Control of Temporal Constraints Based on Dioid Algebra for Timed Event Graphs

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Abstract

We consider a class of controlled timed event graphs subject to strict temporal constraints. Such a graph is deterministic, in the sense that its behavior only depends on the initial marking and on the control that is applied. As it is well-known, this behavior can be modelled by a system of difference equations that are linear in the Min-Plus algebra \((\mathbb{R} \cup \{+\infty\}, \min, \text{plus})\). The temporal constraint is represented by an inequation, that is also linear in the min-plus algebra. Then, a method for the synthesis of a control law ensuring the respect of the constraint is described. Two sufficient conditions are given, in terms of initial tokens and delays along the graph. We give explicit formulas characterizing a control law, which, if the conditions are satisfied, ensures the validity of the temporal constraints. This control law is also defined as a linear system over the Min-Plus algebra. It is a causal state feedback, involving delays. The method is illustrated on a production system.

1. Introduction

We consider in the sequel a class of deterministic controlled processes subject to strict time constraints. Such time critical systems are frequent in the industry, for instance in the case of a thermal or chemical treatment, and also in the food industry. Of course the question is to validate some temporal conditions, which has originated the contribution of many an author, see for instance [1, 4, 5, 6]. In the present contribution, we formulate the question in terms of initial tokens and delays along the graph. We give explicit formulas characterizing a control law, which, if the conditions are satisfied, ensures the validity of the temporal constraints. This control law is also defined as a linear system over the Min-Plus algebra. It is a causal state feedback, involving delays. The method is illustrated on a production system.

In the present paper, we propose a method for the synthesis of a control that permits to validate a given set of time constraints. The control law itself is finally defined by a Min-Plus linear difference equation, involving a finite number of delays. Such an equation corresponds to a timed event graph, too. The paper is organized as follows. In the section 2, some backgrounds are recalled, notably some notations concerning the Min-Plus semiring, the timed event graphs, their Min-Plus linear models, and the concept of state equations. The problem, of finding a causal control verifying critical time constraints, is formulated in Section 3, and we propose in Section 4 a procedure for the control synthesis. We first consider the case of a single temporal constraint. Two conditions are proposed, which are sufficient for ensuring the existence of a solution. A simpler condition, which is satisfied in many practical cases, and is simpler to check, is also provided. Then we extend the method to the case of many different constraints. Section 5 is devoted to an example, and finally Section 6 is devoted to the conclusion.
2. Backgrounds

2.1. Dioid algebra

We begin introducing some general concepts and notations which will be used below. A monoid is a set, say \( D \), endowed with an internal law, noted \( \odot \), which is associative and has a neutral element, denoted \( e \). A semiring is a commutative monoid endowed with a second internal law, denoted \( \otimes \), which is associative, distributive with respect to the first law \( \odot \), has a neutral element, denoted \( e \), and is for which \( e \) is absorbing : \( \forall a \in D, a \odot e = e \odot a = e \). Finally, a dioid is a semiring with an idempotent addition : \( \forall a \in D, a \odot a = a \). The dioid is called commutative if the second law \( \odot \) is commutative.

We shall consider in the sequel the so-called Min–Plus algebra, that is \( (\mathbb{R} \cup \{+\infty\}, \min, +) \). The Min–Plus algebra, denoted \( \mathbb{R}_{\min} \), is a commutative dioid for which the law \( \odot \) is the operation \( \min \), having the neutral element \( e = +\infty \), and the second law \( \otimes \), is the usual addition, with neutral element \( e = 0 \). If \( n \in \mathbb{N} \) and \( p, q \in \mathbb{R}_{\min} \), \( P \in \mathbb{R}_{\min}^{p \times n} \), \( Q \in \mathbb{R}_{\min}^{n \times q} \) are given matrices, \( P \odot Q \) (or just \( PQ \)) will denote the matrix multiplication in \( \mathbb{R}_{\min} \), defined by

\[
(P \odot Q)_{ij} = \min_{k=1}^{n} (P_{ik} \odot Q_{kj} ) = \min_{k=1}^{n} (P_{ik} + Q_{kj}) .
\]

The Kleene star of a square matrix \( M \in \mathbb{R}_{\min}^{n \times n} \), denoted \( M^* \), is defined by

\[
M^* = \bigoplus_{i \in \mathbb{N}} M^i .
\]

One can show that over \( \mathbb{R}_{\min} \), the Kleene star of a matrix with nonnegative entries is given by a finite sum, one has

\[
M^* = I_n \oplus M \oplus M^2 \oplus \ldots \oplus M^{n-1} ,
\]

where \( I_n \) denotes the unit matrix, which entries equal \( e \) on the diagonal, and \( e \) elsewhere. Let us recall that if \( v \in \mathbb{R}_{\min}^n \), then \( x = M^* v \) is the maximal solution of both the inequality

\[
x \leq M \cdot x \oplus v ,
\]

and the equality

\[
x = M \cdot x \oplus v .
\]

2.2. Timed event graphs and Min-Plus linear models

An event graph is an ordinary Petri net where each place has exactly one upstream transition and one downstream transition. A timed event graph is obtained associating delays to the places of an event graph. We shall use the following notations in the sequel. \( P \) will denote the set of places of the considered graph, and \( T \) its set of transitions. The number of transitions having at least one upstream place is denoted \( n \), and \( m \) stands for the number of source transitions, having no upstream place. If \( t_i, t_j \in T \), the unique place relying \( t_j \) to \( t_i \) is denoted \( p_{ij} \), if any, the corresponding delay is denoted \( \tau_{ij} \), and the initial marking of this place is denoted \( m_{ij} \). The maximal delay arising in the graph is denoted \( \tau^{\max} \), i.e.

\[
\tau^{\max} = \max_{p_{ij} \in P} \{ \tau_{ij} \} .
\]

A path \( \alpha \) from transition \( t_i \) to transition \( t_j \) is a sequence of transitions and places, of the form \( (t_i, p_{ik_1}, t_{k_1}, p_{k_1k_2}, t_{k_2}, \ldots, t_j) \), where \( p_{ik_1}, p_{k_1k_2}, \ldots \in P \). We denote \( \tau_\alpha \) the sum of delays along the path \( \alpha \)

\[
\tau_\alpha = \sum_{p_{kl} \in \alpha} \tau_{kl} .
\]

One associates to each transition of the considered timed event graph a function of the time \( t \), corresponding to the cumulated number of firings of the transition at time \( t \). Such a function is called a counter. The counters corresponding to source transitions form the components of a vector \( u(t) \in \mathbb{R}_{\min}^m \), and the ones corresponding to the other transitions form the components of the vector \( \theta(t) \in \mathbb{R}_{\min}^m \). As it is well known (see [3]), the dynamical behaviour of a timed event graph can be expressed by means of a linear equation over \( \mathbb{R}_{\min} \), as follows

\[
\theta(t) = \bigoplus_{\tau=0}^{\tau^{\max}} (A_{\tau} \cdot \theta(t - \tau) \oplus B_{\tau} \cdot u(t - \tau)) , \quad (1)
\]

where, for each value of \( \tau \), \( A_{\tau} \in \mathbb{R}_{\min}^{n \times n} \) is a matrix which entry \( A_{\tau,ij} \) equals \( m_{ij} \), the number of initial tokens in the place \( p_{ij} \), if this place exists and the associated delay is \( \tau \), and \( e \) else. Similarly the entries of the matrices \( B_{\tau} \in \mathbb{R}_{\min}^{n \times m} \) correspond to the initial tokens of the places following source transitions.

Equation (1) is implicit in general. It is worth replacing it by the following explicit equation,

\[
\theta(t) = \bigoplus_{\tau>0} \left( A_{\tau} \cdot \theta(t - \tau) \oplus A_0 \cdot B_{\tau} \cdot u(t - \tau) \right) , \quad (2)
\]

where \( A_0 \) is the Kleene star of \( A_0 \), defined in the previous section.

2.3. Explicit and state equation

For the sake of clarity, we shall now state some simplifying hypothesis. These are the following.
(H1) All the delays equal 0 or 1.

Analogously to the case of usual linear systems, the explicit equation 2 can be brought in state space form, if all the delays in the timed event graph are commensurable to a single delay. We can assume without loss of generality that this elementary delay equals 1, this is the hypothesis (H1).

For obtaining a state–space model, we first expand all the places with delay \( \tau > 1 \) into \( \tau \) places with delays equal to 1. Hence one adds \( \tau - 1 \) intermediate transitions and \( \tau - 1 \) intermediate places. The added intermediate transitions are associated to counters that form the components of a vector \( \theta(t) \in \mathbb{R}^{n'} \), and we denote \( x(t) \) the resulting extended state vector

\[
x(t) = \begin{bmatrix} \theta(t) \end{bmatrix},
\]

which belongs to \( \mathbb{R}^{N_{\min}} \), with \( N := n + n' \). Further one assumes that

(H2) The control acts without delay, i.e. all the delays associated to places with a downstream source transition are equal to 0.

This is not limitative, one can always add intermediate transitions and places so that the resulting extended graph satisfies this assumption. The dynamic of the expanded timed event graph is then described by an equation of the form

\[
x(t) = \hat{A}_0 \cdot x(t) \oplus \hat{A}_1 \cdot x(t-1) \oplus \hat{B} \cdot u(t),
\]

which can be rewritten into the explicit form

\[
x(t) = A \cdot x(t-1) + Bu(t), \tag{3}
\]

with \( A = \hat{A}_0^\ast \cdot \hat{A}_1 \), and \( B = \hat{A}_0^\ast \cdot \hat{B} \).

All these notations permit to point out that the behaviour of a controlled timed event graph is deterministic, depending on the input \( u(t) \) and on some initial conditions. This dependance can be explicitized, and we shall make use of the following formulation:

\[
x(t) = A^\tau \cdot x(t-\tau) \oplus \bigoplus_{k=0}^{\tau-1} A^k \cdot B \cdot u(t-k), \tag{4}
\]

which holds true, for every \( \tau \geq 1 \).

In the following, we shall assume that the input \( u(t) \) is actually a control, which can be arbitrarily assigned. For instance in a production process, the input can correspond to the authorization of performing a certain operation. Typically the beginning of a task performed by a robot, for instance, is subject to such a control input. Finally, we shall restrict our attention to the case of a single control:

(H3) There is only one control, say \( m = 1 \), and there is only one place downstream the unique source transition.

Hence the matrix \( B \) is actually a vector. Just one transition is controllable, say \( t_u \), which means that all the entries of this matrix are equal to \( \epsilon \), except \( B_{u1} \).

**Figure 1. A P-timed event graph**

### 2.4. Example

Let us consider the timed event graph of Figure 1. For this graph, Equation (1) reads

\[
\theta(t) = A_0 \theta(t) + A_1 \theta(t-1) + A_2 \theta(t-2) + A_4 \theta(t-4) + Bu(t),
\]

with

\[
A_0 = \begin{pmatrix} \epsilon & 2 & 2 \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix},
A_1 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 1 \\ \epsilon & \epsilon & \epsilon \end{pmatrix},
A_2 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix},
A_4 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix},
\]

and

\[
B = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \end{pmatrix}.
\]

The explicit equation (2) hence reads

\[
\theta(t) = \begin{pmatrix} \epsilon & \epsilon & 3 \\ \epsilon & \epsilon & 1 \\ \epsilon & \epsilon & \epsilon \end{pmatrix} \theta(t-1) \oplus \begin{pmatrix} 2 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix} \theta(t-2) \oplus \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix} u(t).
\]

Extending the initial graph to get a graph with delays normalized to 0 or 1, one obtains the graph of Figure 2.

Hence, the resulting state equation is

\[
x(t) = \begin{pmatrix} \epsilon & \epsilon & 3 & 2 & \epsilon & \epsilon & 2 \\ \epsilon & \epsilon & 1 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \end{pmatrix} x(t-1) \oplus \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} u(t).
\]
3. Problem formulation

3.1. Temporal constraint

Strict time constraints are frequent in industrial processes. One can for instance consider the example of a production process with a furnace for realizing a thermal treatment. The duration of any treatment in the furnace is fixed, or defined by a time interval. One wants to control the system to respect this constraint. The definition of a timed event graph already takes into account a delay on each place, that corresponds to a minimal holding time. The maximal duration appears as an additional constraint that should be verified. Rather than a verification problem, we formulate the question as a control problem.

Hence $p_{ij}$ is the place subject to a strict constraint, an interval $[\tau_{ij}, \tau_{ij}^{\max}]$ of time is associated to the place $p_{ij}$ where is a strict constraint, with $\tau_{ij}^{\min} = \tau_{ij}$, as in the figure 3.

This constraint is expressed through the following inequality,

$$m_{ij} + x_j(t - \tau_{ij}) \geq x_i(t) \geq m_{ij} + x_j(t - \tau_{ij}^{\max})$$

where $m_{ij}$ is the initial marking of the place $p_{ij}$. The left inequality is already taken into account by the linear model (3), so that the second one, say

$$x_i(t) \geq m_{ij} x_j(t - \tau_{ij}^{\max}) ,$$

where the product is over $\mathbb{R}_{\min}$, is actually the additional constraint to be validated.

3.2. Causal feedback

We consider a process modeled by the equation (3), subject to the additional constraint (5). We want to determine a control $u(t)$ ensuring the respect of (5) for $t > 0$. We shall a priori research this control in the form of a well posed causal feedback of the form

$$u(t) = F \cdot x(t - 1) ,$$

for $t > 1$, where the product is in the sense of the Min–Plus algebra, and $F \in \mathbb{R}_{\min}^{m \times N}$.

Remark 1 A static control of the form $u(t) = Fx(t)$ may result in an implicit loop, actually blocking the system, that is why we prefer using a delayed control law, always leading to a well-posed closed-loop system.

Consider for instance the event graph of Figure 1, and assume that this graph is subject to the additional constraint $x_2(t) \geq x_1(t - 3)$. We remark from the graph equations that $u(t) \geq x_1(t)$, therefore one can choose $u(t) = x_2(t)$, trying to validate the constraint. This control law is of the form $u(t) = Fx(t)$, with $F = [e, e, e]$. One can finally check that, actually, this is an implicit control law, resulting to a blocking of the closed-loop controlled event graph.
4. Control synthesis

4.1. Single constraint

We propose a method for the synthesis of a control law solving our problem, provided that the following additional hypothesis is satisfied.

(H4) There exists a path \( \alpha \) from \( t_u \) to \( t_j \), and we denote \( \tau_\alpha \) the cumulated delay along this path.

Taking \( \tau = \tau_\alpha \) in Equation (4), and from the definition of \( t_u \), we have

\[
x_j(t) \leq A_{j_u}^{\tau_\alpha} x_u(t - \tau_\alpha) ,
\]

and

\[
x_u(t) \leq u(t) ,
\]

from which it is clear that

\[
x_j(t) \leq A_{j_u}^{\tau_\alpha} u(t - \tau_\alpha) .
\] (6)

Applying again Equation (4) with \( \tau = \phi \), we obtain the following explicit expression for \( x_i(t) \)

\[
x_i(t) = \bigoplus_{r=1}^{N} A_{ir}^{\phi} x_r(t - \phi) \bigoplus \phi \left( (A^k B)_i u(t - k) \right) , \] (7)

for every integer \( \phi \geq 1 \), which is the key to obtain the following result.

**Theorem 1** Taking \( \phi = \tau_{ij}^{\max} + \tau_\alpha + 1 \), the inequation

\[
u(t) \leq \bigoplus_{r=1}^{N} (A_{ir}^{\phi} - A_{j_u}^{\tau_\alpha} - m_{ij} x_r(t - 1))
\]
defines causal controls which guarantee that the constraint (5) is satisfied if the two following sets of conditions hold:

(i) \( A_{ir}^{\phi} \geq A_{j_u}^{\tau_\alpha} + m_{ij} \), for \( r = 1 \) to \( N \),

(ii) \( (A^k B)_i \geq A_{j_u}^{\tau_\alpha} + m_{ij} \), for \( k = 0 \) to \( \phi - 1 \).

**Proof**

Taking (7) into account, it appears that the constraint (5) is satisfied if the two following conditions hold,

\[
\bigoplus_{r=1}^{N} A_{ir}^{\phi} x_r(t - \phi + \tau_{ij}^{\max}) \geq m_{ij} x_j(t) ,
\]

and

\[
\phi \left( (A^k B)_i u(t - k + \tau_{ij}^{\max}) \right) \geq m_{ij} x_j(t) .
\]

Further, taking (6) into account, these conditions become

\[
\bigoplus_{r=1}^{N} (A_{ir}^{\phi} - m_{ij}) x_r(t - \phi + \tau_{ij}^{\max}) \geq A_{j_u}^{\tau_\alpha} u(t - \tau_\alpha) ,
\]

and

\[
\bigoplus_{k=0}^{\phi-1} (A^k B)_i u(t - k + \tau_{ij}^{\max}) \geq m_{ij} x_j(t) .
\]

Choosing \( \phi = \tau_{ij}^{\max} + \tau_\alpha + 1 \) as in the theorem, the conditions (i) and (ii) being verified, and the control law satisfying the inequality of the theorem, one can check that the condition (5) is satisfied.

**Corollary 1** There always exists a causal control validating the constraint (5), if the initial markings of the place \( p_{ij} \) and of the places of any path \( \alpha \) from \( t_u \) to \( t_j \) are null. Such a control is given by

\[
u(t) = \bigoplus_{r=1}^{N} A_{ir}^{\phi} x_r(t - 1) .
\]

**Proof** In that case the conditions of the theorem respectively read

(i) \( A_{ir}^{\phi} \geq c \), for \( r = 1 \) to \( N \),

(ii) \( (A^k B)_i \geq c \), for \( k = 0 \) to \( \phi - 1 \).

Both conditions are trivially satisfied, and the expression of the control law follows.

Let us come back to the example of Remark 1. The maximal delay associated to the place \( p_{21} \) is in that case \( \tau_{21}^{\max} = \tau_{21}^{\max} = 3 \) units of time, the minimal delay being equal to 2. This temporal constraint correspond to the inequation

\[
x_2(t) \geq x_1(t - 3) .
\] (8)

The delay of the path from \( t_u \) to \( t_j \) equals \( \tau_\alpha = 2 \). Further, Equation (4) reads in that case

\[
x(t) = A^4 x(t - 4) \oplus \bigoplus_{k=0}^{3} A^k B u(t - k) ,
\]

with

\[
A^4 = \begin{pmatrix}
4 & 2 & e & 3 & 4 & 5 \\
2 & e & e & 1 & 2 & e \\
e & e & e & e & e & e \\
e & e & 5 & 4 & 2 & 3 \\
e & e & 3 & 2 & e & 1 \\
e & e & e & e & e & 1 \\
e & e & e & e & e & e \\
\end{pmatrix} , \quad B = \begin{pmatrix}
e \\
e \\
e \\
e \\
e \\
e \\
e \\
\end{pmatrix}.
\]

The conditions of Theorem 1 are trivially satisfied, and the following control law guarantees the respect of the temporal constraint (8)

\[
u(t) = \bigoplus_{r=1}^{7} A_{ir}^{\phi} x_r(t - 1) ,
\]

that is

\[
u(t) = 2x_1(t - 1) \oplus 1x_5(t - 1) \oplus 2x_6(t - 1) .
\]
4.2. Generalization to the case of multiple constraints

We consider again the case of a timed event graph, having one source transition which is a control, but \( Z \) places are constrained, say \( p_z \), for \( z = 1 \) to \( Z \). For each constrained place \( p_z \), let \( m_z, \tau_z, \tau_z^\text{max} \) respectively denote the initial marking, the minimal and maximal delays. Further, let \( t_z \) and \( t'_z \) respectively denote the input and output transitions of the place, \( x_z \) and \( x'_z \) denote the corresponding counters, and \( \lambda_z \) denote the cumulated delay along a path going from \( t_u \) to \( t_z \).

These constraints are expressed by the inequations

\[
m_z x_z (t - \tau_z) \geq x'_z (t) \geq m_z x_z (t - \tau_z^\text{max}) ,
\]

for \( z = 1 \) to \( Z \). We denote \( u_z (t) \) the control law calculated as in the previous section to satisfy the corresponding constraint (9).

**Theorem 2** The equation

\[
u(t) = \bigoplus_{z=1}^Z u_z(t) ,
\]

with

\[
u_z(t) = \bigoplus_{r=1}^N \left( A^{\phi_z}_{x'r} - A^{\lambda_z}_{x'u} - m_z \right) x_r (t - 1)
\]

and \( \phi_z = \tau_z^\text{max} + \lambda_z + 1 \), for \( z = 1 \) to \( Z \), defines a causal control ensuring the respect of all the constraints (9), if the two following sets of sufficient conditions are all satisfied, for \( z = 1 \) to \( Z \).

(iii) \( A^{\phi_z}_{x'r} \geq A^{\lambda_z}_{x'u} + m_z \), for \( r = 1 \) to \( N \),

(iv) \( (A^k B)_{x'r} \geq A^{\lambda_z}_{x'u} + m_z \), for \( k = 0 \) to \( \phi_z - 1 \).

**Proof** From Theorem 1, the following control law

\[
u_z(t) = \bigoplus_{r=1}^N \left( A^{\phi_z}_{x'r} - A^{\lambda_z}_{x'u} - m_z \right) x_r (t - \phi_z + \tau_z^\text{max} + \lambda_z)
\]

validate the 4th constraint, if are satisfied the conditions (iii) and (iv) of the theorem. In addition, we have, for \( z = 1 \) to \( Z \),

\[
u_z(t) \geq \bigoplus_{z=1}^Z u_z(t) .
\]

It is finally clear that \( u(t) = \bigoplus_{z=1}^Z u_z(t) \) validates the \( Z \) temporal constraints. \( \square \)

5. Example

Consider the timed event graph of Figure 4. It represents a simple production process, which includes a furnace (place \( P_1 \)), and a robot unloading the parts thermally treated (place \( P_4 \) corresponds to the operation, and place \( P_3 \) corresponds to the release). The delays associated to places \( P_1 \), \( P_3 \), and \( P_4 \) are of one time unit, and the input transition of the place representing the furnace is controlled. The problem consists in choosing the control \( u(t) \) so that the holding time in the furnace is exactly one unit of time, that is the nominal duration of the thermal treatment.

The state equation associated with the timed event graph of Figure 4 is

\[
x(t) = \begin{pmatrix} \epsilon & 2 & \epsilon \\ 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \end{pmatrix} x(t - 1) \oplus \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} u(t) ,
\]

where the components of \( x(t) \) are the counter functions associated to the transitions \( t_1 \), \( t_2 \), and \( t_3 \), and \( u(t) \) is the control. Further, the time constraint is expressed in terms of an inequality, say

\[
x_2(t) \geq 1 \cdot x_1(t - 1) ,
\]

for \( t \geq 1 \). We shall then apply Theorem 1 to calculate a control \( u(t) \) which guarantees that the inequation (11) is satisfied.

Notice that in this example, one has \( \tau_{ij}^\text{max} = \tau_{21}^\text{max} = 1 \), and \( \lambda = \epsilon \). The initial marking of place \( P_1 \) is \( m_{ij} = m_{21} = 1 \). Then we choose

\[
\phi = \tau_{21}^\text{max} + \lambda + 1 := 2 .
\]

Because of the property (4), Equation (10) implies

\[
x(t) = \begin{pmatrix} 3 & \epsilon & 3 \\ \epsilon & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix} x(t - 2) 
\]

\[
\oplus \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} u(t) \oplus \begin{pmatrix} \epsilon \\ 1 \\ \epsilon \end{pmatrix} u(t - 1) .
\]

We can check that one has \( A^1_{x'u} + m_{21} = 1 \), and \( A^2_{x'r} = \epsilon \), \( 1 \), \( \epsilon \) respectively, for \( r = 1 \), \( 2 \), \( 3 \), hence the conditions (i)
of Theorem 1 hold. Similarly, we check that $B_2 = \epsilon$, and $(AB)^2 = 1$, so that the conditions $(ii)$ of Theorem 1 hold too. Finally, according to Theorem 1, the control law
\[ u(t) = \bigoplus_{r=1}^{3} \left( A_{2,r}^2 - 1 \right) x_r(t - 1) := x_2(t - 1) \]
guarantees the respect of the time constraint.

6. Conclusions

We have defined two conditions which are sufficient for the existence of a causal control ensuring the satisfaction of a given temporal constraint in a controlled timed event graph. We have also applied this approach to the case of a timed event graph subject to multiple constraints.

This method was illustrated on the example of a production process. We trust that it could be valuable as well in different contexts, notably for the verification and validation of telecommunication processes and real-time software. Another important issue for future works would be considering the case of systems with multivariable control, and studying the necessity and conservativeness of the obtained conditions.

References