Portfolio Insurance Strategies: 
OBPI versus CPPI

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Abstract

We compare performances of the two standard portfolio insurance methods: the Option Based Portfolio Insurance (OBPI) and the Constant Proportion Portfolio Insurance (CPPI). First we examine basic properties of these two strategies and compare them by means of various criteria: comparison of their payoffs, possible property of stochastic dominance, expectations, variances, skewness and kurtosis of their returns, and some of the quantiles of their returns. We prove that the OBPI method can be analyzed as a kind of CPPI where the multiple is allowed to vary. We then study the properties of this varying multiple.

In a second section, we analyze more deeply both method's dynamic properties. We turn our attention to the "dynamics management" involved by these two strategies. Although the pure OBPI do not require any management by the buyer (if the put or call option is available on the market), we can calculate the "greeks" of its call part. We derive the "greeks" of the CPPI and show the very different nature of the dynamic properties of the two strategies.
1 Introduction

A popular and simple strategy of portfolio insurance is the Option Based Portfolio Insurance (OBPI). The simplest way is to consider a portfolio invested in a risky asset $S$ (generally a financial index) covered by a put written on it. Whatever the value of $S$ at the terminal date $T$, the portfolio value will be always greater than the strike $K$ of the put. Of course, if we build a portfolio with quantities $q$ of $S$ and $q$ of the put, the insured amount will be $qK$. To simplify the presentation, we shall assume that $q$ is normalized and set equal to one. At first glance, the goal of the OBPI method is to guarantee a fixed amount only at the terminal date. In fact, as shown in this paper, the OBPI method allows to get a portfolio insurance at any time.

The CPPI method uses a simplified strategy to allocate assets dynamically over time. The investor starts by setting a floor equal to the lowest acceptable value of the portfolio, computes the cushion as the excess of the portfolio value over the floor, and then determines the amount allocated to the risky asset by multiplying the cushion by a predetermined multiple. Both the floor and the multiple are functions of the investor’s risk tolerance and are exogeneous to the model. The total amount allocated to the riskier asset is known as the exposure. The remaining funds are invested in the reserve asset, usually Treasury bills or other liquid money market instruments.

The higher the multiple, the more the investor will participate in a sustained increase in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero too. In continuous time, this keeps portfolio value from falling below the floor. Portfolio value will fall below the floor only when there is a very sharp drop in the market before the investor has a chance to trade.

Advantages of this strategy over other approaches to portfolio insurance are its simplicity and its flexibility (see for example De Vitry and Moulin (1994) and Black and Rouhani (1987)). Initial cushion, multiple, floor and tolerance can be chosen according to the investor’s own objective.

The purpose of this paper is to compare these two strategies of portfolio insurance. In a first section, we recall the basic properties of these two strategies and compare them by means of various criteria: comparison of their payoffs, possible property of stochastic dominance, expectations, variances, skewness and kurtosis of their returns, and some of the quantiles of their returns. We prove that the OBPI method can be analyzed as a kind of CPPI where the multiple is allowed to vary. We then study the properties of this varying multiple.

In a second section, we analyze more deeply both method’s dynamic properties. We turn our attention to the “dynamics management” involved by these two strategies. Although the pure OBPI do not require any management by the buyer (if the put or call option is available on the market), we can calculate the “greeks” of its call part. We derive the “greeks” of the CPPI and show the very different nature of the dynamic properties of the two strategies.
2 Comparison between the standard OBPI and the CPPI at maturity.

2.1 Definition of the two strategies.

The period of time considered is \([0;T]\). Denote \(B_t\) the riskless asset. Assume that it is given by:

\[ dB_t = B_t r dt \]

Assume that the risky asset \(S_t\) is a diffusion process:

\[ dS_t = S_t \left[ a(t; S_t) dt + \frac{1}{2} \sigma(t; S_t) dW_t \right] \]

where \((W_t)_t\) is a standard Brownian motion.

For the OBPI method, introduce the portfolio value \(V_{OBPI}^T\) which is defined at the terminal date by:

\[ V_{OBPI}^T = S_T + (K - S_T)^+ \]

which is also:

\[ V_{OBPI}^T = K + (S_T - K)^+ \]

due to the Put/Call parity.

The value of the portfolio is normally insured only at the final date \(T\). In fact, for all date \(t\) before \(T\), there is a deterministic level of insurance determined by \(Ke^{r(T-t)}\).

Examine now the CPPI method. We consider the case where the floor \(F_t\) is not stochastic. The process \(F_t\) follows the dynamic given by:

\[ dF_t = F_t r dt \]

Obviously, the initial floor \(F_0\) is less than the initial portfolio value \(V_{CPPI}^0\): Denote at time \(t\) \(F_0\) the values of the portfolio and of the cushion respectively by \(V_{CPPI}^t\) and \(C_t\). By definition, the cushion is equal to the difference between the portfolio's and the floor's values:

\[ C_t = V_{CPPI}^t - F_t \]

Denote by \(e_t\) the exposure which is the total amount invested in the risky asset. The standard CPPI method consist in letting \(e_t = mC_t\) where \(m\) is a constant called the multiple. The interesting case is when \(m > 1\) and usually, the multiple \(m\) is less than 10.

The CPPI method is parametrized by \(F_0\) and \(m\). The OBPI has just one parameter, the strike \(K\) of the put, which plays the same role as \(F_0e^{rT}\) in the CPPI model. Assume that the decision criteria of the investor is the amount insured, \(K\), at maturity \(T\). In order to compare the two methods, we first assume that the initial amounts \(V_{OBPI}^0\) and \(V_{CPPI}^0\) are equal. Secondly, we assume that they provide the same guarantee \(K\) at the final date. So, we must take \(F_0 = Ke^{r(T-t)}\), in order to have \(F_T = K\).
3 Comparison of the portfolio values at maturity

An interesting problem is to compare the terminal payoffs of the two strategies as functions of \( S_T \), for all values of the multiple, \( m \) greater than one and for strikes \( K \) at, in and out the money. In fact, we want to know for example if it is possible that a payoff function for one of these two strategies is above the other for all \( S_T \) or at a given level of probability. We can also examine the four first expectations of the returns...

3.1 Comparison of the payoff functions

For both the CPPI and the OBPI, we set the same initial value of the portfolio:

\[
V_0 = V_0(K) = S_0 + P^K_0 = Ke^{rT} + C^K_0;
\]

where \( P^K_0 \) (respectively \( C^K_0 \)) denotes the price at time 0 of an European put (resp. call) written on \( S \) with a strike equal to \( K \). For the OBPI, the time \( T \) value of the strategy is defined by:

\[
V_T = K + (S_T - K)^+ = K \text{ if } S_T < K;
= S_T \text{ if } S_T > K;
\]

The risky part of the OBPI is simply the call value at \( T \).

Remark 1 The amount insured at the final date is often expressed as a percentage of the initial investment \( V_0 \). Since here, this amount is equal to the strike \( K \) itself, some additional constraints must be imposed on the strike like the following ones:

\[
aV_0(K) \cdot K \cdot bV_0(K);
\]

with for example, \( a = 0; 9 \) and \( b = 1; \) This condition is equivalent to:

\[
a(Ke^{rT} + C^K_0) \cdot K \cdot b(Ke^{rT} + C^K_0);
\]

Note that this condition is in fact compatible with market conditions on the ratio "strike on spot" (= \( \frac{K}{S_0} \)) such as:

\[
c \cdot \frac{K}{S_0} \cdot d;
\]

(See numerical examples in the subsection devoted to "quantile" criteria.)

For the CPPI, the time \( T \) value of the strategy is defined by \(^3\):

\[
V_T(m) = K + C_T(m); \quad h \quad h
= K + (V_0 + F_0) \exp \frac{m^2 \sigma^2 T}{2} + r + m(a_i - r)i \quad m^2 \sigma^2 \quad i \quad T;
= K + (C^K_0 \exp \frac{m^2 \sigma^2 T}{2} + r + m(a_i - r) \quad m^2 \sigma^2 \quad T;
\]

\(^3\)Since this result is well known, we do not detail this calculation. We refer to Prigent [12] and [13].
We are interested in comparing the terminal value of the risky exposure of these two strategies knowing that the riskless part are equal (to \( K \)).

Assume for example that the risky asset price follow a geometric Brownian motion as in the Black and Scholes model:

\[
S_T = S_0 \exp \left( \frac{\theta}{2} \right) + \mu \left( \frac{\sqrt{T}}{2} \right).
\]

\[
S_T = S_0 \exp \{ X_T \};
\]

with \( X_T = \frac{\theta}{2} \) \( W_T + \frac{\mu}{2} \frac{\sqrt{T}}{2} \).

Note that we get also:

\[
e^{\theta W_T} = \frac{s_T}{s_0} \exp \left( \frac{\mu}{2} \frac{\sqrt{T}}{2} \right);
\]

or equivalently:

\[
W_T = \frac{1}{\sqrt{T}} \ln \left( \frac{s_T}{s_0} \right) \frac{\mu}{2} \frac{\sqrt{T}}{2}.
\]

The cushion of the CPPI can be written as:

\[
C_T (m; S_T) = \pm S_T^m;
\]

which has the form:

\[
C_T (m; S_T) = \gamma S_T^m;
\]

with

\[
\gamma = \frac{c^K}{s_0} \exp \left( -T \right);
\]

where we note:

\[
\gamma = m \mu \frac{\sqrt{T}}{2} \frac{\sqrt{T}}{2}.
\]

\( \gamma \) is a constant with respect to \( S_T \). \( C_T (m; S_T) \) is a convex function in \( S_T \) as soon as \( m > 1 \), which is the usual assumption.

**Proposition 1** None of the two payoffs is greater than the other, for all terminal values of the risky asset. The two payoffs functions intersect one another.

**Proof.** There are two basic securities on the financial market, the riskless asset and the risky asset (which is supposed to follow a geometric Brownian motion \( W_t \) so the market is arbitrage-free). If \( V_0^{OBPI} = V_0^{CPPI} \), the absence of arbitrage implies that it is impossible to obtain:

\[
8S_T \in [0; 1] \Rightarrow V_T^{OBPI} > V_T^{CPPI} (\text{or} ; 8S_T ; V_T^{OBPI} < V_T^{CPPI});
\]

To illustrate what happens, we consider the following numerical examples with typical values for the financial markets (parameters : \( a, \frac{\sqrt{T}}{2} r \) : \( S_0 = 100; \)
\( a = 10\%, \ \frac{\Delta}{4} = 20\%, \ T = 1, \ K = 100, \ r = 5\%: \) As \( m \) increases, the payo® function becomes more convex.

We can check on this example that the two curves intersect one another for the different values of \( m \) considered (\( m = 2; m = 4; m = 6 \) and \( m = 10 \)).

### 3.2 Comparison with the stochastic dominance criterion.

We now explicitly take into account the risky dimension of the terminal payo® functions for the two methods. Recall that the only source of risk is the risky asset which is assumed to follow a geometric Brownian motion.

We first examine a weaker property on the payo® functions of the two strategies: the rst-order stochastic dominance.

Recall that a random variable \( X \) stochastically dominates a random variable \( Y \) at the rst order (\( X \preceq Y \)) if and only if the distribution function of \( X \), denoted by \( F_X \), is always smaller than the distribution function \( F_Y \) of \( Y \):

**Proposition 2** None of the two strategies stochastically dominates at the rst-order the other.

**Proof.** See appendix.

### 3.3 Comparison of the expectation, variance, skewness and kurtosis.

When dealing with options, the mean-variance approach is not always justified since payo®s are not linear. So we examine simultaneously the rst four moments.

**Case 1: The OBPI method**

Recall that the portfolio return is given by \( R_{T}^{OBPI} = \frac{V_{T}^{OBPI}}{V_{0}^{OBPI}} \). Thus its expectation is given by:
E[R_t^{OBPI}] = \frac{1}{\sqrt{V_{OBPI}}} \cdot \int E[(S_t; K) + s_i K] f_{S_t}(s) ds.

Its variance is equal to:

\text{Var}[R_t^{OBPI}] = \frac{1}{\sqrt{V_{OBPI}}} \cdot \int \left( \int E[(S_t; K)]^2 f_{S_t}(s) ds \right) \cdot \left( \int f_{S_t}(s) ds \right)^2 ds.

The skewness and the kurtosis are given by:

\text{Skew}[R_t^{OBPI}] = \frac{1}{\sqrt{V_{OBPI}}} \cdot \int \left( \int (S_t; K)^3 f_{S_t}(s) ds \right) \cdot \left( \int f_{S_t}(s) ds \right)^3 ds.

\text{Kur}[R_t^{OBPI}] = \frac{1}{\sqrt{V_{OBPI}}} \cdot \int \left( \int (S_t; K)^4 f_{S_t}(s) ds \right) \cdot \left( \int f_{S_t}(s) ds \right)^4 ds.

Case 2: The CPPI method

Recall that the portfolio return is given by R_t^{CPPI} = \frac{V_{CPPI}}{V_0}. Thus its expectation is given by:

E[R_t^{CPPI}] = \frac{1}{\sqrt{V_{CPPI}}} \cdot \int [K + \circ S_t^m] f_{S_t}(s) ds.

Since S_t has a lognormal distribution, we deduce that:

E[R_t^{CPPI}] = \frac{1}{\sqrt{V_{CPPI}}} \cdot \int [K + C_0(r; K) \exp[(r + m(a_i \cdot r))] ds.

Its variance is equal to:

\text{Var}[R_t^{CPPI}] = \frac{1}{\sqrt{V_{CPPI}}} \cdot \int [K + C_0(r; K) \exp[2A(m) + B(m)^2]\exp[B(m)^2]] ds.

with A(m) = (r + m(a_i \cdot r)) \left( \frac{1}{2} m^2 \right) T

and B(m) = m^2 T.

The skewness and the kurtosis are given by:

\text{Skew}[R_t^{CPPI}] = \frac{1}{\sqrt{V_{CPPI}}} \cdot \int \left( \int (S_t^m; K) \exp[(r + m(a_i \cdot r))] ds \right) ds.

\text{Kur}[R_t^{CPPI}] = \frac{1}{\sqrt{V_{CPPI}}} \cdot \int \left( \int (S_t^m; K) \exp[(r + m(a_i \cdot r))] ds \right) ds.

Remark 2 To compare the expectations of the returns, we note that:

E[R_t^{OBPI}] \cdot E[R_t^{CPPI}] = \exp[T C_0(a; K) \cdot C_0(r; K) \exp[(r + m(a_i \cdot r))] T].

If we compare the first two moments (mean-variance analysis), note that for m high, the expectation and variance of the CPPI portfolio are greater than those of the OBPI one and so there is no-dominance with respect to the mean-variance criterion.
Remark 3 Note that $E[(S_T - K)^+]$ is also equal to $e^{aT} C_0(a; K)$ where $C_0(a; K)$ is the Black-Scholes value of the call with strike $K$ where the riskless return $r$ is replaced by $a$.

Remark 4 For some parametrization of the financial markets, there exists at least one value for $m$ such that the OBPI strategy dominates, in a mean-variance sense, the CPPI one.

The following example give an illustration with the same values for the parameters as previously. We choose $m$ such that $E[R_{CPPI}^T] = E[R_{OBPI}^T]$, we find $m = 5.77647$.

<table>
<thead>
<tr>
<th>OBPI</th>
<th>CPPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>expectation</td>
<td>0.0861176</td>
</tr>
<tr>
<td>volatility</td>
<td>0.168625</td>
</tr>
<tr>
<td>skewness</td>
<td>1.49114</td>
</tr>
<tr>
<td>kurtosis</td>
<td>5.4576</td>
</tr>
</tbody>
</table>

We can notice that if we take into account a general utility function, we may find a reverse result due to the skewness and to the kurtosis.

3.4 Comparison of \(quantiles\)

First, we evaluate the probability that the CPPI portfolio value is greater than that of the OBPI. We need to compute:

$$
P \left( V_{OBPI}^T \geq V_{CPPI}^T \right) = \int_{S_T}^{\infty} f(S_T) \left( h_{m}(y) - h_{m}(y) \right) dy$$

where $f(S_T)$ is the density of the risky asset $S_T$ at $T$.

Denote the indicator function of a subset $A$ by $I_A$. Define the function $h_m(y)$ by:

$$h_m(y) = \int_{S_T}^{\infty} \exp\left[ m\left( a - \frac{1}{2} \sigma^2 T \right) + m \sigma T \sqrt{y} \right] I_{S_T}^{\infty} \left( h_{m}(y) - h_{m}(y) \right) dy$$

Finally, we get:

$$P \left( V_{OBPI}^T \geq V_{CPPI}^T \right) = R_{m+1} \left( \frac{h_m(y)}{2\pi} \right) dy$$

We rst compare the two strategies on the whole domain of $S_T$.

As the standard normal distribution lies between $-3$ and $+3$ with probability almost one, $S_T$ is between 59.4521 and 197.388 with probability (almost) one with the parameter values of our previous example.
The plot of the probability $P_\left(V^C_{CPPI}\right) > V^O_{OBPI}$ as a function of $m$ is:

As expected, for reasonable values of $m$ ($3 \leq m \leq 10$), this probability is near 0.5. That means that neither of the strategies dominates the other with respect to this criterion.

We re-engage our analysis and consider now the probability that the CPPI portfolio value is greater than that of the OBPI conditionally on the fact that the value of the risky asset belongs to some particular intervals. This probability has the form:

$$P_\left(V^C_{CPPI} > V^O_{OBPI} \mid aS_0 \leq S_T \leq bS_0\right)$$

We will examine four cases:

- **Case 1 (large drop):** $0 \leq \frac{S_T}{S_0} \leq 0.9$
- **Case 2 (fall):** $0 < \frac{S_T}{S_0} \leq 1$
- **Case 3 (rise):** $1 < \frac{S_T}{S_0} \leq 1.2$
- **Case 4 (large rise):** $1.2 < \frac{S_T}{S_0} \leq 1.4$

### 3.4.1 Case 1: $0.8 \leq \frac{S_T}{S_0} \leq 0.9$

For that range of values for the return of the risky asset, the conditional probability is close to one, meaning that the value of the CPPI portfolio is always above the OBPI portfolio value (as it can be seen on the previous payo® graphics). This result remains valid for the strike, $K$, in the interval $[0.9S_0; 1.1S_0]$.

But as soon as the strike is below $0.9S_0$, we get a different result. For example, with a strike of $0.89S_0$ and for $m$ greater or equal to 9, the conditional probability is no more equal to one and is decreasing in $m$.

### 3.4.2 Case 2: $0.9 < \frac{S_T}{S_0} < 1$

We get the same qualitative conclusions. For $K < S_0$, the conditional probability is always close to one, meaning that the value of the CPPI portfolio is always above the OBPI portfolio value.
For $K$ sufficiently smaller than $S_0$, the conclusion depends on the value of $m$, as shown in the graphic below ($K = 0.9S_0$).

![Graph](Image)

### 3.4.3 Case 3: $1 \cdot \frac{S_T}{S_0} \cdot 1:2$

Consider first the case $K = S_0$.

![Graph](Image)

Depending on the value of the multiple chosen, the conclusion may change. For small $m$, the probability is slightly in favour of the CPPI. The reverse is true as soon as $m$ is greater than 6.

For $K = 1.1S_0$, the conclusion is qualitatively the same but the value of the probabilities are greater (above 0.6).

Finally, for $K = 0.9S_0$, we have also the same conclusion, except the probabilities are now smaller (less than 0.3).

### 3.4.4 Case 4: $1:2 \cdot \frac{S_T}{S_0} \cdot 1:4$

For $K = S_0$, $P \left[ V_t^{CPPI} \leq V_t^{OBPI} \right] = 0$. That is, for large increase in the risky asset, the OBPI portfolio value is above the CPPI portfolio value. The same conclusion apply until $K = 1.13S_0$. The probability is increasing then decreasing, this comes from the fact that the payoff function of the CPPI intersect the
OBPI payoff function in one or two points (in the given range of $S_T$ values) according to the different values that $m$ can take.

Again, for $K = 0.8S_0$, we obtain the same kind of figure:

The explanation is the same.

4 The dynamic behavior of OBPI and CPPI

We now study the dynamic properties of the two strategies. Our purpose is first to examine the OBPI method as a generalized CPPI one. Then, we also analyze the hedging properties of both methods, in particular the behavior of the quantity to invest on the risky asset at any time in the period ("delta" for the option).

4.1 OBPI as a generalized CPPI

In this subsection, we study the dynamics of the two models for any date $t \in [0; T]$. More precisely, we propose the following proposition that establishes the link between the CPPI model and the OBPI model.

Proposition 3 The OBPI method is equivalent to the CPPI method in which the multiple is allowed to vary and is given by $m_{OBPI}^t (t; S_t) = rac{S_t N(d_1^t (t; S_t))}{C_t^K (S_t)}$.

Proof. Recall that:

$$V_{T}^{OBPI} = S_T + (K - S_T)^+ = K + (S_T - K)^+$$

So:

$$V_{t}^{OBPI} = Ke^{r(T-t)} + C_t^K (S_t)$$

where $Ke^{r(T-t)} = F_t$ is the time $t$ value of the floor and $C_t^K (S_t)$ is the cushion at time $t$. By definition, the cushion is defined as $\frac{S_t}{m_t}$. Here, the cushion is
simply the call and the exposure (total amount invested in the risky asset) is $S_t N (d_1 (t; S_t))$. Finally, we obtain the desired result for the multiple:

$$m^{OBPI}_t = \frac{S_t N (d_1 (t; S_t))}{C^*_t (S_t)}.$$ 

For $K = S_0$, with the same values for the parameters as previously, we can illustrate the evolution of $m^{OBPI}_t$ for the lower and the upper bound on $S_t$ and when the risky asset price remains constant.

In this case, we consider the minimum value for $S_t$ (more precisely, it is the value obtained for the standard normal variable equal to $\phi$). We plot the function $m^{OBPI}_t$ until $t = 0.9$.

Indeed, when we are near the maturity of the call, the multiple becomes indeterminate. To see this, consider what happens at maturity $T$:

$$V^{OBPI}_T = K + (S_T - K)^+$$

But as the call is out of the money ($S_T < K$), the cushion and the exposure are nil, which leads to indetermination of the multiple. This result remains true just before the maturity date.

The figure for the upper bound on $S_t$ is given below:
In particular, we verify that in a rising market, the OBPI method prevents the portfolio being over-invested in the risky asset, as the multiple is low and decreasing with time elapsed.

If the value of the risky asset remains stable, $S_t = S_0$, we obtain:

$$m_{OBPI}$$

It is interesting to notice that most of the time (at least for $t \cdot 0.8$), the values for $m_{OBPI}$ are comparable to the usual values used in the CPPI method.

For that purpose, we will analyse the "greeks".

### 4.2 The Delta

The delta of the OBPI is obviously the delta of the call. For the CPPI, we get:

$$\frac{\partial V_{CPPI}}{\partial S_t} = \frac{\partial m_{CPPI}}{\partial S_t} = \frac{\partial m_{OBPI}}{\partial S_t}$$

We plot on the following graphic the delta as a function of $S_t$.

We can observe on the graphic above that delta's behavior of the two strategies is quite different. For the CPPI, we find not surprisingly that the delta becomes more convex with $m$ and that the delta can be greater than one.
For a large range of the risky asset values, the sensitivity of the OBPI is greater than that of the CPPI. Moreover, it happens for the most likely values of the underlying asset (i.e. around the money). In order to be more precise, we calculate the probability that the delta of the OBPI is greater than that of the CPPI for various market parametrizations. We find evidence that, in probability, CPPI is significantly less sensitive to the risky asset than OBPI as shown in the following tables. Notice that this finding has practical implications.

The graphic below shows the evolution of the delta with time. Whatever the level of $S$ compared to the level of the insured level at maturity, $K$, the delta of the CPPI is decreasing with time.

Besides, all other things being equal, we know that delta's call is increasing with time. The opposite is true for the CPPI. This property is due to the fact that the floor of the CPPI is increasing with time.

More surprisingly, we are able to show that the delta of the CPPI is decreasing with volatility as soon as $m > 1$. For the OBPI, the result depends on the moneyness of the option.

### 4.3 The Gamma

The gamma of the CPPI is equal to:

$$\Gamma_{CPPI} = \frac{\partial^2 C_{CPPI}}{\partial t \partial S} = \frac{\partial}{\partial t} (m^{m-1} S^{-2})^2.$$

For the CPPI, it is always for high values of $S$ that the gamma is important. But, the gamma of the CPPI is monotonically decreasing with time, although it does not reach zero at maturity. We know that for a call, the gamma will go to zero as the expiration date approaches if the call is in-the-money or out-of-the-money, but will become very large if it is exactly at-the-money.
4.4 The Vega

The Vega of the CPPI is defined as:

\[
\rho_{CPPI} = \frac{\partial V_{CPPI}}{\partial \sigma} = C_0 K \exp \left[ -\frac{3}{2} \ln \left( \frac{S_t}{S_0} \right) \right] i \left( \frac{1}{2} - \frac{1}{2} i \right) t \left( 1 + \frac{1}{2} i \right)^i t \left( 1 + \frac{1}{2} i \right)^i t
\]

Here, the absence of options features in the CPPI makes possible negative values for vega.

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\(^4\)In the following calculus, we do not take into account the effect of the volatility on \(C_0^K\) because the call enters in the CPPI formula only to insure the compatibility, at time 0, with the OBPI.
5 Conclusion

We have examined the two main portfolio insurance methods: the OBPI and the CPPI. In a first part, we have proved that there is no "dominance" for the standard criteria of portfolio choices. Nevertheless, conditionally to some events on the dynamics of $S$, one method can be preferred to the other one. Then, we have analyzed the dynamic properties of these two methods, showing in particular how the OBPI method can be considered as a generalized CPPI method. Therefore, it suggests to compare OBPI with other CPPI methods where the multiple is some function of the path of the risky asset.

6 Appendix

Lemma 1 Consider two functions $f: A \rightarrow B$ and $g: A \rightarrow B$, where $f$ or $g$ is increasing and the reciprocal of $f$ and $g$ exists. Then:

$$f^{-1} \cdot g^{-1} \cdot g \cdot f$$

Proof. We get the following equivalent relations:

$$f^{-1} \cdot g^{-1} \cdot ( ) \quad 8x \in A; f^{-1}[f(x)] \cdot g^{-1}[f(x)]$$
$$() \quad 8x \in A; x \cdot g^{-1}[f(x)]$$
$$() \quad 8x \in A; g(x) \cdot f(x) \quad (g \text{ increasing})$$

Proof of Proposition 2. We proceed by contradiction. Consider first the following functions

$$f: [S^n; +1] \rightarrow [S^n; K; +1] \text{ such that } f(x) = @x^m;$$
$$g: [S^n; +1] \rightarrow [S^n; K; +1] \text{ such that } g(x) = x_i K;$$

where $f$ and $g$ are the two payo® functions restricted on a particular domain. $S^n$ is the smallest solution of the equation $x_i K = @x^m$ (the other solution will be noted as $S^m$, see the graphic of the payo® functions).

Suppose now that: $V_{T}^{CPPI} \leq V_{T}^{OBPI}$. Then:

$$8z \quad 0; P \left( S_T \cdot K \right)^{+} \cdot z \quad \Pi \quad P \left[ @S^m \cdot z \right];$$

so in particular, $8z \quad S^n \cdot K$.

But, for $z \quad S^n \cdot K$, we get:

$$\frac{1}{2} P \left( (S_T \cdot K)^{+} \cdot z \right) \leq P \left[ S_T \cdot S^n \right] + P \left[ S^n \cdot Z + K \right];$$
$$P \left[ @S^m \cdot z \right] = P \left[ S_T \cdot S^n \right] + P \left[ S_T \cdot S^n \text{ and } @S^m \cdot z \right];$$

Thus, for $z \quad S^n \cdot K$, the relation $V_{T}^{CPPI} \leq V_{T}^{OBPI}$ can be stated as:
\[
\begin{align*}
& \text{P} \{ S_T \geq z \} \leq \text{P} \{ S_T \geq g(S_T) \cdot z \} \leq \text{P} \{ S_T \geq f(S_T) \cdot z \}.
\end{align*}
\]

So, \( 8z, S_T \geq K, g^{-1}(z), f^{-1}(z) \).

By Lemma 1, we get: \( 8x, S_T, g(x) \cdot f(x) \).

This result leads to a contradiction with the definition of the two payoff functions: for any \( x \in ]S^n; S^\infty[ \), \( g(x) > f(x) \). So, the CPPI can never stochastically dominate at the first-order the OBPI strategy.

2 The converse can be proved in the same way. But, we can notice that:

\[
\begin{align*}
V_{OBPI}^T \leq V_{CPPI}^T & \quad (8z > 0; P, S_T \cdot z + K) \\
\quad \text{For any finite values of the parameters (} K; m \text{), the previous inequality is never satisfied at } z = 0. \text{ So, the OBPI can never stochastically dominate at the first-order the CPPI strategy.}
\end{align*}
\]

References


