On the generation of bicliques of a graph

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1 Introduction

Generating all configurations that satisfy a given specification is a well-studied problem in Combinatorics and in Graph theory suggesting many interesting problems. Among them, generating all maximal independent sets of a given graph is one that has attracted considerable attention [3,4,7]. A maximal independent set of a graph $G = (V,E)$ is a subset $V' \subseteq V$ such that no two vertices in $V'$ are adjacent by an edge in $E$, and such that each vertex in $V - V'$ is adjacent to some vertex in $V'$.

Let $s_1, \ldots, s_{|S|}$ and $t_1, \ldots, t_{|T|}$ be two distinct sequences $S$ and $T$, respectively and with $|S| \leq |T|$, of elements of an ordered set. Say that $S$ is lexicographically smaller than $T$, if the first position $i$ with respect to which $S$ and $T$ disagree satisfies: $s_i$ is smaller than $t_i$ or $i > |S|$. Johnson, Yannakakis and

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Papadimitriou [3] showed that there is no polynomial-delay algorithm for generating all maximal independent sets of a given graph in reverse lexicographic order, unless $P = NP$.

We examine the generation of bicliques in a graph. A biclique of a graph $G = (V, E)$ is an inclusion maximal induced complete bipartite subgraph of a graph, i.e., a pair $(U, W)$ of subsets of $V$, such that both $U$ and $W$ are independent sets of $G$, every vertex in $U$ is adjacent to every vertex in $W$, and such that for each vertex $v$ in $V - (U \cup W)$, if $\{v\} \cup U$ is an independent set, then $v$ is nonadjacent to some vertex in $W$, and if $\{v\} \cup W$ is an independent set, then $v$ is nonadjacent to some vertex in $U$. When the requirement that $U$ and $W$ are independent sets of $G$ is dropped, we have a non-induced biclique.

Prisner [6] gives upper bounds on the number of bicliques in bipartite graphs and general graphs, exhibits examples of classes of graphs where the number of bicliques is exponential, and characterizes classes of graphs where the number of bicliques is polynomial in the number of vertices of the graph. The NP-completeness of the weighted maximum edge biclique problem for bipartite graphs is established by Dawanke et al. [2], and more recently for the non-weighted version by Peeters [5]. Alexe et al. [1] describe an algorithm for generating all non-induced bicliques with polynomial delay. The generation of all bicliques of a general graph with polynomial delay is an open problem.

We show it is NP-complete to test whether a subset of the vertices of a graph is part of a biclique. In light of this result, it might not be obvious how to obtain a polynomial-delay algorithm for generating all the (induced) bicliques. We show there is no polynomial-delay algorithm for generating all bicliques in reverse lexicographic order, unless $P = NP$. Nevertheless, we describe a greedy algorithm that generates the lexicographically first biclique of a graph $G = (V, E)$ in time $O(|V|^2)$. In a more complete version of this paper, we describe an algorithm for generating both all the (induced) bicliques and maximal independent sets with polynomial delay. In addition, we also propose specialized efficient algorithms for generating the bicliques of special classes of graphs.

## 2 NP-completeness results

Given a graph $G = (V, E)$, we say that a subset $S \subseteq V$ is part of a biclique $(U, W)$ of $G$, if $S = U$ or $S = W$. In this section we establish the $NP$-completeness of two biclique decision problems defined next, by reducing the $NP$-complete problem satisfiability to each of them.

**satisfiability**

Instance: Set $X = \{x_1, \ldots, x_k\}$ of $k$ Boolean variables, collection $C =$
\{c_1, \ldots, c_n\} \text{ of } n > 1 \text{ clauses over literals of } X.

Question: Is there a truth assignment satisfying \(C\)?

**PART OF BICLIQUE**

Instance: Graph \(G = (V,E)\), subset \(S \subseteq V\).
Question: Is \(S\) part of a biclique?

**LEXICOGRAPHICALLY LAST BICLIQUE**

Instance: Graph \(G = (V,E)\), biclique \(B\), order on \(V\).
Question: Is there a biclique \(B'\) of \(G\) such that \(B'\) is lexicographically larger than \(B\)?

**Theorem 2.1** Given a graph \(G = (V,E)\) and a subset \(S \subseteq V\), it is NP-complete to decide whether \(S\) is part of a biclique.

**Proof.** Problem **PART OF BICLIQUE** is in NP since a short certificate is a biclique \((U, S)\) having subset \(S\) as a part. We can verify in polynomial time that \((U, S)\) induces a complete bipartite subgraph, and in order to verify its maximality, we can verify in polynomial time for each vertex \(v\) in \(V - (U \cup S)\) such that \(\{v\} \cup U\) is an independent set that \(v\) is nonadjacent to some vertex in \(S\), and for each vertex \(v\) in \(V - (U \cup S)\) such that \(\{v\} \cup S\) is an independent set that \(v\) is nonadjacent to some vertex in \(U\).

To show completeness, we sketch a polynomial reduction from satisfiability. Given an instance \((X, C)\) of satisfiability we construct in polynomial time a graph \(G = (V,E)\) and a subset \(S \subseteq V\), such that there is a truth assignment satisfying \(C\) if and only if \(S\) is part of a biclique of \(G\).

Let \(\ell_{ij}\) be the \(j\)-th literal of clause \(c_i\). Vertex set \(V\) is the union \(V = W \cup S \cup Z\), where there is a \(w_{ij} \in W\) corresponding to each \(\ell_{ij}; S = \{v_1, \ldots, v_n\}; Z = \{z_1, \ldots, z_n\}\). Edge set \(E\) is the union \(E = W' \cup S' \cup Z'\), where \(W' = \{(w_{ij}, w_{ip}) : \ell_{ij} = \neg \ell_{ip}\}; S' = \{(v_i, w_{ip}) : v_i \in S, w_{ip} \in W\}; Z' = \{(z_i, w_{ip}) : z_i \in Z, w_{ip} \in W, i \neq p\}\).}

Let \(T\) be a truth assignment satisfying \(C\). We define an independent set \(U \subseteq W\) such that \((U, S)\) is a biclique of \(G\). Define first \(U_1\) as the set of vertices of \(W\) that correspond to literals with value true in \(T\). By definition, \(U_1\) is an independent set of \(G\), containing at least one vertex \(w_{ij}\), for each \(i = 1, \ldots, n\). Now let \(U\) be a maximal independent set of \(G\) containing \(U_1\). Note \(U \subseteq W\). Clearly \((U, S)\) induces a complete bipartite subgraph. In order to verify that \((U, S)\) is a biclique, we recall that \(|C| > 1\) and note that every vertex \(z_i \in Z\) in nonadjacent to a vertex \(w_{ij} \in U_1 \subseteq U\).

Conversely, let \((U, S)\) be a biclique of \(G\). Since \(Z \cup S\) is an independent set of \(G\), we have \(U \subseteq W\). The maximality of \((U, S)\) says that for each \(z_i \in Z\) there exists \(w_{ij} \in U\). Since \(U\) is a stable set, each \(w_{ij} \in U\) corresponds to a literal whose complementary literal does not belong to \(U\). So, for each clause
Corollary 2.2 Given a graph \( G = (V, E) \) and a vertex \( v \in V \), it is \( \text{NP}- \) complete to decide whether \( \{ v \} \) is part of a biclique.

Theorem 2.3 Given a graph \( G = (V, E) \), a biclique \( B \), and an order on \( V \), it is \( \text{coNP} \)-complete to decide whether \( B \) is the lexicographically last biclique.

Proof. Problem lexicographically last biclique is in \( \text{coNP} \) since a short certificate is a biclique \( B' \) that is lexicographically larger than \( B \).

To show completeness, we sketch a polynomial transformation from satisfiability. Given an instance \((X, C)\) of satisfiability we construct in polynomial time a graph \( G = (V, E) \), a biclique \( B \), and an order on \( V \), such that there is a truth assignment satisfying \( C \) if and only if \( B \) is the lexicographically last biclique.

Let \( \ell_{ij} \) be the \( j \)-th literal of clause \( c_i \). Vertex set \( V \) is the union \( V = Y \cup Z \cup W \), where \( Y = \{ y_1, \ldots, y_n, y'_1, \ldots, y'_n \} \), set \( Y \) contains a pair of vertices \( y_i, y'_i \) corresponding to each clause \( c_i \); \( Z = \{ z, z' \} \); set \( W \) contains a pair of vertices \( w_{ij}, w'_{ij} \) corresponding to each \( \ell_{ij} \). Edge set \( E \) is the union \( E = Y^* \cup Z^* \cup W^* \), where \( Y^* = \{ (y_i, y'_i) : i = 1, \ldots, n \} \cup \{ (y_i, w'_{p\ell}) : i = 1, \ldots, n \} \cup \{ (y'_i, w_{p\ell}) : i = 1, \ldots, n \} \cup \{ (y'_i, y_i) : i = 1, \ldots, n \} \cup \{ (w_{ij}, w'_ij) : w_{ij}, w'_ij \in W \} \cup \{ (w_{ij}, w'_{p\ell}), (w'_ij, w_{p\ell}) : i \neq p \) and \( \ell_{ij} \neq -\ell_{p\ell} \}.

Each clause \( c_i \) corresponds to an edge \((y_i, y'_i)\). The graph \( G \) is bipartite, since \( V = R \cup Q \), where \( R = \{ z \} \cup \{ y_1, \ldots, y_n \} \cup \{ w_{ij} \} \) and \( Q = V \setminus R \) are both independent sets. Note that \( B = (z', z \cup \{ y_1, \ldots, y_n \}) \) is a biclique, as \( N(z) = \{ z' \} \), and \( N(z') = \{ z \} \cup \{ y_1, \ldots, y_n \} \). In addition, \( B \) is the only biclique containing vertex \( z \). Every biclique \( B' \neq B \) such that \( z' \in B' \) is such that \( y'_i \in B' \), for some \( i = 1, \ldots, n \). In addition, \( B \) is the only biclique containing \( y_i \) but not \( y'_i \), for some \( i = 1, \ldots, n \).

The structure of the graph \( G \) gives the following equivalence: there is a biclique of \( G \) contained in \( W \) if and only if there is a truth assignment satisfying \( C \). Clearly, a complete bipartite contained in \( W \) that cannot be enlarged by addition of \( y \in Y \) is a complete bipartite that meets all clauses and that consequently gives a truth assignment satisfying \( C \).

Now, the order on \( V \) is defined as follows. Each vertex \( y'_i \) has label \( i \), \( i = 1, \ldots, n \), each vertex \( y_i \) has label \( n + i \), \( i = 1, \ldots, n \); vertices \( z \) and \( z' \) have labels \( 2n + 1 \) and \( 2n + 2 \), respectively; each vertex \( w \in W \) has a label \( j \geq 2n + 3 \).

Clearly, a biclique of \( G \) contained in \( W \) is lexicographically larger than \( B \). Conversely, let \( B' = (U, U') \) be a biclique lexicographically larger than \( B \).
Hence $y_i' \not\in B'$, $i = 1, \ldots, n$. We conclude that $U \cup U' \subseteq W$.  

Corollary 2.4  Given a graph $G = (V,E)$, a biclique $B$, and an order on $V$, it is coNP-complete to test if $B$ is the lexicographically last non-induced biclique.

3  Greedy generation of lexicographically first biclique

Let $G = (V,E)$ be a connected graph, with an order on $V = \{1, \ldots, n\}$. We describe an $O(n^2)$ greedy algorithm to generate the lexicographically first biclique.

Let $B = (U,W)$ be the lexicographically first biclique. Clearly, $1 \in B$, assume $1 \in U$. Hence, $W \subseteq N(1)$. Let $j$ be the smallest element of $N(1)$. Either $j \in B$, which means that $j$ is the smallest element of $W$; or $j \not\in B$, which means that there exists a vertex $i$, with $1 < i < j$ such that $i \not\in N(j)$ and $N(i) \cap N(1) \neq \emptyset$.

The proposed algorithm consists of two phases. Phase 1 finds $j' \geq j$ the smallest element of $W$, and $U' = U \cap \{1, \ldots, (j' - 1)\}$. Phase 2 enlarges subset $U'$ to $U$ and subset $\{j'\}$ to $W$, by scanning vertices $j' < k \leq n$ in the specified order, and by accepting every vertex $k$ that satisfies precisely one of the situations: vertex $k$ is nonadjacent to every vertex already accepted to be in $U$ and vertex $k$ is adjacent to every vertex already accepted to be in $W$; or vertex $k$ is nonadjacent to every vertex already accepted to be in $W$ and vertex $k$ is adjacent to every vertex already accepted to be in $U$.

References


