,\gamma) = \log_2 \left[ 1 + \frac{\gamma}{\beta} - \mathcal{F} \left( \beta, \frac{\gamma}{\beta} \right) \right] + \frac{\log_2 (\epsilon)}{\gamma} \mathcal{F} \left( \beta, \frac{\gamma}{\beta} \right)
$$

with

$$
\mathcal{F}(x, y) = \frac{1}{4} \left[ \sqrt{1 + y(1 - \sqrt{x})^2} - \sqrt{1 + y(1 - \sqrt{x})^2} \right]^2.
$$

Therefore, the ergodic capacity of the SVD system with perfect CSIT is approximated as

$$
C_{\text{SVD}} \approx \mathbb{E} \left[ \log \det \left( I_{N_r} + \frac{\beta \gamma}{N_t} H[n] H^*[n] \right) \right] \approx C_{\text{iso}}(\beta,\beta\gamma).
$$

This approximation is easy to calculate and is valid for the whole SNR range. In addition, it will be shown later that this approximation is very accurate. It will be used to compare with BD to determine the preferred transmission technique in a given scenario.

**B. MU-MIMO: BD Precoding**

With perfect CSIT, BD precoding achieves $H_u[n]T_u[n] = 0, \forall u' \neq u$. The design of the precoding matrix for the $u$-th user is based on the SVD of the aggregated channel matrix of the other users [1]-[3]. To assist the analysis, $T_u[n]$ is constrained to have orthonormal columns, i.e. $T_u^*[n]T_u[n] = I_{N_r}$. In this way, the $u$-th user sees an $N_r \times N_r$ effective point-to-point interference-free channel. Its received signal becomes

$$
y_u[n] = H_u[n]T_u[n]x_u[n] + z_u[n] = H_{\text{eff},u}[n]x_u[n] + z_u[n],
$$

where $H_{\text{eff},u}[n] = H_u[n]T_u[n]$. As $T_u[n]$ is a unitary matrix, which is independent of $H_u[n]$, $H_{\text{eff},u}[n]$ is also a complex Gaussian matrix as $H[n]$, i.e. $H_{\text{eff},u}[n] \sim \mathcal{CN}(0_{N_r \times N_r}, I_{N_r})$.

Assuming the number of data streams for user $u$ is equal to the number of receive antennas, and with equal power...
allocation, the input covariance is \( \Phi_{BD,u} = \frac{\gamma}{N_t} I_{N_r} \). The achievable ergodic rate for the \( u \)-th user is given by
\[
R_{BD,u} = \mathbb{E} \left[ \log_2 \det \left( I_{N_r} + \frac{\gamma}{N_t} H_{eff,u}[n] H_{eff,u}^*[n] \right) \right],
\]
which is similar to (4), but with an \( N_r \times N_r \) effective channel. Therefore, the achievable rate for the BD system with perfect CSIT is given as
\[
R_{BD} = \sum_{u=1}^{U} R_{BD,u} \approx U N_r C_{iso}(1, \gamma/\beta). \tag{9}
\]
The spatial multiplexing gain of BD is \( U N_r \), compared to \( N_r \) for the SVD system.

An exact expression for \( C_{iso} \) in (4) can be found in [21], which can be used to calculate \( R_{BD} \). The asymptotic analysis is adopted in this paper due to its computational efficiency, which is desirable when determining the operating region for different transmission techniques. In addition, asymptotic analysis can be easily extended to analyze the system with delayed CSIT.

C. SU-MIMO vs. MU-MIMO

The asymptotic results are compared with simulations in Fig. 1. We see that the asymptotic approximation is good for the SVD and is slightly loose for the BD system at high SNR, due to the effective low-dimension \( 2 \times 2 \) channels for the BD users. The SU system provides a higher throughput at low SNR due to its array gain; the BD system is preferred at high SNR due to its enhanced spatial multiplexing gain. Therefore, switching between the SU and MU MIMO modes improves the spectral efficiency even with perfect CSIT. Equating (6) and (9), the approximation of the mode switching point can be calculated. For the SU MIMO mode, one user is randomly selected.

IV. IMPACT OF DELAYED CSIT

In this section, we compare the MU-MIMO transmission with BD precoding and the SU-MIMO transmission in the system with CSI delay. The emphasis is on the impact of imperfect CSIT on MU-MIMO. For SU-MIMO, the throughput for the perfect CSIT system serves as an upper bound.

A. SU-MIMO: SVD Transceiver

With CSI delay, the SVD transceiver based on the outdated channel cannot perfectly diagonalize the channel matrix. The receiver performs joint decoding rather than separate decoding which is only possible with perfect CSIT. Therefore, the achievable rate is
\[
R_{SV D}^{(D)} = \mathbb{E} \left[ \log_2 \det \left( I_{N_r} + H[n] \Phi[n - D] H^*[n] \right) \right], \tag{10}
\]
where \( \Phi[n - D] \) is the input covariance based on the outdated CSIT. This rate is difficult to calculate. As shown in [9], the SU-MIMO system with \( N_r = 1 \) is relatively robust to imperfect CSIT, as it does not suffer from residual inter-user interference. We would expect it is also the case for the system with \( N_r > 1 \). Therefore, we will approximate (10) by (6) for the system with perfect CSIT, which will later be verified by numerical results.

To achieve the rate in (10), we assume the base station knows the transmission rate and the receiver knows the input covariance and precoding matrix perfectly. Otherwise, there will be rate mismatch and outage will occur. How to deal with the outage issue is left to our future work.

B. MU-MIMO: BD Precoding

With delayed CSIT, BD precoding matrices cannot perfectly cancel mutual interference, as they only achieve \( \Phi_u[n - D] T^{(D)}_{u'}[n] = 0, \forall u' \neq u \). The received signal for user \( u \) thus becomes
\[
y_u[n] = H_{eff,u}[n] x_u[n] + E_u[n] \sum_{u' \neq u} T^{(D)}_{u'}[n] x_{u'}[n] + z_u[n],
\]
where \( E_u[n] \) is the channel error matrix as in (2). Treating residual interference as noise, the achievable rate for the \( u \)-th user is given by
\[
R_{BD}^{(D)} = \mathbb{E} \left[ \log_2 \det \left( I_{N_r} + \frac{\gamma}{N_t} H_{eff,u}[n] H_{eff,u}^*[n] R_u^{-1}[n] \right) \right], \tag{11}
\]
\footnote{At high SNR, this performs closely to the system employing optimal water-filling, as power allocation mainly benefits at low SNR. This has been verified in [20].}
where $R_u[n]$ is the interference-plus-noise covariance matrix, given by

$$R_u[n] = E_u[n]E_u[n] = \sum_{u' \neq u} T_{u'}^D[n]X_{u'}^D[n]X_{u'}^D[n]^* T_{u'}^D[n]^*$$

$$= E_u[n] \left[ \sum_{u' \neq u} \frac{\gamma}{N_t} T_{u'}^D[n]T_{u'}^D[n]^* \right] E_u[n]^* + I_{N_r}. \quad (12)$$

To calculate (11), we first focus on an upper bound for the rate loss, as did in [10] for the limited feedback system. This will provide a lower bound for (11) and provide useful insights. We will then provide a closed-form approximation through asymptotic analysis, which is accurate and can be used to calculate the SU/MU mode switching points.

The following theorem provides an upper bound for the rate loss of the BD system with delay.

**Theorem 1:** Compared to the system with perfect CSIT, the rate loss for the $u$-th user of the delayed BD system is upper bounded by

$$\Delta R_{BD,u}^{(D)} = R_{BD,u}^{(D)} - R_{BD,u}^{()} \leq N_t \log_2 \left( \frac{N_t - N_r}{N_t} \frac{\gamma^2}{N_t} \epsilon_{u,u}^2 + 1 \right). \quad (13)$$

**Proof:** See Appendix A. This theorem provides several helpful insights:

1. The rate loss increases with $\epsilon_{u,u}^2$, determined by the amount of delay, and $\frac{\gamma}{N_t}$, which is the transmit power of other users.
2. There is a scaling factor $N_t - N_r$ on $\epsilon_{u,u}^2$, which is the number of interfering streams for each data stream. As shown in [9], there are $N_t - 1$ interfering streams for each data stream in the ZF precoding system. Therefore, the rate loss for each data stream of BD is lower than that of ZF, which means BD is more robust to the CSI delay than ZF precoding. Intuitively, this is because the receiver of the BD system can perform joint decoding and there is no interference between data streams for the same user.
3. At high SNR, the rate loss increases with $\gamma$, and the BD system will lose spatial multiplexing gain as the rate loss has the same scaling with $\gamma$ as $R_{BD,u}$.

Next, we provide the asymptotic result for the achievable rate for the $u$-th user, as in the following theorem.

**Theorem 2:** For a BD system with CSI delay, as $N_t, N_r \to \infty$ with $\frac{N_t}{N_r} \to \beta$, the asymptotic results for the achievable rate per receive antenna for the $u$-th user is approximated as

$$\frac{R_{BD,u}^{(D)}}{N_r} \approx \sum_{u' \neq u} \log_2 \left( \frac{1 + N_r \gamma \epsilon_{u,u}^2 \eta_1}{1 + N_r \gamma \epsilon_{u,u}^2 \eta_2} \right) + \log_2 \left( 1 + N_r \gamma \eta_1 \right) + \log_2 \frac{\eta_2}{\eta_1} + (\eta_1 - \eta_2) \log_2(e),$$

with $\eta_1$ and $\eta_2$ given in Appendix B.

**Proof:** See Appendix B.

The approximation in (14) comes from assuming that the interfering streams and the data streams are independent of each other. This result is accurate and can better characterize the impact of delayed CSIT.

**C. SU-MIMO vs. MU-MIMO**

Numerical results for both SVD and BD systems with delayed CSIT are shown in Fig. 2. We see that the asymptotic result from Theorem 2 is accurate especially for $N_t = 4$, and the throughput of the BD system saturates at high SNR. The upper bound for the SVD system is also accurate, which demonstrates the robustness of the SU-MIMO transmission to imperfect CSIT. Due to delay the SU-MIMO transmission is preferred at both low and high SNRs. In addition, increasing $N_t$ provides more performance gain for the MU-MIMO transmission than for the SU-MIMO.

**V. NUMERICAL RESULTS**

In this section, we present some numerical results to show the impact of imperfect CSIT and the application of our results.

**A. Operating Regions**

With the derived approximation (14) for BD precoding and the upper bound (6) for the SVD transceiver, we can determine the preferred transmission technique in a given scenario. The operating regions for different transmission techniques with different delays are plotted in Fig. 3. We see that the BD precoding is preferred with certain requirements on both delay and average SNR. Specifically, for BD to be active, we require $f_d \tau < 0.046$ and $\text{SNR > 9.6 dB}$. On the other hand, the SU-MIMO transmission is preferred at both low and high SNRs.

**B. Comparison of Different Transmission Techniques**

We compare the BD precoding with other transmission techniques in Fig. 4. The CBF (Coordinated Beamforming) algorithm jointly designs precoding and decoding vectors and
provides better performance than BD with perfect CSIT [2], [11], [12]. ZF MMT (Multi-Mode Transmission) is a technique to improve the performance of ZF precoding with imperfect CSIT by adaptively adjusting the number of active users [22]. We keep the total number of data streams the same for BD and CBF. In this case, BD has two users and two data streams for each user, while CBF has four users with a single data stream for each user. For ZF MMT, each antenna element acts as an independent user, and the number of data streams depends on the amount of delay and average SNR, specified in [22]. We see that BD provides a higher throughput than ZF MMT, while CBF outperforms BD especially at low to medium SNR. In addition, both BD and CBF provide a throughput gain over SU-MIMO at medium SNR. However, all of these MU-MIMO transmission techniques reach a rate ceiling at high SNR. It is desirable to switch between SU-MIMO and MU-MIMO transmission, for which an accurate closed-form approximation is derived from asymptotic analysis for the achievable rate of BD precoding. Further investigation is required to improve the performance of the MIMO broadcast channel with imperfect CSIT.

VI. Conclusion

In this paper, we consider a MIMO broadcast channel with multiple receive antennas and with delay in the available CSIT. The impact of such imperfect CSIT on BD precoding is investigated, which shows that BD is more robust to imperfect CSIT than ZF precoding but still suffers from residual inter-user interference at high SNR. It is desirable to switch between SU-MIMO and MU-MIMO transmission, for which an accurate closed-form approximation is derived from asymptotic analysis for the achievable rate of BD precoding. Further investigation is required to improve the performance of the MIMO broadcast channel with imperfect CSIT.
where \( G_1 = [H_{eff,u}[n], G_2] \),
\( G_2 = [G_1, \cdots, G_{u-1}, G_u, \cdots, G_U] \),
\( B_1 = \text{blockdiag} \{ \gamma_u I_{N_r}, B_2 \}, \)
\( B_2 = \text{blockdiag} \{ \gamma_u e_{u,u} I_{N_r}, \cdots, \gamma_{u-1} e_{u,u} I_{N_r}, \gamma_u e_{u,u} I_{N_r}, \cdots, \gamma_{u+1} e_{u,u} I_{N_r} \} \),
and blockdiag\( \{ A_1, \cdots, A_N \} \) denotes the block diagonal matrix with matrices \( A_i, i = 1, \cdots, N \) on the main diagonal.

To get a closed-form expression, we assume the entries of matrices \( B_1 \) and \( B_2 \) are independent. This approximation will be verified by simulation. Then from [23], the following asymptotic results can be derived

\[
\frac{R_1}{N_r} \approx \log_2 \left( 1 + N_r \gamma_u \eta_1 \right) + \sum_{u' \neq u} \log_2 \left( 1 + N_r \gamma_{u'} e_{u,u} \eta_1 \right) + \log_2 \left( \frac{1}{\eta_1} + (\eta_1 - 1) \log_2(e) \right),
\]

\[
\frac{R_2}{N_r} \approx \sum_{u' \neq u} \log_2 \left( 1 + N_r \gamma_{u'} e_{u,u} \eta_2 \right) + \log_2 \left( \frac{1}{\eta_2} + (\eta_2 - 1) \log_2(e) \right),
\]

where \( \eta_1 \) and \( \eta_2 \) are the positive solutions to

\[
\frac{1}{\eta_1} + N_r \gamma_u \eta_1 = 1, \quad \frac{1}{\eta_1} + N_r \gamma_{u'} e_{u,u} \eta_1 = 1,
\]

\[
\frac{1}{\eta_2} + N_r \gamma_u e_{u,u} \eta_2 = 1, \quad \frac{1}{\eta_2} + N_r \gamma_{u'} e_{u,u} \eta_2 = 1.
\]

Then we get the result in (14).

REFERENCES


