Characterization of grid graphs

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Abstract

In this paper we are mainly interested in the characterization of grid graphs i.e. products of paths.

Introduction

A graph is called an \((n\text{-dimensional})\ p_1\cdot p_2\cdot \ldots \cdot p_n\)-grid if it is the product of \(n\) paths \(P_{p_1}, P_{p_2}, \ldots, P_{p_n}\). Those graphs are of special interest because they can be used for practical implementation of parallel algorithms. Fig. 1 shows the \(3\times 4\) grid \(P_3\times P_4\).

A cycle \(C \subseteq G\) is called a gap iff it is an isometric subgraph of \(G\), i.e. for any two of its vertices \(x, y\) their distances \(d_C(x, y)\) and \(d_G(x, y)\) are equal.

Let \(\mathcal{J}_n(G)\) denote the set of the different subgraphs induced (up to isomorphism) in \(G\) by intervals of length \(\leq n\).

2-dimensional grids

Theorem 1. A simple connected graph \(G\) is a 2-dimensional grid or a tree iff

\[
\mathcal{J}_2(G) \subseteq \{\emptyset, \{1, 0\}, \{2, 0\}\}.
\]
(ii) Any edge in $G$ belongs to at most two 4-cycles in $G$.
(iii) In $G$ are no other gaps $G$ than 4-cycles.

**Sketch of the proof.** It starts with easy observations and a lemma:

(a) $\forall u \in G \exists v \in G$ such that $d_G(u, v) = 3$ and $G$ as induced subgraph;
(b) $K_{2,3} \not\subseteq G$;
(c) every edge of $G$ is contained in at least one 4-cycle;
(d) $\forall x \in V(G) \quad d_G(x) \leq 4$.

**Lemma.** If $G$ is a connected graph satisfying (i), (ii) and (iii) then for any cycle $C \subseteq G$ there exist adjacent edges $e$, $e' \subseteq C$ and edges $e'' \in E(G)$ such that $e$, $e'$, $e''$ form a 4-cycle in $G$.

The proof of this lemma is rather long, since many cases have to be considered.

Now suppose $G$ fulfills the conditions of the theorem and consider a maximal tree $T$ such that there are no adjacent edges $e$, $e'$ from the tree contained in the same 4-cycle. We prove that either $G = T$ or there are pairwise disjoint isomorphic trees $T = T_1, T_2, \cdots, T_k$ in $G$ with $V(G) = \bigcup V(T_i)$ and $G = T \square P_n$ where $P_n$ is a path. From (ii) $T$ must be a path too and we are done.

**Theorem 2.** 2-dimensional grids can be recognized in linear time.

**Proof.** Assume the graph is given as a list of $n$ records representing the neighbors of successively all vertices. When reading those data we check that there are just 2, 3 or 4 neighbors in each record, making the input time at most $4n$. If no record corresponds to a degree 2 we are done and $G$ cannot be a grid. The procedure consists in a progressive embedding of $G$ in a grid $\subseteq \mathbb{N} \times \mathbb{N}$. We start with a vertex $s$ of degree 2, which we assign coordinates $(0,0)$. We now determine the unique square $s_0$ containing $s$, assigning its other vertices, on a standard way, the coordinates $(1,0)$, $(0,1)$ and $(1,1)$. If such a square does not exist or is not uniquely determined we return the answer NO, unless $G$ turns out to be a path (this of course can be checked in linear time). This leads us to distinguish a horizontal direction along $(0,0) \rightarrow (1,0)$ and a vertical one along $(0,0) \rightarrow (0,1)$. We can now ‘translate’ $s_0$ horizontally—as long as it is possible—and obtain a ladder $\lambda_0$ having $p$ ‘steps’. In such a translation we assign at each step coordinates...
to two new involved vertices of $G$ and check their degrees and adjacency relations
with the already embedded vertices. The computation time for $\lambda_0$ is bounded by
some $k_1 p + k'_1$. We now translate the ladder vertically—as long as possible—and obtain a $p-q$-grid. Here coordinates are assigned to $p + 1$ new involved
vertices and the corresponding degrees and adjacency relations are to be checked.
The computation time is $k_2 q + k'_2$ where $k_2 = kp + k'$. For the whole computation
we do not need more than $(kp + k')q + k'_2 = kpq + k'q + k'_2 < Cn + C'$. For
connectedness we verify $n = (p + 1)(q + 1)$. □

$n$-dimensional grids

The preceding result can be quite easily extended to 3-dimensional case, giving

**Theorem 1'.** A connected simple bipartite graph $G$ is a 3-dimensional grid iff:

(i) $\mathcal{I}(G) \subseteq \{\ldots, \ldots, \ldots\}$.

(i') $\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

(ii) Any induced square in $G$ 'belongs' to at most two cubes in $G$,

(iii) In $G$ are no other gaps than 4-cycles and 6-cycles.

Various considerations support however the following:

**Conjecture.** For $n \geq 1$ connected simple bipartite graph $G$ is a $n$-dimensional
grid iff

(i) $\{n$-cube$\} \subseteq \mathcal{I}_{n+1}(G) \subseteq \{\text{grids of diameter } \leq n + 1 \text{ excluding the } (n + 1)$-cube$\}$,

(ii) any $(n - 1)$-dimensional induced hypercube in $G$ 'belongs' to at most two
$n$-dimensional hypercubes in $G$,

(iii) in $G$ are no other gaps than $p$-cycles for $p \in [4, 2n]$.

As suggested by Horst Sachs (ii) may be replaced in the last conjecture by:

(ii') Any edge in $G$ belongs to $p$ 4-cycles where $2^{n-2} \leq p \leq 2^{n-1}$.

**Remark 1.** 'Grids of diameter $\leq n + 1$' can be obtained from 'grids of diameter $\leq n$' in adding them to the grids corresponding to the partitions of $n + 1$:
when upgrading from 2 to 3 we have to add to the grid
set \( \{0, 1, 2\} \) the grid corresponding to \( 1 + 1 + 1 = 3 \)

and those corresponding to \( 4 = 4 + 0 = 3 + 1 = 2 + 2 = 2 + 1 + 1 \) but not to \( 1 + 1 +

\[ 1 + 1 + 1 : \]

\[ \{0, 1, 2\} \).

**Remark 2.** The conjecture holds for \( n = 1, 2, 3 \).

**Remark 3.** The procedure presented by the proof of Theorem 2 extends obviously to the \( n \)-dimensional case:

**Theorem 2'.** For a given \( n \), there exists a linear algorithm for deciding if a graph is an (at most) \( n \)-dimensional grid.