**Economics of Femtocells**

Nikhil Shetty  
EECS, UC Berkeley  
Berkeley, California-94720, USA  
Email: nikhils@eecs.berkeley.edu

Shyam Parekh  
Alcatel-Lucent Bell Labs  
Berkeley, USA  
Email: sparekh@alcatel-lucent.com

Jean Walrand  
EECS, UC Berkeley  
Berkeley, California-94720, USA  
Email: wlr@eecs.berkeley.edu

**Abstract**—Femtocells or home base stations are a proposed solution to the problem of degraded indoor service from the macrocell base station in future 4G data networks. In this paper, we study user incentives for the adoption of femtocells and their resulting impact on network operator revenues. We model a monopolist network operator who offers the option of macrocell access or macro+femtocell access to a population of users who possess linear valuations for the data throughput. We compare the revenues from two possible spectrum schemes for femtocell deployment: the split spectrum scheme, where femtocells and macrocells operate on different frequencies and do not interfere, and, the common spectrum scheme, where they operate on the same frequencies (partially or fully) and interfere. Our results suggest that the optimal pricing scheme always charges a higher price for the femtocell service, i.e., the operator does not offer any subsidies for adoption. Yet, at the optimal prices, almost full adoption of femtocells is achieved even for many common spectrum schemes that degrade macrocell capacity. Femtocell deployments provide huge revenue gains when macrocell capacities are low. However, in this range, even common spectrum schemes that heavily degrade the macrocell capacity perform comparably to the split spectrum scheme. Some common spectrum schemes with moderate macrocell degradation yield revenues comparable or higher than the split spectrum scheme at all levels of macrocell congestion.

**I. INTRODUCTION**

4G networks, especially those operating at high frequencies, are expected to face the problem of poor connectivity inside user households. This is mainly due to the high attenuation suffered at these frequencies. To circumvent poor reception inside such built-up areas, tiny base stations for homes called femtocells [1]–[4] have been proposed. These femtocells not only enable high-quality use of mobile devices in the user’s home but also allow the user to seamlessly move his calls and data sessions between the macrocell and his femtocell. From the point of view of the network operator, femtocells appear advantageous since femtocell usage reduces the load on the macrocell network and allows more users to be served, which helps raise revenues. In addition, network operators may be able to price discriminate and extract a higher value from femtocell users. However, an operator’s use of femtocells is not devoid of costs. In this paper, we do not consider an increase in operational costs (like the additional costs of managing an integrated macro-femto network, customer support, etc.) due to the provision of femtocells. We only consider the opportunity costs of the network operator due to the (in)efficient use of spectrum in the hybrid macro-femto network. Note that these opportunity costs would not exist if femtocells were to operate in free spectrum, like those in the 2.4 and 5.8 GHz bands. However, due to the prolific number of devices (that the operator himself does not control) present in these frequencies, no quality of service (QoS) guarantees can be given. To provide QoS, the operator must utilize his own spectrum, adding opportunity costs to his femtocell operations.

Various spectrum deployment options have been proposed for the deployment of femtocells [5]. The authors in [5] suggest 3 possible spectrum schemes - the “separate carrier” deployment where the spectrum is divided into two parts and a dedicated fraction is used for femtocells, the “shared carrier” deployment where the macrocell and the femtocells operate on the same frequencies and the “partially-shared carrier” deployment where the femtocells operate only on a fraction of the spectrum used by the macrocell. In this paper, we will consider two spectrum schemes only. The first scheme - which we term “split spectrum” - will be similar to the separate carrier scheme. The second scheme - which we term “common spectrum” - will model both the shared carrier and the partially-shared carrier cases.

Both femtocell schemes provide gains via increased macrocell capacity due to lower congestion. However, the two schemes impose different costs on the macrocell. With split spectrum, macrocell capacity is not affected due to interference, but, dedicating spectrum for femtocell usage directly reduces the capacity of the macrocell. With common spectrum, there is no loss due to dedicated capacity, but increased adoption of femtocells leads to increased interference for macrocell users and decreases its capacity. Previous research [4], [6]–[9] has focused on how this interference affects macrocell capacity and service quality and have suggested that common spectrum deployments are as feasible as split spectrum ones.

The focus of this paper is to study the impact of the complex interplay of interference and service pricing on user adoption of femtocells. Fig. 1 depicts the high-level model that we analyze in this paper. Analysis of femtocell adoption is not straightforward. With split spectrum, as femtocell adoption increases, the pure macrocell service becomes more attractive to the users. Hence, user incentive to adopt femtocells...
Consider a monopolist wireless network operator who offers mobile services to a population of users \( N \). Assume that this operator has a fixed amount of spectrum to deploy. Assuming that the operator wants to deploy femtocells, he has the following two options: deploy femtocells under a split spectrum scheme or under a common spectrum scheme. In both these cases, we will assume that the operator only provides two service options - a mobile-only service \( m \) that allows the user to access the macrocell only and a mobile-plus-femto service \( f \) that permits the additional usage of a home-based femtocell. Let \( p_m \) and \( p_f \) be the prices charged for the services \( m \) and \( f \) respectively. Once the prices are charged, users are free to choose their preferred service, if any. Let the operator’s objective be to maximize his revenue \( V \) given by:

\[
V = p_m X_m + p_f X_f,
\]

where \( X_m \) and \( X_f \) are the number of users who adopt services \( m \) and \( f \) respectively. Also, define

\[
X = X_m + X_f, \quad x = \frac{X_f}{X} \quad \text{and} \quad \alpha = \frac{X_f}{X}.
\]

Next, we model the user demand for services \( m \) and \( f \). Let \( T_j \) be the instantaneous data throughput received by a user from service \( j = m, f \). We assume that a user derives an instantaneous benefit \( \gamma f(T_j) \) from the service where \( \gamma \) represents the user’s valuation for this throughput and \( f(\cdot) \) is a concave function with \( f' > 0 \) and \( f'' \leq 0 \). Further, we assume that the user population consists of users of type \( \gamma \in (0, \gamma_{\text{max}}] \) and let the cumulative distribution function (cdf) of the user types be \( \Gamma \), satisfying the usual conditions \( \Gamma(\gamma \leq 0) = 0 \) and \( \Gamma(\gamma \geq \gamma_{\text{max}}) = 1 \).

The instantaneous throughput \( T_j \) varies with the user’s position, the time of access, and the congestion in the network (i.e. the access times of other users). We obtain the user’s expected benefit \( \gamma E[f(T_j)] \) from adoption of the service \( j \) by taking the expectation over all possible user trajectories and network access times. Next, we make the simplifying assumption that \( E[f(T_j)] \) is a function of \( x \) and \( \alpha \) only and does not depend upon a specific user. Let this dependence be given by the function \( g_j(\alpha, x) \) for \( j = m, f \). Then, a type \( \gamma \) user’s utility from adopting service \( j = m, f \) will be

\[
U_j = \gamma g_j(\alpha, x) - p_j.
\]

For any given \( x \) and \( \alpha \), we assume

\[
g_j(\alpha, x) > g_m(\alpha, x),
\]

i.e., a user derives a higher benefit from service \( f \) than service \( m \). Hence, if \( p_f \leq p_m, \alpha = 1 \), i.e., users choose to buy the service \( f \) only, if any. Note that \( \alpha \) may be 1 even when \( p_f \) is higher than \( p_m \).

From (3) and (4), for \( j = m, f \), if for some \( \tilde{\gamma} \), \( U_j^{\tilde{\gamma}} > 0 \), then, \( U_j^{\gamma} > 0 \) for all \( \gamma > \tilde{\gamma} \). This implies that there is a threshold user type beyond which all users (with a higher valuation for the throughput) adopt some service. Define \( \gamma_m \) to be the critical user type beyond which all users buy some service. Then, the fraction of users who adopt some service will be:

\[
x = \left[1 - \Gamma(\gamma_m)\right].
\]

Next, suppose \( \alpha < 1 \). From (3) and (4), if for some \( \tilde{\gamma} \), \( U_j^{\tilde{\gamma}} > U_m^{\tilde{\gamma}} \), then, \( U_j^{\gamma} > U_m^{\gamma} \) for all \( \gamma > \tilde{\gamma} \). This implies that all users with type greater than a certain critical user type adopt service \( f \) over service \( m \). Define \( \gamma_f \) to be the critical user type beyond which all customers buy service \( f \). Then, given \( x \), the fraction of customers who adopt service \( f \) will be:

\[
\alpha = \frac{1 - \Gamma(\gamma_f)}{x}.
\]

Fig. 2 depicts \( \gamma_m, \gamma_f, \alpha \) and \( x \).

**Theorem 2.1:** For any given \( p_m \) and \( p_f \), the values of \( x \) and \( \alpha \) in equilibrium are determined from (5) and (6) with \( \gamma_m \) and

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978-1-4244-4148-8/09/$25.00 ©2009
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$\gamma_f$ given as below:

$$\gamma_m = \begin{cases} \frac{p_f}{g_f(x)} & \alpha = 1 \\ \frac{p_m}{g_m(x)} & \alpha < 1 \text{, and} \end{cases}$$

$$\gamma_f = \frac{p_f - p_m}{g_f(x) - g_m(x)} \text{, if } \alpha < 1.$$  (8)

**Proof:** See Appendix.

**Corollary 2.2:** If \( \frac{g_f(x)}{g_m(x)} + \frac{g_f(1, x)}{g_m(1, x)} \geq \frac{g_f(1, x)}{g_m(1, x)} \) \( \forall \alpha < 1 \), then, for any \( x \), \( \alpha = 1 \Leftrightarrow \frac{p_f}{p_m} = \frac{g_f(1, x)}{g_m(1, x)} \).

**Proof:** See Appendix.

From Corollary 2.2, in equilibrium, if \( \alpha = 1 \), the operator must charge \( p_f = \frac{g_f(1, x)}{g_m(1, x)} p_m \) to maximize his revenue. Hence, even when \( \alpha = 1 \) in equilibrium, \( p_f > p_m \), i.e., service \( f \) will be costlier than service \( m \). Substituting the values of \( x \) and \( \alpha \) from Theorem 2.1 in (1) and optimizing w.r.t. \( p_m \) and \( p_f \), we obtain the monopolist’s optimal choice.

**A. Model for \( g(\cdot) \)**

In this section, we present an approximation to the function \( g \). First, we assume that the user’s benefit \( f(T) \) is proportional to the instantaneous throughput \( T \) that he receives [11]:

$$f(T) = kT,$$

where \( k \) is the constant of proportionality. We discuss the implication of this assumption in Section V. Thus, the utility for the user type \( \gamma \) if he adopts service \( j = m, f \) becomes

$$U^*_j = \gamma kE[T_j] - p_j,$$

which, from (3), gives us

$$g_j(x) = kE[T_j], \ j = m, f.$$  (9)

In the rest of the discussion, we will restrict our attention to the throughput obtained in the downlink only. A similar analysis for the uplink may be performed.

Next, let users spend a fraction \( f_i (f_o = 1 - f_i \text{ and } f_o < f_i) \) of their time inside their home. Then, we assume

$$E[T_f] = f_i E[T_b] + f_o E[T_m],$$  (10)

where \( T_b \) is the throughput obtained by the user from his broadband connection via the femtocell. We assume that \( E[T_b] \) is fixed and independent of the user’s position in his home, the interference from femtocells and macrocell users in the neighborhood (see Fig. 3 for the only interference that is modeled) and the spectrum scheme being employed. \( E[T_b] \) depends on the congestion in the wired network and is not the maximum supportable data rate of the femtocell. It is conceivable that \( E[T_b] > E[T_m] \) for the near future and this gives us our desired condition: \( g_m(x, \alpha) < g_f(\alpha, x) \). Note that (10) underestimates \( E[T_f] \). Since femtocell adopters use macrocell services only when they are outside, they may receive better expected throughput \( E[T_m] \) than pure macrocell users. Yet, this impact will be low since \( f_i > f_o \) and \( E[T_b] > E[T_m] \).

\( T_m \) depends upon the position-varying and time-varying channel conditions, interference from the femtocells and the congestion in the network. Analyzing data rate variations due to channel conditions is beyond the scope of this paper. Hence, to simplify, we define a macrocell data rate \( R(\alpha x) \), that captures all channel variations and depends only upon the fraction of population that has adopted femtocells. \( R(\alpha x) \) can be conceived as an average throughput received from the macrocell if exactly one user were to move around the macrocell, both outside and inside his home, in the presence of \( \alpha x \) fraction of femtocells. Note that \( R(\alpha x) \) depends on the spectrum scheme being employed (see Section II-B). With \( R(\alpha x) \) thus defined, the throughput \( T_m \) depends upon the congestion in the network only.

To determine \( E[T_m] \), we model the congestion in the macrocell network as follows. First, we assume that user population is distributed identically across all cells. Then, if \( X_{cell} \) are the adopters and \( N_{cell} \) are the number of users in any macrocell, we let \( x = \frac{X_{cell}}{N_{cell}} = \frac{x}{\gamma} \). Next, let users generate i.i.d. requests for downloads following a Poisson process of rate \( \lambda_o \) when they are outside their homes and rate \( \lambda_i \) when they are inside. Define the activity ratio \( \beta \) as

$$\beta = \frac{\lambda_o}{\lambda_i}.$$  (11)

Next, assume that the file lengths are exponentially distributed. Then, if this download is served at rate \( R(\alpha x) \), it would take random exponential amount of time of mean \( 1/\mu \), where

$$\mu = \frac{R(\alpha x)}{\text{Mean File Length}}.$$  (12)

When \( l \geq 0 \) downloads are simultaneously active, and each download shares the macrocell data rate equally, each download will be served in random time given by an exponential distribution with mean \( 1/\mu \). With this, we can now generate a Markov chain of the number of active downloads in the system. In this Markov chain, at any state \( l \), let \( \lambda_i \) be the rate at which new downloads are added and \( \mu \) be the rate at which downloads are removed from the system. The Markov chain thus generated is identical to a processor-sharing queue and is depicted in Fig. 4. Next, we determine \( \lambda_i \) and \( \mu \).

Fig. 5 depicts how \( \lambda_i \) is modeled. Suppose \( f_x X_{cell} \) users are outside their homes, and they generate download requests for the macrocell at rate \( \lambda_o \) irrespective of whether they have adopted service \( m \) or \( f \). Of the remaining \( f_x X_{cell} \) users who are inside their homes, a fraction \( \alpha \) have adopted the femtocell and do not generate any requests for the macrocell while the rest generate requests at a rate \( \lambda_i \). Note that this does
not capture correlated behavior (like peak hour) but only an average sense of the traffic load. Then, we have
\[
\lambda_l = [f_l(1 - \alpha) + \beta f_o] x \lambda_l N_{cell}.
\]  
(13)
When there are \( l \) downloads in parallel for the Markov chain, each taking i.i.d. exponential amount of time with mean \( \frac{1}{\tau} \), the rate at which the system exits from state \( l \) is given by

\[
\mu_l = \frac{\mu}{l} \times l = \frac{R(\alpha x)}{\text{Mean File Length}}.
\]  
(14)

From (14), since \( \mu_l \) depends on \( R(\alpha x) \), we note that \( \tau \) also depends upon the spectrum scheme being used. Henceforth, we use the subscripts \( s \) and \( c \) to denote the quantities specific to the split and common spectrum schemes respectively.

\section*{B. Model for \( R(\alpha x) \)}

Assume that the operator has a total spectrum availability of 1.2\( W \). We will assume that the macrocell data rate is proportional to the employed spectrum. Accordingly, if the operator employs this complete spectrum for the macrocell, let him obtain the macrocell data rate 1.2\( R_0 \). Let the corresponding service rate as defined in (14) be 1.2\( \mu_0 \).

**Split Spectrum:** In this scheme, assume that the operator chooses to split his spectrum as follows - \( W \) for the macrocell and 0.2\( W \) for the femtocells.\(^3\) In this case, we assume there will be no interference due to the femtocells and the macrocell data rate will be \( R_0 \) (correspondingly \( \mu_0 \)).

\[
R^s(\alpha x) = R_0.
\]  
(16)

**Common Spectrum:** In this scheme, the operator chooses to operate both the macrocell and the femtocell in the same 1.2\( W \) MHz spectrum. In this case, when no femtocells are adopted, the macrocell data rate will be 1.2\( R_0 \) (correspondingly 1.2\( \mu_0 \)). As femtocell adoption rises, interference from the femtocell downlink reduces throughput for macrocell users by affecting the downlink macrocell rate (see Fig. 3). Next, assuming that all femtocell users contribute equally to degradation of the macrocell data rate, we let the macrocell data rate decrease linearly in the number of users who adopt the femtocell.

\[
R^c(\alpha x) = \max\{1.2 R_0 (1 - d \alpha x), 0\},
\]  
(17)
where \( d > 0 \) is the coefficient of degradation and \( x = \sum_{\text{cell}} = \frac{N_{cell}}{\lambda_l} \). For any network, \( d \) may be estimated as follows. If \( R(\alpha) \neq 0 \) for \( \alpha < 1 \), i.e., if the macrocell rate does not go to 0 before every user adopts the femtocell, then \( d = 1 - \frac{R(\alpha)}{R(0)} \leq 1 \). Else, \( d = \frac{1}{\alpha_{min}} > 1 \) where \( \alpha_{min} = \arg\min_{R(\alpha)=0} \alpha \).

**Model for \( \gamma \)**

\( \Gamma \) gives us the distribution for the user valuations. For \( \gamma \leq 0 \), \( \Gamma(\gamma) = 0 \) and for \( \gamma > \gamma_{max} \), \( \Gamma(\gamma) = 1 \). In this paper, we will assume only a uniform distribution of users.

\[
\Gamma(\gamma) = \frac{\gamma}{\gamma_{max}}, \quad \gamma \in [0, \gamma_{max}].
\]  
(18)

\section*{III. OPERATOR REVENUES}

To simplify expression, we define the broadband rate factor \( b \) and the macrocell capacity \( c_0 \) as

\[
b = \frac{E[T_m]}{R_0}, \quad c_0 = \frac{\mu_0}{\lambda_l N_{cell}}.
\]  
(19)
and normalize the values of \( p_m, p_f, g_m, g_f \) w.r.t. \( k R_0 \). From (1) and (7), revenue has the same units as \( \gamma g(\cdot) X \sim \gamma k R_0 x N \). Henceforth, we normalize the revenues w.r.t. \( k R_0 N \).

\(^3\)This closely models the solution proposed by Clearwire/Sprint where 5 MHz will be reserved for femtocells and 30 MHz for the macrocell.
A. Femtocell: Split Spectrum

From (9), (15) and (16), we have (normalizing \( g_m^a \) and \( g_f^a \))
\[
g_m^a = r^a, \quad g_f^a = (f_b + f_o r^a),
\]
(20)
\[
\tau^a = (1 - p_f^a) \frac{\log (1 - p_m^a)}{p_f^a} \quad \text{where} \quad p_f^a = \frac{f_o (1 - \alpha) + \beta f_m^c}{c_0}
\]
Solving (5), (6), (7) and (8) using (18) and (20), we can obtain the values of \( x \) and \( \alpha \) for any given \( p_m \) and \( p_f \). Substituting these values in (1) and optimizing w.r.t. \( p_m \) and \( p_f \), we get the monopolist’s optimal choice.

B. Femtocell: Common Spectrum

From (9), (15) and (17), we have (normalizing \( g_m^c \) and \( g_f^c \))
\[
g_m^c = 1.2 r^c [1 - d x]^4, \quad g_f^c = (f_b + f_o g_m^c),
\]
(21)
\[
\tau^c = (1 - p_f^c) \frac{\log (1 - p_m^c)}{p_f^c} \quad \text{where} \quad p_f^c = \frac{f_o (1 - \alpha) + \beta f_m^c}{c_0}
\]
Solving (5), (6), (7) and (8) using (18) and (21), we can obtain the values of \( x \) and \( \alpha \) for any given \( p_m \) and \( p_f \). Substituting these values in (1) and optimizing w.r.t. \( p_m \) and \( p_f \), we get the monopolist’s optimal choice.

C. Base case: No Femtocell

In this case, the entire spectrum is used for the macrocell and no femtocells are deployed. Hence, this is similar to the common spectrum femtocell deployment with \( \alpha = 0 \). From (5), (7) and (21), \( \gamma_m \) is a solution of
\[
\gamma = \frac{p_m}{g_m^c (0, 1 - \Gamma (\gamma))} = \frac{p_m}{1.2 k R_0 \tau^c}.
\]
(22)
Normalized operator revenue is \( \frac{p_m X^c}{R_0} = 1.2 \gamma m \tau^c [1 - \Gamma (\gamma_m)] \).

IV. RESULTS

We carried out the numerical analysis⁴ in MATLAB®. For each scenario, we varied \( p_m \), the price of service \( m \), and \( p_f \), the price of service \( f \). For each such pair \( (p_m, p_f) \), we determined the values of \( x \) and \( \alpha \) in equilibrium using the fixed point approach. Table I lists the parameter values⁵ used for the numerical analysis. Unless otherwise specified (or varied), the parameter values used in the numerical analysis are the ones listed in Table I.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_o = 1 - f_t )</td>
<td>Fraction outside</td>
<td>0.5</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Activity Ratio ((\frac{x}{x}))</td>
<td>1</td>
</tr>
<tr>
<td>( b )</td>
<td>Broadband factor ((\frac{f_b}{R_0}))</td>
<td>2</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>Macrocell capacity ((\frac{R_0}{N_{cell}}))</td>
<td>0.5</td>
</tr>
<tr>
<td>( d )</td>
<td>Degradation coefficient</td>
<td>0.5</td>
</tr>
<tr>
<td>( \gamma_{max} )</td>
<td>Maximum User Type</td>
<td>1</td>
</tr>
</tbody>
</table>

Figures 6(a), 6(b) and 6(c) depict the optimal values of the normalized revenues, \( x \) and \( \alpha \) varying with the network capacity \( c_0 \) (defined in (19)). For each value of \( c_0, p_m \) and \( p_f \) have been chosen optimally. From Fig. 6(a), all femtocell schemes yield much higher revenues than with no femtocells. Further, this revenue gain is relatively more pronounced when the macrocell capacity is low. However, at these low levels of capacity, even the common spectrum scheme with \( d = 1 \) earns revenues comparable to the split spectrum deployment. As the degradation coefficient \( (d) \) increases, revenues from common spectrum deployments strictly decrease as expected. However, the common spectrum scheme with \( d = 0.1 \) earns higher revenues than the split spectrum scheme for all \( c_0 \), which confirms that a common spectrum scheme with low enough \( d \) may be superior to the split spectrum scheme.

V. CONCLUSION

In this paper, we provide an economic framework for the analysis of adoption of femtocells. We compared the economic viability of two spectrum schemes - split spectrum and common spectrum - for deployment of femtocells in a 4G network. We assumed that a single monopolist network operator sets prices for both the mobile-only service and the femto+mobile service. Users were assumed to possess linear utility for data throughput and have different valuations for data throughput. Our results suggest that the optimal pricing scheme always charges a higher price for the femtocell service. Further, at the optimal prices, almost full adoption of femtocells is achieved in most cases. As expected, if the degradation coefficient is sufficiently low, the revenues from the common spectrum scheme are always higher than with the split spectrum scheme. However, interestingly, when the macrocell capacity is low, though all femtocell deployments bring in higher revenues, the revenues from common spectrum schemes are comparable to the split spectrum even when they heavily degrade the macrocell capacity.

Though we assumed that the user benefit is linear in throughput, in reality, we expect it to be concave. The linearity impacts our results in two ways, yet we will argue that it does not markedly change our results. One, high femtocell throughput does not result in proportionally higher revenues from femtocells. Including this effect will reduce the viability of all spectrum schemes equally, without affecting the relative performance. Two, users lose utility if the throughput varies considerably during usage. Though it appears that the common spectrum scheme suffers more from this effect than the split spectrum scheme due to the random interference from femtocells, this may not necessarily be true. Since the macrocell
Proof of Theorem 2.1

The user with type $\gamma_m$ will be indifferent between buying some service and not buying, i.e., $U_m^f = 0$ or $U_f^m = 0$ depending upon the value of $\alpha$.

If $\alpha = 1$, the critical user type $\gamma_m$ prefers to buy service $f$ over service $m$, and he is indifferent between buying service $f$ and not buying anything:

$$\gamma_m g_m(1, x) - p_f = 0,$$
which gives us the desired result: $\gamma_m = \frac{p_f}{g_f(1, x)} \leq \frac{p_m}{g_m(1, x)}$.

If $\alpha < 1$, the critical user type $\gamma_m$ is indifferent between buying service $m$ and not buying anything and his utility from buying service $f$ is strictly lower:

$$\gamma_m g_m(\alpha, x) - p_m = 0,$$and $\gamma_m g_f(\alpha, x) - p_f < 0$,
which gives us the desired result: $\gamma_m = \frac{p_m}{g_m(\alpha, x)} < \frac{p_f}{g_f(\alpha, x)}$. Note that if $\gamma_m > \gamma_{m\max}$, then $\alpha = 0$.

For any given $x$, if $\alpha < 1$, the critical user type $\gamma_f$ is indifferent between service $m$ and service $f$, i.e., $U_f^m = U_f^f$:

$$\gamma_f g_m(\alpha, x) - p_m = \gamma_f g_f(\alpha, x) - p_f$$
which gives us the desired result: $\gamma_f = \frac{p_f - p_m}{g_f(\alpha, x) - g_m(\alpha, x)}$. Note that if $\gamma_f > \gamma_{f\max}$, then $\alpha = 0$.

Proof of Corollary 2.2

If $\alpha = 1$, for any $x$, $\gamma_m = \frac{p_f}{g_f(1, x)} \leq \frac{p_m}{g_m(1, x)}$. Then, it must hold in equilibrium that $p_f \leq g_f(\alpha, x)$. If $\alpha < 1$, for any $x$, $\gamma_m = \frac{p_m}{g_m(\alpha, x)} < \frac{p_f}{g_f(\alpha, x)}$. Then, it must hold in equilibrium that $p_f > g_f(\alpha, x)$. If $p_f \leq g_f(\alpha, x)$, $\alpha \neq 1$, which gives us $\alpha = 1$ and the desired result.

REFERENCES