Energy Constrained Trajectory Generation for ADAS

Jérémie Daniel, Abderazik Birouche, Jean-Philippe Lauffenburger and Michel Basset

Abstract—This paper presents a new constrained trajectory generation method dedicated to Advanced Driver Assistance Systems (ADAS). Based on the information provided by the digital map database of a navigation system, and considering different constraints related to the road profile, the vehicle and the driver, a convex optimization algorithm generates specific Spline-based trajectories. Characteristically, these trajectories are safe, they stay within the traffic lane borders, at the same time minimize an energy criterion along the path, and finally, they are curvature continuous. The present solution has been tested on several roads and the results show the efficiency of the energy constrained trajectory generation method.

I. INTRODUCTION

Trajectory generation has been widely used in different domains as it can provide a set of continuous information which is helpful for control-oriented applications. In addition, most of the path generation solutions include constraints which help to define the optimal or suboptimal trajectory [1]. There are numerous trajectory generation solutions, based on the use of several mathematical models. Parametric Cubic Splines [2] are used here, because they represent a straightforward interpolation, they have a curvature continuity property, and they are adapted to all road contexts (bend, straight line, roundabout, etc.), contrary to the other models.

This paper presents a new constrained trajectory generation approach to control-oriented ADAS, based on the information provided by a digital map database. Considering the digital map limitations [3], the proposed solution is a map-based continuous trajectory generation formulated as an optimization problem. Contrary to recent research works such as [4] which focuses on road curve reconstruction and curvature estimation, the presented study is dedicated to the definition of a curvature-continuous trajectory for ADAS control applications, such as Lane Departure Avoidance, Longitudinal Control, etc. Taking account of the geometric constraints related to the road profile which keep the vehicle in a specified driving area, the constrained trajectories generated guarantee robustness regarding to the inaccuracies of the data provided by the digital map [3].

The main contribution of this paper lies in the integration of different constraints related to the road, to the vehicle and to the driver. This constrained generation is formulated as a convex optimization problem tending to minimize a given cost criterion. If most optimizations minimize the time required to cover the trajectory or the trajectory distance, here the strain energy of the trajectory is minimized. As this criterion is directly linked to the curvature, its minimization leads to the generation of smooth trajectories which can be of great help for trajectory tracking. Indeed, the proposed trajectories can be used as references for such systems thus replacing the usual trajectory represented by the road lane centreline. To check the validity of the present solution, tests were carried out using real road data. The results have shown that the new method leads to the generation of safe and smooth curvature continuous lane trajectories.

After the description of the work context in Section II, Section III describes the trajectory generation approach. Section IV then presents the results obtained with the trajectory generation process, finally followed by the conclusion in Section V.

II. WORK CONTEXT

A. Problem Statement

The solution proposed here is devoted to a trajectory generation approach for control-oriented ADAS taking account of constraints limiting the behaviour of a Wheeled Rolling System (WRS) and thus the control inputs to be generated. The objective is to provide a reference trajectory which can be followed by a WRS satisfying several constraints linked to the vehicle, the road to be followed and finally the driver. Giving a starting configuration \( q_0 = (x_0, y_0, \theta_0, k_0, \kappa_0) \in \mathbb{D}_0 \subset \mathbb{R}^5 \) defined by the WRS’s Centre of Gravity (CoG) position \( (x_0, y_0) \), the WRS’s orientation \( (\theta_0) \), an initial curvature \( k_0 \) of the path to be followed by the CoG, its respective derivative \( k_0 \) and a final configuration \( q_n = (x_n, y_n, \theta_n, k_n, \kappa_n) \), an optimization technique defines the reachable path regarding the optimization criterion considered. A trajectory is a continuous sequence of achievable configurations \( (x, y, \theta, \kappa, \kappa) \) defined with respect to the constraints to be verified. In the approach described, the road is considered to be flat and only 2-dimensional parametric paths \( (x(t), y(t)) \) defined in \( \mathbb{R}^2 \) are computed. The sequence of reachable configurations is defined, based on constraints of a different nature (geometric, dynamic and kinematic).

B. Multiple Constrained Path Generation

WRS-like cars are known to be non-holonomic systems. These systems are subject to limitations in the way they move: not all the solutions of the configuration space are possible and limitations in the directions of motion have to be processed [5].

This study considers three types of constraints. The first of these limitations are geometric constraints linked, on the one hand, to the configuration space in which the WRS moves (to keep the vehicle in a prescribed driving area) and on
a continuous-curvature trajectory is necessary. So, the gen-
nuities and since the steering angle is directly dependent on
the speed profile along the trajectory. The main
dynamic behaviour of the steering system is upper bounded.

Concerning the mechanical limitations of WRS, the lim-
itations of the steering system are considered, which imply
a minimum break radius of the vehicle \( R_{\text{min}} \). This turning
radius is lower-bounded and consequently, the trajectory
instantaneous curve radius must be constantly greater than
the lower bound vehicle turning radius:

\[
\kappa_{\text{trajectory}} = \frac{1}{R_{\text{trajectory}}} < \kappa_{\text{max}} = \frac{1}{R_{\text{min}}} \quad (1)
\]

Finally, in order to avoid any steering function disconti-
nuities and since the steering angle is directly dependent on
the instantaneous curvature of the trajectory to be followed,
a continuous-curvature trajectory is necessary. So, the gen-
erated trajectories must be \( C^2 \) continuous.

2) Kinematic Constraints: The kinematic constraints de-
pend on the speed profile along the trajectory. The main
kinematic constraint to be considered for WRS is the lim-
itation of the steering velocity. It is well known that the
dynamic behaviour of the steering system is upper bounded.

Since the steering velocity is related to the derivative of
the curvature, this implies that the trajectory instantaneous
curvature is upper-bounded:

\[
\kappa_{\text{trajectory}} < \kappa_{\text{max}} \quad (2)
\]

3) Dynamic Constraints: These constraints are due to the
limited and often nonlinear dynamic behaviour of the
WRS and its subsystems (bounded acceleration capabilities,
variable ground/wheel interaction, etc.). They mainly influ-
ence the longitudinal and lateral accelerations and thus the
velocities of the WRS. In order to provide safe trajectories,
the maximum centrifugal acceleration allowed for curve
negotiation is considered here. In the literature, different
models linking the centrifugal acceleration to the trajectory
curvature are available. This paper focuses on a simplified
model given by:

\[
\kappa_{\text{max}} = \frac{\Gamma_{\text{max}}}{\nu^2} \quad (3)
\]

with \( \Gamma_{\text{max}} \) the maximum allowed lateral acceleration and
\( \nu \) the vehicle speed. Note that in the automotive domain,
the driver is sensitive to the accelerations and mainly to the
lateral acceleration. Considering a maximum value of accel-
eration \( \Gamma_{\text{max}} \), driver-dependent factors are taken in account.
This relation implies that the path curvature must be upper
bounded in order to ensure a limited centrifugal acceleration:

\[
\kappa_{\text{trajectory}} < \kappa_{\text{max}} \quad (4)
\]

D. Energy Cost Criterion

An interesting solution to generate constrained trajectories
is to formalize the problem as an optimization problem.
Indeed, optimizers are powerful tools into which constraints
can be easily integrated and generate optimal trajectories, re-
garding these constraints. Furthermore, the different types of
optimization algorithms [7], mostly allow the minimization
of a cost criterion which can be defined, regarding various
elements or parameters. Common criteria are no criterion,
Trajectory length, Trajectory cover time, etc. In the proposed
constrained trajectory generation, the strain energy criterion
has been chosen. The latter is expressed according to the
trajectory curvature \( \kappa \) as follows [8]:

\[
E = \int \kappa^2 ds \quad (5)
\]

This criterion is in accordance with the aforementioned
constraints, especially with the constraints of a curvature-
dependent expression. In addition, if this energy is only
linked to the trajectory geometry, its minimization implies
smoothing the trajectory curvature which is directly linked
to the energy consumption of the vehicle.

III. NAVIGATION-BASED CONSTRAINED TRAJECTORY
GENERATION

A. Strategy

Fig.2 shows how the trajectory generation interacts with
the other components of the present ADAS control applica-
tion structure. The different elements of a navigation system
can be found: a GPS Receiver, a Map-Matching Algorithm, a Digital Map Database and an Electronic Horizon Provider. The Electronic Horizon (EH) is used as the source of information for the three main components of this structure: the Situation Classification, the Road Model Estimation (which is also used for the constraints definition) and the Trajectory Generation. Note that this paper only describe the trajectory generation process.

The strategy adopted for constrained trajectory generation, presented in Fig.3, is divided in two parts: the Road Model Estimation and the Trajectory Generation. The Road Model Estimation uses the road centerline shape points which are extracted from the EH. Based on these EH points, the first step is to use a transformation algorithm which generates the road boundary points. These points are then used by a Spline algorithm which provides continuous boundaries and gives the required information to the Trajectory Generation process. The latter uses the road model and the vehicle width to define the trajectory validity area (cf. Fig.1). This validity area is then used as the template for the optimization process which also includes the other aforementioned constraints (cf. Section.II-C) and which minimizes the trajectory energy.

### B. Mathematical model

Among all the different mathematical models, the Parametric Cubic Splines model [2] has been chosen. Indeed, this mathematical model, mostly used in computer graphics, provides smooth curvature-continuous trajectories contrary to other solutions such as arc-circle methods for instance. In addition, parametrization allows the computation of nearly all two-dimensional trajectories. A Cubic Spline is a piecewise polynomial interpolation. Contrary to basic interpolation methods, it avoids the use of large degree polynomials, which leads to trajectory oscillations. The two-dimensional Parametric Cubic Spline used here, with \( t \) the parameter, is of the following form:

\[
\begin{align*}
  f_i(t) &= a_{f_i} t^3 + b_{f_i} t^2 + c_{f_i} t + d_{f_i} \\
  g_i(t) &= a_{g_i} t^3 + b_{g_i} t^2 + c_{g_i} t + d_{g_i} \\
  t \in [t_i, t_{i+1}], \ i = 1, 2, \ldots, (n-1) \quad \text{with: } t_1 < t_2 < \ldots < t_n
\end{align*}
\]  

\( n \) being the number of interpolated points.

To provide the curvature continuity along the trajectory, first and second derivative continuity at each interpolated point is necessary. This is obtained by solving a linear system for each Cartesian coordinate. However, for one set of interpolated points, an infinite number of Splines is possible. These Splines are defined depending on two major elements: boundary conditions and parameter values. As they can have strong effects on the Spline shape, they must be correctly pre-defined. Information about the conditions and the parameter values selection can be found in [9].

#### C. Optimization Problem Formulation

This section describes the integration of the different constraints (cf. Section.II-C) into the selected mathematical model. Considering that Splines have polynomial expressions, the goal of the present trajectory generation process is to find the optimal coefficients \( a_{f_i}, b_{f_i}, c_{f_i}, d_{f_i} \) and \( a_{g_i}, b_{g_i}, c_{g_i}, d_{g_i} \) with:

\[
S_i = [x_i, y_i]^T, \ i = 1, 2, \ldots, n-1 \text{ for the constrained trajectory } f_i(t) \text{ (cf. } 6) \text{ with } t \text{ defined in } [0, 1].
\]

1) Inequality Constraints: Let \( g_i(t) \) and \( e_i(t) \) be the two boundary curves of the validity area. The optimization algorithm must find a Spline \( f_i(t) \) such that:

\[
\begin{align*}
  e_i(t) &\leq f_i(t) \leq g_i(t) \quad \text{with:} \\
  g_i(t) &= g_S(t) = a_{g_S} t^3 + b_{g_S} t^2 + c_{g_S} t + d_{g_S} \\
  e_i(t) &= e_S(t) = a_{e_S} t^3 + b_{e_S} t^2 + c_{e_S} t + d_{e_S}
\end{align*}
\]

To generate a Spline which is constricted by the validity area boundaries, the difference between the desired Spline and the upper and lower bounds must be negative or positive respectively:

\[
\begin{align*}
  f_S(t) - g_S(t) &\leq 0 \quad \text{and} \quad f_S(t) - e_S(t) \geq 0
\end{align*}
\]

Remember that the optimization goal is to find the optimal set of Spline coefficients. The current requirement is also to
find Spline positivity conditions regarding the coefficients. In the literature, several studies have been carried out on the positivity of cubic polynomials. It has been shown that positivity conditions for a cubic polynomial $f(t) = at^3 + bt^2 + ct + d$ are described with $t \in [0, 1]$ by two sets of inequalities [10]. The present study focuses on the following set of inequalities:

$$
f(t) \geq 0 \Rightarrow (a, b, c, d) \in A, \ \text{with:}
A = \{a + b + c + d \geq 0, b + 2c + 3d \geq 0, c + 3d \geq 0, d \geq 0\} \tag{9}
$$

The adaptation of (9) to the present context gives the set of linear inequalities presented here with $\Gamma = \{e_{S_i}, f_{S_i}, g_{S_i}\}$:

$$
\begin{align*}
\alpha_{e_{S_i}} &\leq \alpha_{f_{S_i}} \leq \alpha_{g_{S_i}} \\
\beta_{e_{S_i}} &\leq \beta_{f_{S_i}} \leq \beta_{g_{S_i}} \\
\gamma_{e_{S_i}} &\leq \gamma_{f_{S_i}} \leq \gamma_{g_{S_i}} \\
\delta_{e_{S_i}} &\leq \delta_{f_{S_i}} \leq \delta_{g_{S_i}}
\end{align*}
\tag{10}
$$

2) Equality Constraints: In addition to the inequalities, the optimization process must fulfill the $C^0$, $C^1$ and $C^2$ continuity requirements. For Parametric Cubic Splines, this is expressed by:

$$
\begin{align*}
s_f(t_{i+1}) &= s_f(t_{i+1}) \\
f_f(t_{i+1}) &= f_f(t_{i+1}) \\
s_f(t_{i+1}) &= s_f(t_{i+1}) \\
\Rightarrow & \quad \begin{align*}
\alpha_f &= \alpha_f + b_f + c_f + d_f = d_{f_{i+1}} \\
3\alpha_f + 2b_f + c_f &= c_{f_{i+1}} \\
6a_f + 2b_f + c_f &= 2b_{f_{i+1}}
\end{align*}
\tag{11}
\end{align*}
$$

It can be noted that geometric constraints (limitations of the configuration space and continuity of the trajectory) are explicitly formulated in the optimization through (10) and (11) and that the kinematics and dynamics constraints are implicitly described by the criterion to be minimized (the minimization of the strain energy should grant low curvature variations and values). However, a post-checking of these hypotheses is performed after the optimization. Note that the checking of the curvature derivative $\kappa$ is based on [5] and is expressed in a conservative way by:

$$
\kappa = \frac{\ddot{y}}{v} \tag{12}
$$

with $v$ a constant velocity, $b$ the vehicle wheelbase and $\phi$ the wheel angle.

D. Cost Criterion

As mentioned previously, the selected cost criterion is the trajectory strain energy which is directly linked to the curvature $\kappa$:

$$
E = \int \kappa^2 ds \tag{13}
$$

with:

$$
\kappa(x(t), y(t), t) = \frac{\ddot{y}(t) \dot{x}(t) - \ddot{x}(t) \dot{y}(t)}{(\dot{x}^2(t) + \dot{y}^2(t))^{3/2}} \tag{14}
$$

A curve minimizing this bending energy is known as a minimal energy Spline [11]. Nevertheless, replacing $x(t)$ and $y(t)$ functions into their formal expressions, involve (13) to become non-linear. The optimal solution minimizing (13) is also hard to formulate and to calculate. To overcome this problem, a suboptimal solution is determined. Considering the expressions of $x = f_s(t)$ and $y = f_f(t)$, the objective is to simultaneously minimize the curvature of each parametric curve $\kappa_s$ and $\kappa_f$ given by:

$$
\kappa_s(x(t), t) = \frac{\ddot{x}(t)}{(1 + \dot{x}^2(t))^2} \quad \text{and} \quad \kappa_f(y(t), t) = \frac{\ddot{y}(t)}{(1 + \dot{y}^2(t))^2} \tag{15}
$$

Considering that $\ddot{y}(t)^2$ and $\ddot{x}(t)^2$ are small compared with 1 as in [12] and [13], the energy of the suboptimal solution, corresponding to the cubic interpolation Spline with $C^2$ continuity, is assumed to be:

$$
E = \int_{t_0}^{t_f} (\ddot{x}^2 + \ddot{y}^2) \ dt \tag{16}
$$

Considering the Spline expression and its continuity properties, the continuous expression of the suboptimal energy $E$ can be discretized using the classic Euler method with $h_i = t_{i+1} - t_i$ such that [13]:

$$
\begin{align*}
\dot{E} &= \sum_{i=1}^{n-2} \frac{4h_i}{3} \left( b_{f_{i+1}}^2 + b_{f_{i+1}} b_{f_{i+1}} + b_{f_{i+1}}^2 \right) \\
&\quad + \sum_{i=1}^{n-2} \frac{4h_i}{3} \left( b_{f_{i+1}}^2 + b_{f_{i+1}} b_{f_{i+1}} + b_{f_{i+1}}^2 \right) \tag{17}
\end{align*}
$$

E. Optimization

The trajectory generation process presented in this paper and formulated as an optimization problem has to fulfill the aforementioned conditions: linear inequality constraints (cf. (10)), linear equality constraints (cf. (11)) and a discretized quadratic cost criterion (cf. (17)). There are several optimization techniques which help to solve such systems (cf. [7] for additional information). However, considering the different types of constraints and the fact that they all have a linear or quadratic expression, the convex quadratic programming optimization approach has been chosen:

$$
\Theta^* = \min_{\Theta} \frac{1}{2} H \Theta + J^T \Theta \quad \text{such that:} \quad \begin{align*}
A \Theta &= B \\
C \Theta &\leq D
\end{align*} \tag{18}
$$

with $\Theta \in \mathbb{R}^{2n \times (n-1)}$ the optimal coefficients of $f_i(t)$ interpolating $n$ points such that:

$$
\Theta = [a_{f_s_1}, b_{f_s_1}, c_{f_s_1}, d_{f_s_1}, ..., a_{f_s_{n-1}}, b_{f_s_{n-1}}, c_{f_s_{n-1}}, d_{f_s_{n-1}}]^T \tag{19}
$$

It is clear that this algorithm is well suited for the present problem since:

- The equality and inequality constraints can respectively be written in the $A \Theta = B$ and $C \Theta \leq D$ matrix form,
- The energy criterion, due to its discrete formulation, is only expressed using the quadratic terms of the different
\[ b_{f_1} \text{ and } b_{f_2} \text{ coefficients. It can also be easily expressed in the matricial expression } \Theta^T H \Theta \text{ but, as there are only quadratic elements, } f^T \Theta = 0, \]

- This optimization is convex quadratic, so there is only one global minimum,
- This algorithm does not need long calculation time which may coincide with real-time constraints.

IV. Testing Results

A. Testing Conditions

Validation tests of the trajectory generation process were carried out using data collected from previous tests. These data correspond to the EH shape points of classic country roads. Stored shape points are used as the basic information for the road estimation and for the Trajectory Generation process (cf. Fig.3). The results presented in this Section were obtained using two different trajectory generation processes. The first process only includes the geometric constraints while the second one presents the results obtained with the addition of the Spline strain energy minimization. The aim is to clearly show the impact of the energy minimization on the generated trajectories. Results are described in Fig.5 and for clarity reasons, the road boundaries have been removed from the figure. Only the validity area boundaries and the two trajectories are presented. Fig.6 presents the associated curvatures: the road centerline curvature (which is not represented in Fig.5) and the curvatures of both constrained trajectories. A 7m width road and a 2m width vehicle have been taken for this test.

B. Results

Fig.5 presents the trajectory generation results for a left bend which is not a constant-curvature bend. Globally the differences between the trajectories cannot be distinguished easily, due to the narrow validity area. However, it can be noticed that the trajectory which minimizes the energy (dash-dotted line) uses the lateral space available in the validity area more efficiently; it successively uses the external and the internal parts of this area, contrary to the other trajectory (dashed line) which stays close to the validity area centre. It can therefore be guessed that the curvature shape for the trajectory which does not minimize the energy will be very close to the road centreline curvature.

This guess is confirmed in Fig.6. Indeed, the trajectory which does not minimize the energy presents a shape similar to that of the road centreline. In addition, it is clear that the trajectory minimizing the energy smoothes the curvature.

A consequence of this smoothing is that the trajectory is granting the strict constraint on the curvature presented in (1). Indeed, the maximal curvature value passes from 0.11m\(^{-1}\) to 0.07m\(^{-1}\) which is lower than the car maximum curvature of 0.09m\(^{-1}\) (taking an average turn radius of 11m). Considering the constraint described by (3) and (4), the maximum curvature of the best trajectory which is 0.07m\(^{-1}\) and a legal speed limit of 30km.h\(^{-1}\) (given by the map for this situation when following the trajectory), a lateral acceleration of 4.9m.s\(^{-2}\) is obtained. This is larger than the common lateral acceleration value used for comfortable driving (3m.s\(^{-2}\)). A solution to this problem could be to reduce the vehicle speed. Finally the constraints put on the curvature derivative (2) is granted as the generated trajectory has a maximum curvature derivative value of 0.029m\(^{-2}\).s\(^{-1}\) which is lower than the threshold of 0.038m\(^{-2}\).s\(^{-1}\) (obtained by using (12) with a wheelbase of 2.5m, a speed of 30km.h\(^{-1}\), and a maximum steering angle speed of 15.7rad.s\(^{-1}\) achievable with an electrically driven steering wheel).

C. Energy Comparison

Table.I presents the different energies computed respectively without (\(E_{NME}\)) and with (\(E_{WME}\)) the energy minimization in the trajectory generation process. In addition, the reduction efficiency is described (\(Efficiency = \frac{E_{NME} - E_{WME}}{E_{NME}}\)). Note that the first test of this table corresponds to the test presented in the previous section. Five additional tests were also carried out using other real data measurements. It must first be noted that the integration of the energy criterion has a positive impact on the smoothness of the trajectories (the energy is reduced in all cases). Then, the energy reduction is variable as it passes from 0.8% to 31.9%. The variation of the minimization efficiency is mainly due to the initial smoothness of the road geometry.

V. Conclusion

Considering the different requirements for safe, comfortable driving, this paper has proposed a new solution for the generation of constrained trajectory through the integration of constraints into a Parametric Cubic Spline model and through the formulation of the problem as an optimization problem. The information provided by the digital map
database of a navigation system helps to generate the tra-
djectory validity area (corresponding to the limit boundaries
of the reachable trajectories) which is useful in the formal-
ization of the constraints as a convex optimization problem.
In addition, the minimization of the trajectory strain energy
has been implemented to generate suboptimal trajectories.
The test results have shown the efficiency of the solution as
the trajectories were always located in the validity area and
minimized the energy. This prepares the way for numerous
applications in the automotive domain, but also in others as
on-ground aeronautics or wheeled robotics.

In the future, the proposed solution will be improved with
the explicit integration of the dynamic and kinematic con-
straints presented in (2) and (4) directly in the optimization
formulation, or considering additional constraints related to
the vehicle (wheel/road interaction, etc.), to the road profile
(elevation), and to the driver. Another research line can be
related to the modification of the cost criterion (distance
dependent, etc.).

REFERENCES

curvature path generation based on B-spline curves for parking ma-
1978.
sensor - dynamic road curve reconstruction for a curve speed assistant.
[5] T. Fraichard and A. Scheuer. From reeds and shepp’s to continuous-
curvature paths. IEEE Transactions on Robotics and Automation,
trajectory library for wheeled mobile robots. Journal of dynamic systems,
[8] H. Delingette, M. Hebert, and K. Ikeuchi. Trajectory generation with
curvature constraint based on energy minimization. In International
Workshop on Intelligent Robots and Systems (IROS), Osaka, Japan,
trajectory generation for advanced driver assistance systems applica-
tions. In International Forum On Strategic Technologies (IFOST09),
[10] J.W. Schmidt and W. Heß. Positivity of cubic polynomials on inter-
vals and positive spline interpolation. BIT Numerical Mathematics,
Series Analysis, Signal Processing, and Dynamics, pages 293 – 322.