INFLUENCE AND INTERACTION INDEXES FOR PSEUDO-BOOLEAN FUNCTIONS: A UNIFIED LEAST SQUARES APPROACH

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Abstract. The Banzhaf power and interaction indexes for a pseudo-Boolean function (or a cooperative game) appear naturally as leading coefficients in the standard least squares approximation of the function by a pseudo-Boolean function of a specified degree. We first observe that this property still holds if we consider approximations by pseudo-Boolean functions depending only on specified variables. We then show that the Banzhaf influence index can also be obtained from the latter approximation problem. Considering certain weighted versions of this approximation problem, we introduce a class of weighted Banzhaf influence indexes, analyze their most important properties, and point out similarities between the weighted Banzhaf influence index and the corresponding weighted Banzhaf interaction index.

1. Introduction

Let $f: \{0,1\}^n \to \mathbb{R}$ be an $n$-variable pseudo-Boolean function and let $S$ be a subset of its variables. Define the influence of $S$ over $f$ as the expected value, denoted $I_f(S)$, of the highest variation of $f$ when assigning values independently and uniformly at random to the variables not in $S$ (see [11] for a normalized version of this definition). That is,

$$I_f(S) = \frac{1}{2^{n-|S|}} \sum_{T \subseteq N \setminus S} \left( \max_{R \subseteq S} f(T \cup R) - \min_{R \subseteq S} f(T \cup R) \right),$$

where $N = \{1, \ldots, n\}$. This notion was first introduced for Boolean functions $f: \{0,1\}^n \to \{0,1\}$ by Ben-Or and Linial [1] (see also [9]). There the influence $I_f(S)$ was (equivalently) defined as the probability that, assigning values independently and uniformly at random to the variables not in $S$, the value of $f$ remains undetermined. Since its introduction, this concept has found many applications in discrete mathematics, cooperative game theory, theoretical computer science, and social choice theory (see, e.g., the survey article [10]).

When the function $f$ is nondecreasing in each variable, the formula above reduces to

$$I_f(S) = \frac{1}{2^{n-|S|}} \sum_{T \subseteq N \setminus S} (f(T \cup S) - f(T)).$$

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Throughout we identify Boolean vectors $\mathbf{x} \in \{0,1\}^n$ and subsets $T \subseteq N$ by setting $x_i = 1$ if and only if $i \in T$. We thus use the same symbol to denote both a pseudo-Boolean function $f: \{0,1\}^n \to \mathbb{R}$ and the corresponding set function $f: 2^N \to \mathbb{R}$ interchangeably.
The latter expression has an interesting interpretation even if \( f \) is not nondecreasing. In cooperative game theory for instance, where \( f(T) \) represents the worth of coalition \( T \) in the game \( f \), this expression is precisely the average value of the marginal contributions \( f(T \cup S) - f(T) \) of coalition \( S \) to outer coalitions \( T \subseteq N \setminus S \). Thus, it measures an overall influence (which can be positive or negative) of coalition \( S \) in the game \( f \). In particular, when \( S = \{i\} \) is a singleton it reduces to the Banzhaf power index

\[
I_f(\{i\}) = \frac{1}{2^{n-|i|}} \sum_{T \subseteq N \setminus \{i\}} (f(T \cup \{i\}) - f(T)).
\]

Thus, the expression in (1) can be seen as a variant of the original concept of influence that simply extends the Banzhaf power index to coalitions. We call it the Banzhaf influence index and denote it by \( \Phi_B(f, S) \). Actually, this index was introduced, axiomatized, and even generalized to weighted versions in [12].

The Banzhaf interaction index [16], another index which extends the Banzhaf power index to coalitions, is defined for a pseudo-Boolean function \( f: \{0, 1\}^n \to \mathbb{R} \) and a subset \( S \subseteq N \) by

\[
I_B(f, S) = \frac{1}{2^{n-|S|}} \sum_{T \subseteq N \setminus S} (\Delta_S f)(T),
\]

where \( \Delta_S f \) denotes the \( S \)-difference (or discrete \( S \)-derivative) of \( f \). When \( |S| \geq 2 \), this index measures an overall degree of interaction among the variables of \( f \) that are in \( S \). When \( f \) is a game, it measures an overall degree of interaction among the players of coalition \( S \) in the game \( f \) (see, e.g., [4, 5, 6]).

It is known that the Banzhaf power and interaction indexes can be obtained from the solution of a standard least squares approximation problem for pseudo-Boolean functions (see [5, 8]). Weighted versions of this approximation problem recently enabled us to define a class of weighted Banzhaf interaction indexes having several nice properties (see [13]). However, we observe that there is no such least squares construction for the Banzhaf influence index in the literature.

In this paper we fill this gap in the following way. In Section 2 we first show that the Banzhaf interaction index can be obtained from a different, more natural least squares approximation problem. Specifically, \( I_B(f, S) \) appears as the leading coefficient in the multilinear expansion of the best approximation \( f_S \) of \( f \) by a pseudo-Boolean function that depends only on the variables in \( S \). We then prove that the Banzhaf influence index \( \Phi_B(f, S) \) can be obtained from the same approximation problem simply by considering the difference \( f_S(S) - f_S(\emptyset) \). In Section 3 we introduce a class of weighted Banzhaf influence indexes from the solution of a weighted version of this approximation problem. We show that these indexes define a subclass of the family of generalized values, give their most important properties, and point out similarities between the weighted Banzhaf influence index and the corresponding weighted Banzhaf interaction index. Finally, in Section 4 we give a couple of concluding remarks.

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2The differences of \( f \) are defined as \( \Delta_S f = f \), \( \Delta_{\{i\}} f(x) = f(x \mid x_i = 1) - f(x \mid x_i = 0) \), and \( \Delta_S f = \Delta_{\{i\}} \Delta_{S \setminus \{i\}} f \) for \( i \in S \).
2. Interactions, influences, and least squares approximations

In this section we recall how the Banzhaf interaction index can be obtained from the solution of a standard least squares approximation problem and we show how a variant of this approximation problem can be used to define the Banzhaf influence index.

It is well known (see, e.g., [7]) that any pseudo-Boolean function \( f: \{0,1\}^n \to \mathbb{R} \) can be represented by a multilinear polynomial function
\[
f = \sum_{T \subseteq N} a(T) u_T,
\]
where \( u_T(x) = \prod_{i \in T} x_i \) is the unanimity function (or unanimity game) for \( T \subseteq N \) (with the convention \( u_{\emptyset} = 1 \)) and the set function \( a: 2^N \to \mathbb{R} \), called the Möbius transform of \( f \), is defined through the conversion formulas (Möbius inversion formulas)
\[
(3) \quad a(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f(T) \quad \text{and} \quad f(S) = \sum_{T \subseteq S} a(T).
\]

By extending formally any pseudo-Boolean function \( f: \{0,1\}^n \to \mathbb{R} \) to the unit hypercube \([0,1]^n\) by linear interpolation, Owen [14, 15] introduced the multilinear extension of \( f \), i.e., the multilinear polynomial \( \tilde{f}: [0,1]^n \to \mathbb{R} \) defined by
\[
\tilde{f}(x) = \sum_{S \subseteq N} a(S) \prod_{i \in S} x_i,
\]
where \( a \) is the Möbius transform of \( f \).

Denote by \( \mathcal{F}^N \) the set of pseudo-Boolean functions on \( N \) (i.e., with variables in \( N \)). Recall that the Banzhaf interaction index [6, 16] is the mapping \( I_B: \mathcal{F}^N \times 2^N \to \mathbb{R} \) defined in Eq. (2). Extending the \( S \)-difference operator \( \Delta_S \) to multilinear polynomials on \([0,1]^n\), we can show the following identities (see [6, 15])
\[
I_B(f, S) = (\Delta_S \tilde{f})(\frac{1}{2}) = \int_{[0,1]^n} \Delta_S \tilde{f}(x) \, dx,
\]
where \( \frac{1}{2} \) stands for \( \frac{1}{2}, \ldots, \frac{1}{2} \). Since the \( S \)-difference operator has the same effect as the \( S \)-derivative operator \( D_S \) (i.e., the partial derivative operator with respect to the variables in \( S \)) when applied to multilinear polynomials on \([0,1]^n\), we also have
\[
I_B(f, S) = (D_S \tilde{f})(\frac{1}{2}) = \int_{[0,1]^n} D_S \tilde{f}(x) \, dx.
\]

We now recall how the index \( I_B \) can be obtained from an approximation problem. For \( k \in \{0, \ldots, n\} \) define
\[
V_k = \text{span}\{u_T: T \subseteq N, |T| \leq k\},
\]
that is, \( V_k \) is the linear subspace of all multilinear polynomials \( g: \{0,1\}^n \to \mathbb{R} \) of degree at most \( k \), i.e., of the form
\[
g = \sum_{\substack{T \subseteq N \, \mid \, |T| \leq k}} c(T) u_T, \quad c(T) \in \mathbb{R}.
\]
The best \( k \)th approximation of a function \( f: \{0,1\}^n \to \mathbb{R} \) is the function \( f_k \in V_k \) that minimizes the squared distance
\[
(5) \quad \sum_{x \in \{0,1\}^n} (f(x) - g(x))^2 = \sum_{T \subseteq N} \left( f(T) - g(T) \right)^2
\]
among all functions \( g \in V_k \).

The following proposition, which was proved in [5] (see [3] for an earlier work), expresses the number \( I_B(f, S) \) in terms of the best \( |S| \)th approximation \( f_{|S|} \) of \( f \).

**Proposition 2.1** ([5]). For every \( f: \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), the number \( I_B(f, S) \) is the coefficient of \( u_S \) in the multilinear expansion of the best \( |S| \)th approximation \( f_{|S|} \) of \( f \).

An alternative (and perhaps more natural) approach to measure the influence on \( f \) of its \( i \)th variable consists in considering the coefficient of \( u_{\{i\}} \) in the best approximation of \( f \) by a function of the form

\[
g = c(\emptyset) u_S + c(\{i\}) u_{\{i\}}
\]

(instead of a function in \( V_1 \)), as classically done for linear models in statistics. More generally, for every \( S \subseteq N \) define \( V_S = \{ u_T : T \in S \} \), that is, \( V_S \) is the linear subspace of all multilinear polynomials \( g: \{0,1\}^n \to \mathbb{R} \) that depend only on the variables in \( S \), i.e., of the form

\[
g = \sum_{T \in S} c(T) u_T, \quad c(T) \in \mathbb{R}.
\]

The best \( S \)-approximation of a function \( f: \{0,1\}^n \to \mathbb{R} \) is then the function \( f_S \in V_S \) that minimizes the squared distance [5] among all functions \( g \in V_S \).

We now show that \( I_B(f, S) \) is also the coefficient of \( u_S \) in the multilinear expansion of \( f_S \). On the one hand, \( f_S \) is the orthogonal projection of \( f \) onto \( V_S \) with respect to the inner product

\[
\langle f, g \rangle = \frac{1}{2^n} \sum_{T \subseteq N} f(T) g(T). \tag{6}
\]

On the other hand, it is easy to prove that the \( 2^n \) functions \( v_T(\mathbf{x}) = \prod_{i \in T} (2x_i - 1) \) \((T \subseteq N)\) form an orthonormal set with respect to this inner product. Thus, the best \( k \)th- and \( S \)-approximations of \( f \) are respectively given by

\[
f_k = \sum_{T \subseteq N \mid |T| \leq k} \langle f, v_T \rangle v_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} \langle f, v_T \rangle v_T. \tag{7}
\]

These formulas enable us to prove the following simple but important result, which expresses the number \( I_B(f, S) \) in terms of the best \( S \)-approximation \( f_S \) of \( f \).

**Proposition 2.2.** For every \( f: \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), the number \( I_B(f, S) \) is the coefficient of \( u_S \) (i.e., the leading coefficient) in the multilinear expansion of the best \( S \)-approximation \( f_S \) of \( f \).

**Proof.** Since \( I_B(f, S) \) is the coefficient of \( u_S \) in the multilinear expansion of \( f_{|S|} \), from the first equality in (7) we obtain

\[
I_B(f, S) = 2^{|S|} \langle f, v_S \rangle. \tag{8}
\]

We then conclude by the second equality in (7). \( \square \)

Thus, combining Proposition 2.2 with Eq. (8), we immediately see that the number \( I_B(f, S) \) can be expressed in terms of the approximation \( f_S \) as

\[
I_B(f, S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} f_S(T). \tag{9}
\]

Note that the multiplicative normalization of the inner product does not change the projection problem.
Recall that the Banzhof influence index \( [12] \) is the mapping \( \Phi_B : \mathcal{F}^N \times 2^N \to \mathbb{R} \) defined by

\[
\Phi_B(f,S) = \frac{1}{2^{n-|S|}} \sum_{T \supseteq N \setminus S} (f(T \cup S) - f(T)).
\]

Since the map \( f \mapsto \Phi_B(f,S) \) is linear for every \( S \subseteq N \), it can be expressed by means of the inner product \( (6) \). To this aim, consider the function \( g_S : \{0,1\}^n \to \mathbb{R} \) defined by

\[
g_S(x) = 2^{|S|} \left( \prod_{i \in S} x_i - \prod_{i \in S} (1-x_i) \right).
\]

**Proposition 2.3.** For every \( f : \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), we have \( \Phi_B(f,S) = (f,g_S) \).

**Proof.** Using \( (6) \), we obtain

\[
(f,g_S) = \frac{1}{2^{n-|S|}} \left( \sum_{T \supseteq S} f(T) - \sum_{T \subseteq N \setminus S} f(T) \right),
\]

which is precisely the right-hand side of \( (9) \). \( \square \)

From Proposition 2.3 we can easily derive an explicit expression for \( \Phi_B(f,S) \) in terms of the Banzhof interaction index \( I_B \). This expression was already found in \( [11] \). We first consider a lemma.

**Lemma 2.4.** For every \( S \subseteq N \), we have \( g_S = 2 \sum_{T \in S, |T| \text{ odd}} v_T \).

**Proof.** Since the functions \( v_T \) \( (T \subseteq N) \) form an orthonormal basis for \( \mathcal{F}^N \), we have \( g_S = \sum_{T \in N} (g_S,v_T) v_T \). Using \( (8), (10) \), and then \( (11) \), we obtain

\[
(g_S,v_T) = 2^{|--T|} I_B(g_S,T) = 2^{|--T|} (D_T \tilde{g}_S) \left( \frac{1}{2} \right).
\]

The result then follows directly from the computation of the derivative \( D_T \tilde{g}_S \). \( \square \)

**Proposition 2.5 (11, Proposition 4.1).** For every \( f : \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), we have

\[
\Phi_B(f,S) = \sum_{|T| \text{ odd}} \frac{1}{2^{|--T|}} I_B(f,T).
\]

**Proof.** By Proposition 2.3 and Lemma 2.4 we obtain

\[
\Phi_B(f,S) = \langle f,g_S \rangle = 2 \sum_{T \supseteq S \; |T| \text{ odd}} \langle f,v_T \rangle.
\]

We then conclude by \( (8) \). \( \square \)

The following proposition gives an expression for \( \Phi_B(f,S) \) in terms of the best \( S \)-approximation \( f_S \) of \( f \). This proposition together with Proposition 2.2 show that the indexes \( I_B(f,S) \) and \( \Phi_B(f,S) \) are actually two facets of the same construction, namely the best \( S \)-approximation of \( f \).

**Proposition 2.6.** For every \( f : \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), we have

\[
\Phi_B(f,S) = f_S(S) - f_S(\emptyset).
\]
Proof. By (7), we have
\[ f_S(S) - f_S(\emptyset) = \sum_{T \subseteq S} (f, v_T)(v_T(S) - v_T(\emptyset)) = \sum_{T \subseteq S} (f, v_T)(1 - (-1)^{|T|}). \]
Using (11), we see that the latter expression is precisely \( \Phi_B(f, S) \).

Proposition 2.6 is actually one of the key results of this paper. Indeed, as we will now see, it will enable us to define weighted Banzhaf influence indexes from a weighted version of the approximation problem in complete analogy with the way the weighted Banzhaf interaction index was defined in [13].

3. Weighted influences defined by least squares

In [13] we investigated weighted versions of the best kth approximation problem for pseudo-Boolean functions (e.g., to allow nonuniform assignments of the variables). This study enabled us to define a class of weighted Banzhaf interaction indexes. In the present section we show that the corresponding weighted version of the best S-approximation problem described in Section 2 not only yields the same weighted Banzhaf interaction index but also provides a natural definition of a weighted Banzhaf influence index.

Given a weight function \( w : \{0, 1\}^n \to [0, \infty[ \) and a pseudo-Boolean function \( f : \{0, 1\}^n \to \mathbb{R} \), we define the best S-approximation of \( f \) as the unique multilinear polynomial in \( V_S \) that minimizes the squared distance
\[
\sum_{x \in \{0, 1\}^n} w(x)(f(x) - g(x))^2 = \sum_{T \subseteq N} w(T)(f(T) - g(T))^2
\]
among all functions \( g \in V_S \).

Assuming without loss of generality that \( \sum_{T \subseteq N} w(T) = 1 \), we see that \( w \) defines a probability distribution over \( 2^N \). Considering the cooperative game theory context, we can interpret \( w(T) \) as the probability that coalition \( T \) forms, that is, \( w(T) = \Pr(C = T) \), where \( C \) represents a random coalition.

We also assume that the players behave independently of each other to form coalitions, which means that the events \( (C \ni i) \ (i \in N) \) are independent. Setting \( p_i = \Pr(C \ni i) = \sum_{S \ni i} w(S) \), we then have
\[
w(S) = \prod_{i \in S} p_i \prod_{i \in N \setminus S} (1 - p_i),
\]
which implies \( 0 < p_i < 1 \). Thus, the probability distribution \( w \) is completely determined by the n-tuple \( \mathbf{p} = (p_1, \ldots, p_n) \in ]0, 1[^n \).

We now provide an explicit expression for the best S-approximation of a pseudo-Boolean function. On the one hand, the squared distance (12) is induced by the weighted Euclidean inner product
\[
(f, g) = \sum_{x \in \{0, 1\}^n} w(x)f(x)g(x).
\]
On the other hand, as observed in [2] the functions \( v_{T, \mathbf{p}} : \{0, 1\}^n \to \mathbb{R} \ (T \subseteq N) \) defined by
\[
v_{T, \mathbf{p}}(x) = \prod_{i \in T} \frac{x_i - p_i}{\sqrt{p_i(1 - p_i)}}
\]

\[ 4 \text{In Section 2 we give a justification for this independence assumption.} \]
are pairwise orthogonal and normed. This provides the following immediate solution to the weighted approximation problem.

**Proposition 3.1.** The best $S$-approximation of $f: \{0, 1\}^n \to \mathbb{R}$ is given by

\[
 f_{S,p} = \sum_{T \subseteq S} (f, v_{T,p}) v_{T,p}.
\]

From Proposition 3.1 we immediately deduce that the coefficient of $u_S$ (i.e., the leading coefficient) in the multilinear expansion of $f_{S,p}$ is given by

\[
 I_{B,p}(f, S) = \frac{(f, u_{S,p})}{\prod_{i \in S} \sqrt{p_i(1 - p_i)}},
\]

which is precisely the weighted Banzhaf interaction index introduced in [13] by means of the corresponding $k$th approximation problem. In the non-weighted case (i.e., when $p = \frac{1}{2}$), Eq. (16) reduces to (8).

By analogy with Proposition 2.6 we now propose the following definition of weighted Banzhaf influence index.

**Definition 3.2.** Let $\Phi_{B,p}: \mathcal{F}^N \times 2^N \to \mathbb{R}$ be defined as

\[
 \Phi_{B,p}(f, S) = f_{S,p}(S) - f_{S,p}(\emptyset).
\]

We now provide various explicit expressions for $\Phi_{B,p}(f, S)$ in terms of the weighted Banzhaf interaction index, the Möbius transform of $f$, and the $f$ values.

We start with the following result, which is the weighted counterpart of Proposition 2.5.

**Proposition 3.3.** For every $f: \{0, 1\}^n \to \mathbb{R}$ and every $S \subseteq N$, we have

\[
 \Phi_{B,p}(f, S) = \sum_{T \subseteq S} I_{B,p}(f, T) \left( \prod_{i \in T} (1 - p_i) - (-1)^{|T|} \prod_{i \in T} p_i \right).
\]

**Proof.** Using Definition 3.2 and Eqs. (15) and (14), we obtain

\[
 \Phi_{B,p}(f, S) = \sum_{T \subseteq S} (f, v_{T,p}) \left( \prod_{i \in T} \frac{1 - p_i}{\sqrt{p_i(1 - p_i)}} - (-1)^{|T|} \prod_{i \in T} \frac{p_i}{\sqrt{p_i(1 - p_i)}} \right).
\]

We then conclude by (16). \square

Using the expression of the weighted Banzhaf interaction index in terms of the Möbius transform of $f$, that is,

\[
 I_{B,p}(f, S) = \sum_{T \subseteq S} a(T) \prod_{i \in T \setminus S} p_i
\]

(see [13]), we can obtain the corresponding expression for the weighted Banzhaf influence index. To this extent, recall the binomial product formula

\[
 \sum_{T \subseteq N} \prod_{i \in T} a_i \prod_{i \in N \setminus T} b_i = \prod_{i \in N} (a_i + b_i).
\]

**Proposition 3.4.** For every $f: \{0, 1\}^n \to \mathbb{R}$ and every $S \subseteq N$, we have

\[
 \Phi_{B,p}(f, S) = \sum_{T \cap S \neq \emptyset} a(T) \prod_{i \in T \setminus S} p_i.
\]
Proof. Combining (17) with (19), we obtain

\[ \Phi_{B,p}(f, S) = \sum_{R \subseteq S} \sum_{T \supseteq R} a(T) \prod_{i \in T \setminus R} p_i \left( \prod_{i \in R} (1 - p_i) - \prod_{i \in R} (-p_i) \right) \]

(22)

Using the binomial product formula (20), we see that the inner sum in (22) becomes

\[ 1 - \prod_{i \in T \cap S} (p_i - p_i) = 1 \]

This completes the proof of the proposition.

Interestingly, Eqs. (19) and (21) show that both \( I_{B,p}(f, S) \) and \( \Phi_{B,p}(f, S) \) are independent of those \( p_i \) such that \( i \in S \).

A generalized value [12] is a mapping \( G: \mathcal{F}^N \times 2^N \rightarrow \mathbb{R} \) defined by

\[ G(f, S) = \sum_{T \subseteq N \setminus S} p_T^S (f(T \cup S) - f(T)) \]

(23)

where the coefficients \( p_T^S \) are real numbers for every \( S \subseteq N \) and every \( T \subseteq N \setminus S \).

The following lemma gives an expression for \( G(f, S) \) in terms of the Möbius transform of \( f \). The proof is given in Appendix A.

Lemma 3.5. A mapping \( G: \mathcal{F}^N \times 2^N \rightarrow \mathbb{R} \) of the form

\[ G(f, S) = \sum_{R \subseteq N} q_R^S a(R) \]

(24)

where \( a \) is the Möbius transform of \( f \), defines a generalized value if and only if the coefficients \( q_R^S \) depend only on \( R \) and \( R \setminus S \). In this case, the conversion between (23) and (24) is given by

\[ q_R^S = \sum_{T \supseteq R \subseteq N \setminus S} p_T^S \quad \text{and} \quad p_T^S = \sum_{R \subseteq T \subseteq N \setminus S} (-1)^{|T| - |R|} q_R^S \]

The following proposition shows that the weighted Banzhaf influence index \( \Phi_{B,p} \) is a particular generalized value.

Proposition 3.6. For every \( f: \{0, 1\}^n \rightarrow \mathbb{R} \) and every \( S \subseteq N \), we have

\[ \Phi_{B,p}(f, S) = \sum_{T \subseteq N \setminus S} p_T^S (f(T \cup S) - f(T)) \]

(25)

where the coefficients

\[ p_T^S = \prod_{i \in T} p_i \prod_{i \in N \setminus (S \cup T)} (1 - p_i) \]

satisfy the conditions \( p_T^S \geq 0 \) and \( \sum_{T \subseteq N \setminus S} p_T^S = 1 \).

Proof. Proposition 3.4 and Lemma 3.5 show that \( \Phi_{B,p} \) is a generalized value with \( q_R^S = \prod_{i \in R \setminus S} p_i \). By Lemma 3.5 we then have

\[ p_T^S = \sum_{R \subseteq T \subseteq N \setminus S} (-1)^{|R| - |T|} \prod_{i \in R} p_i \prod_{i \in T \setminus R \subseteq N \setminus S} \prod_{i \in R \setminus T} (-p_i) \]

The result then follows by the binomial product formula (20).
The coefficients $p^n_i$ given in (25) coincide with those of the corresponding expression for the weighted Banzhaf interaction index (see [13, Theorem 10]). Therefore, we immediately derive the following interpretations of these coefficients (see [13, Proposition 11]). For every $S \subseteq N$ and every $T \subseteq N \setminus S$, we have

$$p^n_i = \Pr(T \subseteq C \subseteq S \cup T) = \Pr(C = S \cup T \mid C \supseteq S) = \Pr(C = T \mid C \subseteq N \setminus S),$$

where $C$ denotes a random coalition.

For every $S \subseteq N$, define the linear operator $\sigma_S$ for functions on $\{0, 1\}^n$ or $[0, 1]^n$ by

$$\sigma_S f(x) = f(x \mid x_i = 1 \forall i \in S) - f(x \mid x_i = 0 \forall i \in S).$$

For instance, when applied to the unanimity game $u_T$ ($T \subseteq N$), we obtain

$$\sigma_{S \setminus T} u_T = \begin{cases} u_T \setminus S, & \text{if } S \cap T \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}$$

The next result gives various expressions for $\Phi_B(p, f, S)$ in terms of the function $\sigma_S f$. Recall first that, for every function $f: \{0, 1\}^n \to \mathbb{R}$, we have

$$(27) \quad \hat{f}(p) = \sum_{x \in \{0, 1\}^n} w(x) f(x) = E[f(C)],$$

where $C$ denotes a random coalition (see [15] or [13, Proposition 4]).

**Proposition 3.7.** For every $f: \{0, 1\}^n \to \mathbb{R}$ and every $S \subseteq N$, we have

$$\Phi_B(p, f, S) = (\sigma_S \hat{f})(p) = \sum_{x \in \{0, 1\}^n} w(x) \sigma_S f(x) = E[(\sigma_S f)(C)],$$

where $C$ denotes a random coalition.

**Proof.** The first equality immediately follows from Eqs. (21) and (26). The other equalities immediately follow from (27). \hfill \Box

Interestingly, (28) shows a strong analogy with the identities (see [13, Propositions 4 and 9])

$$(29) \quad I_B(p, f, S) = (D_S \hat{f})(p) = \sum_{x \in \{0, 1\}^n} w(x) \Delta_S f(x) = E[(\Delta_S f)(C)].$$

We also have the following expression for $\Phi_B(p, f, S)$ as an integral. We omit the proof since it follows exactly the same steps as in the proof of the corresponding expression for $I_B(p, f, S)$ (see [13, Proposition 12]).

**Proposition 3.8.** Let $F_1, \ldots, F_n$ be cumulative distribution functions on $[0, 1]$. Then

$$\Phi_B(p, f, S) = \int_{[0, 1]^n} (\sigma_S \hat{f})(x) dF_1(x_1) \cdots dF_n(x_n)$$

for every $f: \{0, 1\}^n \to \mathbb{R}$ and every $S \subseteq N$ if and only if $p_i = \int_0^1 x dF_i(x)$ for every $i \in N$.

We now generalize Proposition 2.3 to the weighted case. To this aim, consider the function $g_{S, p}: \{0, 1\}^n \to \mathbb{R}$ defined by

$$g_{S, p}(x) = \prod_{i \in S} \frac{1}{p_i} - \prod_{i \in S} \frac{1 - x_i}{1 - p_i}.$$
Proposition 3.9. For every \( f: \{0,1\}^n \to \mathbb{R} \) and every \( S \subseteq N \), we have

\[
\Phi_{B,p}(f,S) = (f,g_{S,p}) = \sum_{x \in \{0,1\}^n} w(x) f(x) \left( \prod_{i \in S} x_i p_i - \prod_{i \in S} (1 - p_i) \right)
\]

and

\[
\Phi_{B}(f,S) = \sum_{x \in \{0,1\}^n} f(x) \frac{g_{S}(x)}{2^{|S|}} \prod_{i \in N \setminus S} p_i^{x_i} (1 - p_i)^{1-x_i}.
\]

Proof. On the one hand, by substituting (14) into (18), we obtain \( \Phi_{B,p}(f,S) = (f,g_{S,p}) \), where

\[
g'_{S,p}(x) = \sum_{T \subseteq S} \left( \prod_{i \in T} x_i p_i - (-1)^{|T|} \prod_{i \in T} x_i (1 - p_i) \right).
\]

Using the binomial product formula (20), we immediately see that \( g'_{S,p} = g_{S,p} \), which proves (30).

On the other hand, we have

\[
g_{S,p}(x) w(x) = g_{S,p}(x) \prod_{i \in N} p_i^{x_i} (1 - p_i)^{1-x_i} = \frac{g_{S}(x)}{2^{|S|}} \prod_{i \in N \setminus S} p_i^{x_i} (1 - p_i)^{1-x_i},
\]

which, when combined with (30), immediately leads to (31). \(\Box\)

We end this section by giving an interpretation of the Banzhaf influence index \( \Phi_B \) as a center of mass of weighted Banzhaf influence indexes \( \Phi_{B,p} \).

As already mentioned, the index \( \Phi_B \) can be expressed in terms of \( \Phi_{B,p} \) simply by setting \( p = \frac{1}{2} \). However, by Proposition 3.6 we also have the following expression

\[
\Phi_B(f,S) = \int_{[0,1]^n} \Phi_{B,p}(f,S) dp.
\]

This formula can be interpreted in the game theory context in the same way as the corresponding formula for the interaction index (see [13 §5.1]). We have assumed that the players behave independently of each other to form coalitions, each player \( i \) with probability \( p_i \in ]0,1[ \). Assuming further that this probability is not known a priori, to define an influence index it is then natural to consider the average (center of mass) of the weighted indexes over all possible choices of the probabilities \( p_i \). Eq. (32) then shows that we obtain the non-weighted influence index \( \Phi_B \).

The Shapley generalized value [11,12] for a function \( f: \{0,1\}^n \to \mathbb{R} \) and a coalition \( S \subseteq N \) is defined by

\[
\Phi_{Sh}(f,S) = \sum_{T \subseteq N \setminus S} \frac{a(T)}{|T \setminus S| + 1},
\]

where \( a \) is the Möbius transform of \( f \). Using (21) we obtain the following expression for \( \Phi_{Sh} \) in terms of \( \Phi_{B,p} \), namely

\[
\Phi_{Sh}(f,S) = \int_0^1 \Phi_{B,(p,...,p)}(f,S) dp.
\]

Here the players still behave independently of each other to form coalitions but with the same probability \( p \). The integral in (33) simply represents the average of the weighted indexes over all the possible probabilities.
We now end our investigation with three important observations. We first give a justification for our independence assumption. Then we introduce a normalized influence index and derive tight upper bounds on influences. Finally, we discuss the issue of representing pseudo-Boolean functions in terms of Banzhaf influence indexes.

4.1. Independence assumption. We have made the important assumption that the players behave independently of each other to form coalitions. From this assumption we derived condition (13). Let us now show that this assumption is rather natural.

For every probability distribution \( w \) such that \( p_i = \sum_S w(S) \in ]0,1[ \), the best \( \{i\} \)-approximation of \( f: \{0,1\}^n \rightarrow \mathbb{R} \) with respect to the squared distance (12) associated with \( w \) is given by

\[
I_{(i)} = \langle f, v_{(i)}, p \rangle v_{(i)} + \langle f, 1 \rangle ,
\]

where \( v_{(i)}, p = (x_i - p_i)/\sqrt{p_i(1-p_i)} \). Therefore, we can define the power/influence index associated with \( w \) by

\[
I_w(f, \{i\}) = \frac{\langle f, v_{(i)}, p \rangle}{\sqrt{p_i(1-p_i)}} = \sum_{T \subseteq N \setminus \{i\}} \left( \frac{w(T \cup \{i\})}{p_i} f(T \cup \{i\}) - \frac{w(T)}{1-p_i} f(T) \right).
\]

However, we know from the literature on cooperative game theory (see, e.g., [3, 17]) that “good” power indexes should be of the form

\[
I(f, \{i\}) = \sum_{T \subseteq N \setminus \{i\}} c_T^i \Delta_{(i)} f(T), \quad c_T^i \in \mathbb{R}.
\]

It follows that the index \( I_w(\cdot, \{i\}) \) is of the form (34) if and only if \( \frac{w(T \cup \{i\})}{p_i} = \frac{w(T)}{1-p_i} \) for every \( T \subseteq N \setminus \{i\} \). Thus, we have proved the following result.

**Proposition 4.1.** The index \( I_w(\cdot, \{i\}) \) is of the form (12) for every \( i \in N \) if and only if (13) holds.

4.2. Normalized index and upper bounds on influences. Since the index \( \Phi_{B_p} \) is a linear map, it cannot be considered as an absolute influence index but rather as a relative index constructed to assess and compare influences for a given function.

If we want to compare influences for different functions, we need to consider an absolute, normalized influence index. Such an index can be defined as follows. Considering again \( 2^N \) as a probability space with respect to the measure \( w \), we see that, for every \( S \subseteq N \) the number \( \Phi_{B_p}(f, S) \) is the covariance \( \text{cov}(f, g_{S,p}) \) of the random variables \( f \) and \( g_{S,p} \). In fact, denoting the expectation of \( f \) by \( E[f] = f(p) \) (see (27)), we have

\[
\Phi_{B_p}(f, S) = \langle f, g_{S,p} \rangle = \langle f - E[f], g_{S,p} - E[g_{S,p}] \rangle = \text{cov}(f, g_{S,p})
\]

since \( E[g_{S,p}] = g_{S,p}(p) = 0 \) and \( \langle E[f], g_{S,p} \rangle = \Phi_{B_p}(E[f], S) = 0 \).

To define a normalized influence index, we naturally consider the Pearson correlation coefficient instead of the covariance.\(^5\) First observe that, for every nonempty

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\(^5\)Indeed, the functions 1 and \( v_{(i)} \) form an orthonormal basis for \( V_{(i)} \).

\(^6\)This approach was also considered for the interaction index (see [19] §5).
subset $S \subseteq N$, the standard deviation of $g_{S,p}$ is given by

\begin{equation}
\sigma(g_{S,p}) = \sqrt{\prod_{i \in S} \frac{1}{p_i} + \prod_{i \notin S} \frac{1}{1-p_i}}.
\end{equation}

In fact, since $g_{S,p} \in V_S$, we have

$$
\sigma^2(g_{S,p}) = \text{cov}(g_{S,p}, g_{S,p}) = \Phi_{B,p}(g_{S,p}, S) = g_{S,p}(S) - g_{S,p}(\emptyset),
$$

which immediately leads to (35).

**Definition 4.2.** The *normalized influence index* is the mapping

$$
r: \{f: \{0,1\}^n \to \mathbb{R} : \sigma(f) \neq 0\} \times (2^N \setminus \{\emptyset\}) \to \mathbb{R}
$$

defined by

$$
r(f, S) = \frac{\text{cov}(f, g_{S,p})}{\sigma(f) \sigma(g_{S,p})} = \frac{\Phi_{B,p}(f, S)}{\sigma(f) \sigma(g_{S,p})}.
$$

By definition the normalized influence index remains unchanged under interval scale transformations, that is, $r(af + b, S) = r(f, S)$ for all $a > 0$ and $b \in \mathbb{R}$. Thus, it does not depend on the “size” of $f$ and therefore can be used to compare different functions in terms of influence.

Moreover, as a correlation coefficient, the normalized influence index satisfies the inequality $|r(f, S)| \leq 1$, that is,

$$
\frac{|\Phi_{B,p}(f, S)|}{\sigma(f)} \leq \sigma(g_{S,p}).
$$

The equality holds if and only if there exist $a, b \in \mathbb{R}$ such that $f = ag_{S,p} + b$.

Interestingly, this property shows that (35) is a tight upper bound on the influence of a normed function $f/\sigma(f)$. Thus, for every nonempty subset $S \subseteq N$, those normed functions for which $S$ has the greatest influence are of the form $f = (\pm g_{S,p} + c)/\sigma(g_{S,p})$, where $c \in \mathbb{R}$.

**4.3. Representations of pseudo-Boolean functions.** It is well known that the values $I_B(f, S)$ ($S \subseteq N$) of the non-weighted Banzhaf interaction index for a function $f: \{0,1\}^n \to \mathbb{R}$ provide an alternative representation of $f$ (see [5]). This observation still holds in the weighted case. Indeed, combining the Taylor expansion formula with (29) yields (see [13, Eq. (16)])

$$
f(x) = \sum_{S \subseteq N} I_{B,p}(f, S) \prod_{i \in S} (x_i - p_i).
$$

Thus, the map $f \mapsto \{I_{B,p}(f, S) : S \subseteq N\}$ is a linear bijection.

We observe that the Banzhaf influence index does not share this nice property because there exist linear relations among the values $\Phi_B(f, S)$ ($S \subseteq N$). For instance, for every $i, j \in N$ we have $g_{\{i,j\}} = g_{\{i\}} + g_{\{j\}}$, which, by Proposition 2.3 translates into

$$
\Phi_B(f, \{i,j\}) = \Phi_B(f, \{i\}) + \Phi_B(f, \{j\}), \quad i, j \in N.
$$

Interestingly, in the weighted case we can show that this degeneracy disappears in general. However, we always have $\Phi_{B,p}(f, \emptyset) = 0$, which shows that the weighted Banzhaf influence index provides at most $2^n - 1$ functionals $\Phi_{B,p}(\cdot, S)$ ($\emptyset \neq S \subseteq N$) on $\mathcal{F}^N$ and it is therefore not possible to determine a unique function $f$ with prescribed influences.
Determining the class of pseudo-Boolean functions \( g: \{0,1\}^n \to \mathbb{R} \) corresponding to a given set of influences \( \Phi_{R,p}(f,S) \) \((\varnothing \neq S \subseteq N)\) remains an interesting open problem.

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**Appendix A. Proof of Lemma 3.5**

*Proof of Lemma 3.5.* Using the definition of the Möbius transform in (23), we obtain

\[
G(f,S) = \sum_{T \subseteq N \setminus S} p_T^S \left( \sum_{R \subseteq T \setminus S} a(R) - \sum_{R \subseteq T} a(R) \right) = \sum_{R \subseteq N} a(R) \left( \sum_{T : R \setminus S \subseteq T \subseteq N \setminus S} p_T^S - \sum_{T : R \setminus S \subseteq T \subseteq N \setminus S} p_T^S \right),
\]

which shows that \( G \) has the form (24) with the prescribed \( q_R \).

Conversely, substituting (3) into (24) and assuming \( S \neq \varnothing \), we obtain

\[
\sum_{R \subseteq N \setminus S \neq \varnothing} q_R^S a(R) = \sum_{R \subseteq N \setminus S \neq \varnothing} q_R^S \sum_{T \subseteq R} (-1)^{|R|-|T|} f(T) = \sum_{T \subseteq N} f(T) \sum_{R \subseteq T} \sum_{R' \subseteq R \cap S} (-1)^{|R'|-|R'\cap S|} q_{R'\cup R'}.
\]

Partitioning every \( R \) into \( R' = R \setminus S \) and \( R'' = R \cap S \), the latter expression becomes

\[
\sum_{T \subseteq N} f(T) \sum_{T' \subseteq R \setminus S \neq \varnothing} \sum_{R' \subseteq R \cap S} (-1)^{|R'|-[R'\cap S]} q_{R'\cup R''}.
\]

Since our assumption on the coefficients \( q_R^S \) implies \( q_{R'\cup R''} = q_{R'\cup S}^S \), the latter expression becomes

\[
\sum_{T \subseteq N} f(T) \sum_{T' \subseteq R \setminus S \neq \varnothing} \sum_{R' \subseteq R \cap S} (-1)^{|R'|-[R'\cap S]} q_{R'\cup S} \sum_{R'' \subseteq R \cap S} (-1)^{|R''|-|R'\cap S|},
\]

where the inner sum equals \((1-1)^{|S\setminus T|}\), if \( T \cap S \neq \varnothing \), and \(-1\), otherwise. Setting \( T' = T \setminus S \) for every \( T \) containing \( S \), the latter expression finally becomes

\[
\sum_{T \subseteq N \setminus S} \left( \sum_{T' \subseteq R \setminus S \neq \varnothing} \sum_{R' \subseteq R \cap S} (-1)^{|R'|-|T'|} q_{R'\cup S}^S \right) \left( f(T' \cup S) - f(T') \right),
\]

which completes the proof of the lemma. \( \square \)

**References**


[10] G. Kalai and S. Safra. Threshold phenomena and influence: Perspectives from mathematics, 
computer science, and economics. In: A.G. Percus, G. Istrate, C. Moore (Eds.), Computational 
Complexity and Statistical Physics. Santa Fe Institute Studies on the Sciences of Complexity, 


[16] M. Roubens. Interaction between criteria and definition of weights in MCDA problems. In 
Proc. 44th Meeting of the Eur. Working Group “Multiple Criteria Decision Aiding”, pp. 693– 


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