Abstract

For a Jordan domain with smooth boundary, the boundary rotation $\theta$ is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. The domain is said to have bounded turning or rotation if $\theta < \pi$ and the functions under such mappings are called functions of bounded boundary turning or rotation. We define a generalization of this concept and examine some of its properties. A few open problems are also stated for further explorations.

Keywords: Close-to-convex and bounded boundary turning functions

1. Introduction

For $\alpha \geq 0$, $\beta \geq 0$, and real $\gamma$, let $\mathcal{H}(\alpha, \beta; \gamma)$ denote the class of functions $f$ of the form $f(z) = 1 + a_1z + a_2z^2 + \cdots$ which are analytic and non-zero in the open unit disk $D = \{z: |z| < 1\}$ and satisfy the condition

$$-\alpha \pi \leq \int_{\theta_1}^{\theta_2} \Re \left( \frac{zf''}{f'} - \gamma \right) \leq \beta \pi,$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$.

This class is closely related to functions of bounded boundary turning or rotation. For a Jordan domain with smooth boundary, the boundary rotation $\theta$ is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit (see [4]). Thus $\theta \geq 2\pi$, with equality if and only if the domain is convex. If $\theta \leq 4\pi$, the domain is close-to-convex. A function $f$ in $\Delta$ is close-to-convex of order $\alpha$ if and only if $f' \in \mathcal{H}(\alpha, \alpha + 2; -1)$. If we let $\alpha = 1$, then $\mathcal{H}(1, 3; -1)$ consists of functions $f'$ so that $f$ is close-to-convex univalent in $\Delta$, that is,

$$\int_{\theta_1}^{\theta_2} \Re \left( 1 + \frac{zf''}{f'} \right) \geq -\pi.$$
A geometric characterization of close-to-convex univalent functions is that a function \( f \) is close-to-convex univalent in \( A \) if and only if none of the curves \( f(|z|) = r \) makes a reverse hairpin turn. The class \( \mathcal{K}(1, 3; -1) \) was first defined by Kaplan [8] and independently by Biernacki [2]. The class \( \mathcal{K}(x, \beta; (x - \beta)/2) \) was first introduced and studied by Sheil-Small [11]. Then after, a number of authors, including [5–7, 9, 10, 12], studied the class \( \mathcal{K}(x, \beta; (x - \beta)/2) \). In this note we examine some properties of the class \( \mathcal{K}(x, \beta; \gamma) \) and state a few open problems which hopefully will prompt further research on this topic.

2. Main results

Our first result proves the closure of the class \( \mathcal{K}(x, \beta; \gamma) \) under multiplication.

**Theorem 2.1.** If \( f \in \mathcal{K}(x_1, \beta_1; \gamma_1) \) and \( g \in \mathcal{K}(x_2, \beta_2; \gamma_2) \) then

\[
f \cdot g \in \mathcal{K}(x_1 + x_2, \beta_1 + \beta_2; \gamma_1 + \gamma_2).
\]

**Proof.** Suppose that \( f \in \mathcal{K}(x_1, \beta_1; \gamma_1) \) and \( g \in \mathcal{K}(x_2, \beta_2; \gamma_2) \). Then, according to (1), we must have

\[
-\pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{zf'}{f} - \gamma_1 \right\} d\theta \leq \beta_1 \pi
\]

and

\[
-\pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{zg'}{g} - \gamma_2 \right\} d\theta \leq \beta_2 \pi.
\]

Combining the above two inequalities, we obtain

\[
-(x_1 + x_2) \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{zf'}{f} - \gamma_1 + \frac{zg'}{g} - \gamma_2 \right\} d\theta = \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{zgf' + zfg'}{fg} - (\gamma_1 + \gamma_2) \right\} d\theta \leq (\beta_1 + \beta_2) \pi.
\]

Therefore, by (1), \( f \cdot g \in \mathcal{K}(x_1 + x_2, \beta_1 + \beta_2; \gamma_1 + \gamma_2) \). \( \square \)

**Corollary 2.2.** Let \( n \) be a positive integer. Then \( f \in \mathcal{K}(x, \beta; \gamma) \) if and only if \( f^n \in \mathcal{K}(nx, n\beta; n\gamma) \).

**Proof.** Suppose that \( f \in \mathcal{K}(x, \beta; \gamma) \). Upon noting \( f^n = f \cdot f \cdots f \) and applying Theorem 2.1, we obtain \( f^n = f \cdot f \in \mathcal{K}(2nx, 2n\beta; 2n\gamma) \). Now, applying Theorem 2.1 a total of \( n - 1 \) times, yield \( f^n \in \mathcal{K}(nx, n\beta; n\gamma) \). Conversely, suppose that \( f^n \in \mathcal{K}(nx, n\beta; n\gamma) \). Then, by (1), we must have

\[
-nx \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{z^n f'}{f} - \gamma \right\} d\theta \leq nx \pi.
\]

Dividing by \( n \) we obtain

\[
-x \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{z^n f'}{f} - \gamma \right\} \leq x \pi
\]

and so, again by (1), \( f \in \mathcal{K}(x, \beta; \gamma) \). \( \square \)

**Corollary 2.3.** Let \( n \) be a positive integer. Then \( f \in \mathcal{K}(x, \beta; \gamma) \) if and only if \( f^{-n} \in \mathcal{K}(n\beta, nx; -n\gamma) \).

**Proof.** Upon using the required condition (1), we obtain

\[
f \in \mathcal{K}(x, \beta; \gamma) \iff -x \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{zf'}{f} - \gamma \right\} \leq \beta \pi \iff -n\beta \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{z^n f'}{f} - (-n\gamma) \right\}
\]

\[
\leq nx \pi \iff -n\beta \pi \leq \int_{\alpha_1}^{\beta_1} \Re \left\{ \frac{z^n f'}{f} - (-n\gamma) \right\} \leq nx \pi \iff f^{-n} \in \mathcal{K}(n\beta, nx; -n\gamma). \quad \square
\]

Our next theorem proves an inclusion property for the class \( \mathcal{K}(x, \beta; \gamma) \).
Theorem 2.4. Let \( x_1 \leq x_2 \) and \( \beta_1 \leq \beta_2 \). Then \( \mathcal{K}(x_1, \beta_1; \gamma_1) \subset \mathcal{K}(x_2, \beta_2; \gamma_2) \) if
\[
2 \leq \frac{\beta_2 - \beta_1}{\gamma_2 - \gamma_1}
\]
or
\[
2 \leq \frac{x_2 - x_1}{\gamma_1 - \gamma_2}.
\]

Proof. Let \( f \in \mathcal{K}(x_1, \beta_1; \gamma_1) \). Then
\[
-x_1 \pi + \gamma_1(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} g(z) f' d\theta \leq \beta_1 \pi + \gamma_1(\theta_2 - \theta_1).
\]
For \( f \) to be in \( \mathcal{K}(x_2, \beta_2; \gamma_2) \) we need, by definition, to have
\[
-x_2 \pi + \gamma_2(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} g(z) f' d\theta \leq \beta_2 \pi + \gamma_2(\theta_2 - \theta_1).
\]
In order to have (4) imply (5) we need to have either
\[
-x_2 \pi + \gamma_2(\theta_2 - \theta_1) \leq -x_1 \pi + \gamma_1(\theta_2 - \theta_1)
\]
or
\[
\beta_1 \pi + \gamma_1(\theta_2 - \theta_1) \leq \beta_2 \pi + \gamma_2(\theta_2 - \theta_1).
\]
Since \( 0 \leq \theta_2 - \theta_1 \leq 2\pi \), the above two inequalities reduce to those two required conditions (2) and (3). This completes the proof. \( \square \)

3. Open problems

3.1. Convolution problem

The convolution or Hadamard product of two power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is defined by \( f(z) * g(z) = (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \).

For \( \alpha \geq 1 \) and \( \beta \geq 1 \) define
\[
Q_n(z; \theta) = (1 + z)^{-1} \prod_{j=1}^{n+1} (1 + z e^{(\theta + n-2)\beta}), \quad \theta = \frac{\pi}{n + \beta - \alpha},
\]
and \( Q^{(-1)}_n(z; \theta) \) so that \( Q^{(-1)}_n(z; \theta) * Q_n(z; \theta) = 1/(1 - z) \).

It was conjectured in [5] that if the polynomials \( p(z) = \prod_{k=1}^{n}(1 + z e^{\theta_k}) \) and \( q(z) = \prod_{k=1}^{n}(1 + z e^{\theta_k}) \) belong to \( \mathcal{K}(x, \beta; (x - \beta)/2) \) where \( 1 \leq \alpha \leq n, \alpha \leq \beta \), and \( \alpha \) and \( n \) are positive integers, then the convolution polynomial \( R_n(z) = (p * q)(z) * Q^{(-1)}_n(z; \theta) \) also belongs to \( \mathcal{K}(x, \beta; (x - \beta)/2) \).

This conjecture is proved for the case \( x = n - 1 = 2 \) in [5, Theorem 4] for the case \( 1 = x \leq \beta \) in [13, Theorem 5], and for the case \( 2 \leq x \leq 4 \) and \( x \leq \beta \) in [7, Theorem 1.3].

Can we determine polynomials \( Q_n(z; \theta) \) so that the convolution polynomial \( R_n(z) = (p * q)(z) * Q^{(-1)}_n(z; \theta) \) belongs to \( \mathcal{K}(x, \beta; \gamma) \) for \( \gamma \) real and the polynomials \( p \) and \( q \) in \( \mathcal{K}(x, \beta; \gamma) \)?

3.2. Majorization problem

For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) we write \( f \ll g \) if \( |a_n| \leq |A_n| \). Aharonov and Friedland [1] proved that \( f(z) \ll (1 + z)^{\theta}(1 - z)^{\beta} \) for \( f \in \mathcal{K}(x, 2 + x; -1) \). Later, Brannan [3] showed that if \( \alpha \geq 1 \) and \( \beta \geq 1 \) then \( f(z) \ll (1 + z)^{\theta}(1 - z)^{\beta} \) for \( f \in \mathcal{K}(x, \beta; (x - \beta)/2) \). Sheil-Small [12, Theorem 1] proved that if \( 0 \leq \alpha \leq 1 \) and \( \alpha + \beta \geq 2 \) then \( f(z) \ll (1 + z)^{\theta}(1 - z)^{\beta} \) for \( f \in \mathcal{K}(\alpha, \beta; (x - \beta)/2) \). Can we find functions \( g(z) \) so that \( f \ll g \) for \( f \in \mathcal{K}(\alpha, \beta; \gamma) \) and \( \gamma \) real?
3.3. Coefficient problem

For certain values of $a$ and $b$, \( (1 + z)^a / (1 / C_0 z)^b \) gives sharp coefficient bounds for \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{K}(\alpha; \beta; (\alpha - \beta) / 2) \). For example, if $0 < \alpha < 1$ and $f \in \mathcal{K}(\alpha; \alpha; 0)$ then $|a_n| \leq 2\alpha$. This bound is sharp (see [12, Theorem 3]) for $f(z) = (1 + z)^c / (1 - z)^\beta$. It is to be noted that $(1 + z)^c / (1 - z)^\beta$, in general, is not extremal for the coefficients of functions in $\mathcal{K}(\alpha; \beta; (\alpha - \beta) / 2)$. However, it is conjectured by Sheil-Small [12] that the weaker Rogozinski dominance condition $\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |A_k|^2$ holds for $f \in \mathcal{K}(\alpha; \beta; (\alpha - \beta) / 2)$ where $A_n = A_n(x, \beta)$ are the coefficients of $(1 + z)^c / (1 - z)^\beta$. Can we determine functions $g(z) = \sum_{n=0}^{\infty} a_n z^n$ so that $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{K}(\alpha; \beta; \gamma)$ and $\gamma$ real?

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References