Extended conjugate points in the calculus of variations

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This paper concerns a characterization of second-order optimality conditions for the fixed-endpoint problem in the calculus of variations. The key new concept is a set $S(x)$ with the property that $S(x) = \emptyset$ if and only if the second variation with respect to $x$, independently of non-singularity assumptions, is non-negative along admissible variations. We show that, for this set of points, it may be much easier (and never more difficult) to prove its non-emptiness than directly finding variations that make the second variation negative. Earlier Loewen and Zheng, and Zeidan, introduced related sets $C_1(x)$ and $C_2(x)$, applicable to certain optimal control problems, whose non-emptiness has been established merely as a sufficient condition for the existence of negative second variations. These sets, when reduced to the problem we are considering, are related according to $C_1(x) \subset C_2(x) \subset S(x)$. Contrary to the behaviour of $S(x)$, verifying membership of $C_1(x)$ or $C_2(x)$ may be more difficult than verifying directly if the second-order condition holds. We provide several examples for which it is straightforward to prove that $S(x) \neq \emptyset$, but determining the sets $C_1(x)$ or $C_2(x)$ may be a very difficult or perhaps even a hopeless task.

Keywords: calculus of variations; Jacobi’s necessary condition; generalized conjugate points; non-singular extremals.

1. Introduction

The theory of conjugacy has played a fundamental role in characterizing second-order conditions for problems in the calculus of variations. In particular, for the simple fixed-endpoint problem, it is well-known that, if $H$ denotes the set of trajectories for which the second variation is non-negative along admissible variations and $x$ is a non-singular extremal, then $x \in H$ if and only if $x$ satisfies Legendre’s condition and there are no conjugate points to $t_0$ with respect to $x$ in $(t_0, t_1)$, the underlying open time interval (Jacobi’s necessary condition).

Several attempts have been made to generalize this result to an optimal control setting. In particular, Zeidan & Zezza (1988) considered a class of optimal control problems with one fixed-endpoint and equality constraints in the control. Their notion of ‘conjugate points’ generalizes the classical one in the sense that, if an optimal process $(x, u)$ satisfies the corresponding strengthened condition of Legendre or Clebsch (and therefore is non-singular) together with certain normality conditions (always satisfied in the context of calculus of variations) then there are no conjugate points to $t_0$ with respect to $(x, u)$ in $(t_0, t_1)$.

An entirely different approach was followed in 1994 by Loewen & Zheng (1994), applicable to certain optimal control problems (which include that of Zeidan & Zezza, 1988) with one fixed-endpoint and involving equality and inequality constraints in the
control. With respect to a given extremal, they introduced a set of 'generalized conjugate points' whose non-emptiness in the open time interval implies the existence of an admissible variation for which the second variation is negative. Dealing with conjugacy with respect to the endpoint \( t_0 \), this set is contained in \((t_0, t_1]\). Let us denote it, for any admissible process \((x, u)\), by \( C_1(x, u) \). The corresponding 'Jacobi condition' in terms of this set is obtained by an application of second-order conditions for these problems: if \((x, u)\) is a normal solution, then the second variation with respect to \((x, u)\) is non-negative along admissible variations and, therefore, \( C_1(x, u) \cap (t_0, t_1) \) is empty.

This notion of conjugacy does not generalize the classical one as in Zeidan & Zezza (1988) since the non-existence of such points in \((t_0, t_1)\) is necessary for optimality even if the process under consideration is singular. However, if the problem considered in Loewen & Zheng (1994) is reduced to the fixed-endpoint problem in the calculus of variations and the extremal under consideration is non-singular, then the classical set of conjugate points, in the open time interval, is contained in that of generalized conjugate points. Thus this new notion 'extends' that of conjugacy not only for optimal control problems but also in the classical theory of calculus of variations. Several examples given in Loewen & Zheng (1994) illustrate the usefulness of this set: they deal with singular processes, so that the classical theory of Jacobi or that of Zeidan & Zezza (1988) fail to detect non-optimality, while this fact follows by proving the existence of generalized conjugate points.

For the class of optimal control problems considered in Loewen & Zheng (1994), the initial endpoint is fixed and a convexity assumption on the control set is required. For problems where both endpoints vary and the control set is not necessarily convex, Zeidan (1996) introduced in 1996 yet another set of 'generalized coupled points', say \( C_2(x, u) \), containing \( C_1(x, u) \), and shows that a necessary condition for optimality, for normal extremals, is again the non-existence of such points in \((t_0, t_1)\). As in Loewen & Zheng (1994), this result is a consequence of the non-negativity of the second variation along admissible variations under normality assumptions.

Although the above theory seems to be successful for certain quite general classes of optimal control problems, these sets of 'extended conjugate points' present two undesirable features. First of all, their non-emptiness has been established merely as a sufficient condition for the existence of negative second variations. In other words, it is not clear if there are problems for which, with respect to a normal extremal, these sets are empty but the second variation is negative along an admissible variation (implying non-optimality of the extremal). Second, one should bear in mind that the main objective of introducing a characterization of a second-order condition should be, of course, to obtain a simpler way of verifying it. However, even for simple problems in the calculus of variations, one can easily find examples for which to solve the question of non-emptiness of these sets may be much more difficult than verifying directly if that condition holds. Hence, by using the theories of Loewen and Zheng, or Zeidan, one may fail to achieve the main objective of introducing a characterization of the second-order necessary condition.

With the purpose of solving these two problems, we introduced in Berlanga & Rosenblueth (2002) a third set of points, say \( R(x) \), applicable to the fixed-endpoint problem in the calculus of variations. This set is such that, if \( x \) is any admissible trajectory, then \( x \in H \) if and only if \( R(x) \) is empty. Further, this set enjoys the property that, if there exists an admissible variation \( y \) for which the second variation is negative, then \( y \) satisfies the conditions defining membership of \( R(x) \). In other words, it can never be more difficult to
check non-emptiness of $R(x)$ than to prove directly that $x \not\in H$. Finally, with respect to the sets defined in Loewen & Zheng (1994) and Zeidan (1996), we established in Rosenblueth (2002) the relations $C_1(x) \subset C_2(x) \subset R(x)$ (implying, in particular, that the conditions $C_1(x) = \emptyset$ and $C_2(x) = \emptyset$ are necessary for optimality in the entire half-open interval $(t_0, t_1)$). This set was later generalized in Rosenblueth (2003) to certain linear optimal control problems.

In this paper we shall move further in the direction initiated in Berlanga & Rosenblueth (2002). The idea underlying the definition of $R(x)$ is simple. Given a trajectory $x$, consider the bilinear form

$$J(z, y) = \int_{t_0}^{t_f} \{ \dot{z}(t), L_{\ddot{z}z} y(t) + L_{\ddot{z}t} \dot{y}(t) \} + \{ z(t), L_{xx} y(t) + L_{xt} \dot{y}(t) \} \, dt$$

where the partial derivatives of $L$ are evaluated at $(t, x(t), \dot{x}(t))$. A point $s \in (t_0, t_1]$ belongs to $R(x)$ if there exist piecewise $C^1$ functions $y$ on $[t_0, s]$ with $y(t_0) = y(s) = 0$ and $J(y, y) \leq 0$, and $u$ on $[t_0, t_1]$ with $u(t_0) = u(t_1) = 0$ and $J(u, y) \neq 0$. If $R(x) \neq \emptyset$ then (extending $y$ to the whole interval by zero) the choice of an admissible variation $u + \alpha y$ makes the second variation along $x$ strictly negative for some $\alpha$ of appropriate sign. Conversely, if the second variation along $x$ is negative for some variation $y$ then, by choosing $u \equiv y$, the endpoint $t_1$ belongs to $R(x)$.

Now, at first sight, the reader might have reservations that the use of $R(x)$ is too general to be useful in the sense that checking that there exist such points is of the same difficulty as directly finding variations that make the second variation negative. In fact, if $J(y, y) < 0$ for some $y$ on $[t_0, s]$ with $y(t_0) = y(s) = 0$, then automatically $x \not\in H$. The interesting case is, however, when $J(y, y) = 0$ and, for a wide range of problems, it may be trivial to exhibit a function $y \neq 0$ with this property. This function does not satisfy the strict inequality and may even yield a contradiction in the definition of $C_1(x)$ or $C_2(x)$.

Once a non-trivial $y$ with $y(t_0) = y(s) = 0$ and $J(y, y) = 0$ is found, one needs to find $u$ on $[t_0, t_1]$ with $u(t_0) = u(t_1) = 0$ and $J(u, y) \neq 0$. With the idea of simplifying the conditions defining membership of $R(x)$ we shall prove that $u$ as above exists under mild conditions imposed on $y$. These conditions motivate the introduction of a new set $S(x)$, containing $R(x)$, for which also $x \in H \iff S(x) = \emptyset$, and to verify membership of $S(x)$ may be even simpler than for $R(x)$.

On the other hand, if $s$ belongs to $C_1(x)$ or $C_2(x)$, both sets require, apart from other conditions, the existence of a piecewise $C^1$ function $y$ on $[t_0, s]$ with $y(t_0) = y(s) = 0$, and a constant $c \in \mathbb{R}^n$, such that $(\dot{y}(t), z(t)) \geq 0$ for all $t \in [t_0, s]$, where

$$z(t) = c + \int_{t_0}^{t} \left[ L_{\ddot{z}z} y(\tau) + L_{\ddot{z}t} \dot{y}(\tau) \right] \, d\tau - \left[ L_{xx} y(t) + L_{xt} \dot{y}(t) \right].$$

It is a simple fact to show that, if this holds, then

$$J(y, y) = -\int_{t_0}^{t_f} (\dot{y}(t), z(t)) \, dt \leq 0$$

and the first condition of $R(x)$ is satisfied (the other conditions on $C_1(x)$ or $C_2(x)$ imply that $s \in R(x)$). The converse, however, may not occur. One can easily pose problems for
which there exist piecewise $C^1$ functions $y$ on $[t_0, s]$ with $y(t_0) = y(s) = 0$ and $J(y, y) \leq 0$ (or even $J(y, y) < 0$), but there does not exist $c \in \mathbb{R}^n$ satisfying $(\dot{y}(t), z(t)) \geq 0$ for all $t \in [t_0, s]$.

An example of this nature was given in Berlanga & Rosenblueth (2002). The problem is to minimize $\int_{t_0}^{t_1} t [\dot{x}^2(t) - x^2(t)]dt$ subject to $x(0) = x(a) = 0$. In Berlanga & Rosenblueth (2002) we show that, if $a > \pi$, simple admissible variations with a piecewise constant derivative imply that $J(y, y) < 0$, but these functions cannot be used to show non-emptiness of $C_1(x)$ or $C_2(x)$. If $a = \pi$, the variation $y(t) = \sin t$ is such that $J(y, y) = 0$ and one can easily show that $\pi \in R(x)$. Again, the use of this function contradicts the definition of $C_1(x)$ and $C_2(x)$. In Berlanga & Rosenblueth (2002) we left open the question of whether the three sets coincide, but laid emphasis on the fact that, even if they do, the conditions defining membership of $R(x)$ may be of a much simpler nature than those of $C_1(x)$ or $C_2(x)$. In this paper we study this example in more detail. In particular, we obtain the interval of minimum length for which the three sets are non-empty.

The question of whether there exist problems for which $C_1(x)$ or $C_2(x)$ are empty, but not $R(x)$ or $S(x)$, remains open in the general case. However, we shall illustrate by means of several examples the simplicity and usefulness of $S(x)$, compared with the other sets. In all the examples provided, it is straightforward to prove that $S(x) \neq \emptyset$, but determining the sets $C_1(x)$ or $C_2(x)$ may be a very difficult or perhaps even a hopeless task.

2. The problem

This paper deals with the simple fixed-endpoint problem in the calculus of variations. A full account of the results given in this section can be found, for example, in Ewing (1985) and Hestenes (1966).

To state the problem, suppose that we are given an interval $T := [t_0, t_1]$ in $\mathbb{R}$, two points $\xi_0, \xi_1$ in $\mathbb{R}^n$, a (relatively) open set $A$ in $T \times \mathbb{R}^n \times \mathbb{R}^n$, and a function $L$ mapping $T \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. Let $X$ be the space of piecewise $C^1$ functions mapping $T$ to $\mathbb{R}^n$, set

$$X(A) := \{ x \in X \mid (t, x(t), \dot{x}(t)) \in A \ (t \in T) \},$$

$$X_e(A) := \{ x \in X(A) \mid x(t_0) = \xi_0, x(t_1) = \xi_1 \}.$$

and consider the functional $I : X \to \mathbb{R}$ given by

$$I(x) := \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt \quad (x \in X).$$

The problem, which we label (P), is that of minimizing $I$ over $X_e(A)$.

Elements of $X$ are called trajectories, and $x \in X_e(A)$ solves (P) if $I(x) \leq I(y)$ for all $y \in X_e(A)$. For any trajectory $x$ we use the notation $(\hat{x}(t))$ to represent $(t, x(t), \dot{x}(t))$, and we shall assume throughout that $L$ is continuous on $A$ and $C^2(A)$ with respect to $x$ and $\dot{x}$.

For all $x \in X$ consider the second variation of $I$ along $x$ given by

$$I''(x; y) := 2\int_{t_0}^{t_1} \Omega(t)(\dot{y}(t))dt \quad (y \in X)$$

where, for all $(t, y, \dot{y}) \in T \times \mathbb{R}^n \times \mathbb{R}^n$,

$$2\Omega(t, y, \dot{y}) := \langle y, L_{xx}(\hat{x}(t))y \rangle + 2\langle y, L_{x\dot{x}}(\hat{x}(t))\dot{y} \rangle + \langle \dot{y}, L_{\dot{x}\dot{x}}(\hat{x}(t))\dot{y} \rangle.$$
and define
\[ H := \{ x \in X \mid I''(x; y) \geq 0 \text{ for all } y \in Y \} \]
where \( Y = \{ y \in X \mid y(t_0) = y(t_1) = 0 \} \) is the set of admissible variations. It is well-known that any solution to \((P)\) belongs to \( H \).

To state Jacobi’s necessary condition, let us introduce the following notation. For any \( s \in (t_0, t_1) \), set \( T_s := [t_0, s] \) and \( Y_s := \{ y \in X_s \mid y(t_0) = y(s) = 0 \} \) where \( X_s \) denotes the space of piecewise \( C^1 \) functions mapping \( T_s \) to \( \mathbb{R}^n \). Whenever we are given \( x \in X \) and \( y \in Y_s \), we shall consider the functions \( v, w : T_s \to \mathbb{R}^n \) (depending on both \( x \) and \( y \)) defined by
\[
\begin{align*}
v(t) &:= \Omega^x_0(\tilde{y}(t)) = L_{x\tilde{y}}(\tilde{x}(t))y(t) + L_{x\tilde{x}}(\tilde{x}(t))\tilde{y}(t) \\
w(t) &:= \Omega^y_0(\tilde{y}(t)) = L_{y\tilde{y}}(\tilde{x}(t))y(t) + L_{y\tilde{x}}(\tilde{x}(t))\tilde{y}(t).
\end{align*}
\]

**Definition 2.1** For any \( x \in X \) let \( C_0(x) \) be the set of points \( s \in (t_0, t_1] \) for which there exists \( y \in Y_s \) with \( y \neq 0 \) such that \( \dot{v}(t) = w(t) \) \( (t \in T_s) \).

Elements of \( C_0(x) \) are called points conjugate to \( t_0 \) on \( x \). Let
\[
L' := \{ x \in X \mid L_{x\tilde{x}}(\tilde{x}(t)) > 0 \ (t \in T) \}
\]
be the set of trajectories satisfying Legendre’s strengthened condition. The main result concerning Jacobi’s condition, in terms of conjugate points, can be stated as follows.

**Theorem 2.2** Let \( x \in X(A) \cap C^1 \cap L' \). If \( x \in H \) then \( C_0(x) \cap (t_0, t_1] = \emptyset \).

This theorem implies that, if there exists a conjugate point to \( t_0 \) on \( x \) strictly less than \( t_1 \), then \( x \) cannot be a solution to the problem. The assumption of non-singularity of \( x \) (that is, \( |L_{x\tilde{x}}(\tilde{x}(t))| \neq 0 \) for all \( t \in T \)) is essential in the theorem, and we are interested in characterizing \( H \) even if this condition does not hold. In this event, as mentioned in the introduction, three sets related to \( H \) have been recently proposed (in Berlanga & Rosenblueth, 2002; Loewen & Zheng, 1994 and Zeidan, 1996), and some of their properties are summarized in the following section.

### 3. Extended conjugate points

In Loewen & Zheng (1994), Loewen and Zheng introduced a set of ‘generalized conjugate points’, applicable to certain classes of optimal control problems, and showed that its non-emptiness in the open time interval implies the existence of an admissible variation for which the second variation is negative. When reduced to the problem we are considering (and conjugacy is reversed in terms of \( t_0 \) instead of \( t_1 \)), it corresponds to the following set (recall the definitions of \( v, w \) given in Section 2).

**Definition 3.1** For any \( x \in X \) let \( C_1(x) \) be the set of points \( s \in (t_0, t_1] \) for which there exist \( y \in Y_s \) and \( q \in X_s \) such that
\[
\begin{align*}
(i) \quad & \dot{q}(t) = w(t) \ (t \in T_s) . \\
(ii) \quad & q(s) \neq 0 . \\
(iii) \quad & \langle \dot{y}(t), q(t) - v(t) \rangle \geq 0 \ (t \in T_s) .
\end{align*}
\]
and either (a) or (b) holds:

(a) \[ \langle \dot{y}(t), q(t) - v(t) \rangle > 0 \]

on a set of positive measure.

(b) There exists \( u \in Y \) such that

(i) \[ \langle u(s), q(s) \rangle < 0. \]

(ii) \[ \langle \dot{u}(t), q(t) - v(t) \rangle \geq 0 \quad (t \in T_s). \]

The main result obtained in Loewen & Zheng (1994) (for problem (P)), relating this set to \( H \) and to the classical set of conjugate points, can be stated as follows.

**Theorem 3.2** Let \( x \in X(A) \). Then the following holds

(a) \( x \in H \Rightarrow C_1(x) \cap (t_0, t_1) = \emptyset. \)

(b) \( x \in L' \Rightarrow C_0(x) \cap (t_0, t_1) \subset C_1(x). \)

In Zeidan (1996), Zeidan introduced another set of points which is applicable to certain optimal control problems more general than those treated in Loewen & Zheng (1994). When reduced to (P) (and, as before, conjugacy is reversed in terms of \( t_0 \) instead of \( t_1 \)), it is defined as follows.

**Definition 3.3** For any \( x \in X \) let \( C_2(x) \) be the set of points \( s \in (t_0, t_1) \) for which there exist \( y \in Y_s \) and \( q \in X_s \) such that

(i) \[ \dot{q}(t) = w(t) \quad (t \in T_s). \]

(ii) \[ \langle \dot{y}(t), q(t) - v(t) \rangle \geq 0 \quad (t \in T_s). \]

(iii) If the inequality in (ii) is equality for all \( t \in T_s \) and \( q(t) - v(t) = \alpha \) for some \( \alpha \in \mathbb{R}^n \) and all \( t \in T_s \), then there exists \( u : [s, t_1] \rightarrow \mathbb{R}^n \) piecewise \( C^1 \) with \( u(t_1) = 0 \) and \( \langle u(s), q(s) - \alpha \rangle < 0. \)

Relating this set to \( H \) and \( C_1(x) \), one obtains from Zeidan (1996) (applied to (P)) the following result.

**Theorem 3.4** Let \( x \in X(A) \). Then the following holds

(a) \( x \in H \Rightarrow C_2(x) \cap (t_0, t_1) = \emptyset. \)

(b) \( C_1(x) \subset C_2(x). \)

Now, as mentioned in the introduction, though these sets give some answer to the question of how to generalize Jacobi’s theory both for singular extremals and for certain optimal control problems, they present two unpleasant features. First of all, the question of the converse of 3.2(a) or 3.4(a) remains open (this question is not even posed in those papers). In other words, it is not known if there exist extremals \( x \) with negative second variations for which \( C_1(x) \) or \( C_2(x) \) are empty. Second, there are simple problems, even in the calculus of variations, for which determining the sets \( C_1(x) \) or \( C_2(x) \) may be extremely difficult, but one can trivially exhibit admissible variations for which the second variation is negative. With the purpose of solving these difficulties, we introduced in Berlanga & Rosenblueth (2002) the following set.

**Definition 3.5** For any \( x \in X \) let \( R(x) \) be the set of points \( s \in (t_0, t_1) \) for which there exists \( y \in Y_s \) such that
In view of the remarks of the previous section, it is of interest to see if the conditions defining $R$ are assured. Note that, if the inequality in 3.5(i) is strict, condition 3.5(ii) holds by setting $\gamma := \int_{t_0}^t \langle \dot{u}(t), v(t) \rangle \, dt \neq 0$.

From Berlanga & Rosenblueth (2002) and Rosenblueth (2002) we obtain the following result.

**Theorem 3.6** Let $x \in X(A)$. Then the following holds

(a) $x \in H \iff R(x) = \emptyset$.

(b) $C_2(x) \subset R(x)$.

With this new set, it turns out that the above difficulties are solved. First of all, its non-emptiness is not only a sufficient condition for the existence of negative second variations, but also necessary. Also, it can never be more difficult to check non-emptiness of $R(x)$ than to prove directly that $x \notin H$, since any $y \in Y$ with $I''(x; y) < 0$ satisfies the conditions defining membership of $R(x)$. The same applies, in terms of $R(x)$, with respect to the sets defined in 3.1 and 3.3 since, if there exist $s, y, q$ and $u$ satisfying the conditions of any of these sets, then $s \in R(x)$. Note also that a simple corollary to Theorem 3.6 is that the conditions $C_1(x) = \emptyset$ and $C_2(x) = \emptyset$ are necessary for optimality in the entire interval $(t_0, t_1]$ and not only in the open interval, as stated in Loewen & Zheng (1994) and Zeidan (1996).

**4. A new set**

In view of the remarks of the previous section, it is of interest to see if the conditions defining $R(x)$ can be simplified. In particular, one would like to know under what assumptions the existence of $u \in Y$ satisfying $\gamma \neq 0$ (that is, condition 3.5(ii)) can be assured. Note that, if the inequality in 3.5(i) is strict, condition 3.5(ii) holds by setting $u(t) := \gamma(t)$ for $t \in [t_0, s]$ and $u(t) := 0$ for $t \in [s, t_1]$. On the other hand, if the inequality in 3.5(i) is equality, we shall provide simple conditions implying the existence of $u \in Y$ for which $\gamma \neq 0$. To do so, let us introduce the following set.

**Definition 4.1** For any $x \in X$ let $S(x)$ be the set of points $s \in (t_0, t_1]$ for which there exists $y \in Y_x$ such that

(i) $\int_{t_0}^s \langle \dot{y}(t), v(t) \rangle + \langle y(t), w(t) \rangle \, dt \leq 0$.

(ii) If $\dot{v}(t) = w(t)$ ($t \in T_x$), then $s < t_1$ and $L_{s+t} \langle \tilde{x}(s) \rangle \dot{y}(s) \neq 0$.

Let us prove that this set of points, just like $R(x)$, characterizes $H$.

**Theorem 4.2** Let $x \in X(A)$. Then $x \in H \iff S(x) = \emptyset$.

**Proof.** $\Rightarrow$: Suppose $x \in H$ and there exists $s \in S(x)$. Let $y$ be as in Definition 4.1 and define $z(t) := y(t)$ for $t \in [t_0, s]$ and $z(t) := 0$ for $t \in [s, t_1]$. If the value of the integral in 4.1(i) is negative, we reach a contradiction since $z \in Y$ and $I''(x; z) < 0$. If it vanishes, then $z$ solves the ‘accessory problem’ of minimizing $I''(x; \cdot)$ over $Y$ and, therefore, $\dot{v}(t) = w(t)$ ($t \in T_x$). By (ii), $s < t_1$ and $L_{s+t} \langle \tilde{x}(s) \rangle \dot{y}(s) \neq 0$. Let $u(t) := (t - t_0)(t - t_1) v(s)$ for all $t \in T$ and define

$$\gamma := \int_{t_0}^t \langle \dot{u}(t), v(t) \rangle + \langle u(t), w(t) \rangle \, dt.$$
Observe that \( \gamma = (s - t_0)(s - t_1)|v(s)|^2 \neq 0 \). Set \( k := I''(x; u), \alpha := -(\gamma + k/2\gamma), y_a := u + \alpha z \). Then
\[
I''(x; y_a) = \int_{t_0}^{t_1} 2\Omega(\tilde{y}_a(t))\,dt = k + \alpha^2 I''(x; z) + 2\alpha \gamma = -2\gamma^2 < 0
\]
which contradicts the assumption \( x \in H \).

\( \leftarrow \) If \( x \not\in H \), there exists \( y \in Y \) such that \( I''(x; y) < 0 \), implying that \( t_1 \in S(x) \). \( \square \)

In view of this result, if there exist \( s \in (t_0, t_1) \) and \( y \in Y_s \) such that the integral in 4.1(i) vanishes, we can conclude that \( x \not\in H \) simply by checking if \( s < t_1 \) and \( L_{x\tilde{x}}(\tilde{x}(s))s(s) \neq 0 \). If one of these relations fails, the same conclusion follows if \( y \) does not satisfy Jacobi’s equation \( \dot{v}(t) = w(t) \). This new set of points is related to \( R(x) \) as follows.

**Proposition 4.3** For any \( x \in X, R(x) \subset S(x) \) and, if we define \( \tilde{S}(x) \) as the set of points \( s \in (t_0, t_1) \) for which there exists \( y \in Y_s \) satisfying the conditions of 4.1 and \( \psi(t) := w(t) - \dot{v}(t) \), the result follows as in the proof of 4.2 by setting \( u(t) := (t - t_0)(t - t_1)v(s) \) for all \( t \in T \). Finally, suppose the inequality in 4.1(i) is equality but \( \dot{v}(\tau) \neq w(\tau) \) for some \( \tau \in T \). Without loss of generality, \( \tau \in (t_0, s) \). Since \( \psi \) is continuous, there exists \( \epsilon > 0 \) such that \( \psi(t) \neq 0 \) for all \( t \in (\tau - \epsilon, \tau + \epsilon) \subset (t_0, s) \). Let \( \mu \) be any continuous function defined in \( T \), supported in \( [\tau - \epsilon, \tau + \epsilon] \), and positive in its interior, and define \( u(t) := \mu(t)\psi(t) \). Then \( u(t) := 0 \) for all \( t \in [s, t_1] \).

Let us turn now to the problem of finding functions \( y \) on \( [t_0, s] \) for which the integral in 4.1(i) vanishes. We shall next prove, for certain problems, a simple criterion in this respect. In the remainder of this section we assume that \( n = 1 \) and that all the functions considered are continuous.

Given \( x \in X \) and \( s \in (t_0, t_1) \), let
\[
K(y) := \int_{t_0}^{s} \{ \dot{y}(t)u(t) + y(t)w(t) \} \,dt \quad (y \in Y_s).
\]

From the definition, and integrating \( \int L_{x\tilde{x}}(\tilde{x}(t))y(t)\dot{y}(t)\,dt \) by parts, we have
\[
K(y) = \int_{t_0}^{s} \{ L_{x\tilde{x}}(\tilde{x}(t))\dot{y}(t) + 2L_{x\tilde{x}}(\tilde{x}(t))y(t)\dot{y}(t) + L_{x\tilde{x}}(\tilde{x}(t))y^2(t) \} \,dt
= \int_{t_0}^{s} \{ p(t)\dot{y}^2(t) - r(t)y^2(t) \} \,dt
\]
where

\[ p(t) = L_{\ddot{x}}(\ddot{x}(t)) \quad \text{and} \quad r(t) = \left[ \frac{d}{dt} L_{x\ddot{x}}(\ddot{x}(t)) \right] - L_{xx}(\ddot{x}(t)) \quad (t \in T_a). \]

Integrating again by parts, we have

\[ \int_{t_0}^s p(t) \dot{y}^2(t) \, dt = -\int_{t_0}^s y(t) [p(t) \ddot{y}(t) + \dot{p}(t) \dot{y}(t) + r(t)y(t)] \, dt \]

and therefore

\[ K(y) = -\int_{t_0}^s y(t) [p(t) \ddot{y}(t) + \dot{p}(t) \dot{y}(t) + r(t)y(t)] \, dt. \]

Integrating once more by parts, now \( \int \dot{p}(t) \ddot{y}(t) \, dt \), we obtain

\[ K(y) = -\int_{t_0}^s y(t) [p(t) \ddot{y}(t) + \dot{p}(t) \dot{y}(t) + r(t)y(t)] \, dt \]

where

\[ \sigma(t) = r(t) - \frac{\dot{p}(t)}{2}. \]

Let us consider the two second-order differential equations

\[ p(t) \ddot{y}(t) + \dot{p}(t) \dot{y}(t) + r(t)y(t) = 0 \quad (A) \]
\[ p(t) \ddot{y}(t) + \sigma(t)y(t) = 0. \quad (B) \]

Note that equation (A) is precisely Jacobi’s differential equation and that, if \( y \in Y_s \) satisfies any of these equations, then \( K(y) = 0 \).

5. The answer to an open question

In Berlanga & Rosenblueth (2002) we provided a simple problem for which one can easily show that \( R(x) \neq \emptyset \) but the question of non-emptiness of \( C_1(x) \) and \( C_2(x) \) was left unsolved. This example deserves some special attention. It corresponds to the problem of minimizing

\[ I(x) = \frac{1}{2} \int_0^a t(\dot{x}^2(t) - x^2(t)) \, dt \]

subject to \( x(0) = x(a) = 0 \). In this case \( n = 1, T = [0, a], \xi_0 = \xi_1 = 0, A = T \times \mathbb{R} \times \mathbb{R}, \) and \( L(t, x, \dot{x}) = t(\dot{x}^2 - x^2)/2 \).

Note first that any trajectory is singular since \( L_{\ddot{x}}(\ddot{x}(t)) = t \) for all \( t \in [0, a] \) and, thus, the classical theory of conjugate points, as presented in Ewing (1985) and Hestenes (1966), cannot be applied. The second variation with respect to any trajectory \( x \) is given by

\[ I''(x; y) = \int_0^a t(\dot{y}^2(t) - y^2(t)) \, dt \]
and so $H \neq \emptyset$ if and only if $x_0 \equiv 0$ solves (P). At first sight it is not obvious how to solve this problem. In Berlanga & Rosenblueth (2002) we tested simple functions which, for different intervals of integration, make the second variation negative. If, for example,

$$y(t) := \begin{cases} t & t \in [0, s/2] \\ s-t & t \in [s/2, s] \\ 0 & t \in [s, a] \end{cases}$$

then $I''(x; y) = s^2/2 - s^4/24$, which is negative if and only if $s^2 > 12$. Thus, if $a > \sqrt{12}$, then $H = \emptyset$. If

$$y(t) := \begin{cases} t & t \in [0, s/4] \\ (s-t)/3 & t \in [s/4, s] \\ 0 & t \in [s, a] \end{cases}$$

then $I''(x; y) < 0$ if and only if $s^2 > 32/3$. These bounds were improved in Berlanga & Rosenblueth (2002) by an application of $R(x)$. If $y(t) := \sin t$ for all $t \in [0, \pi]$, then $y \in Y_\pi$ and $\int_0^\pi t(\dot{y}(t)^2 - y(t)^2)dt = 0$ showing that condition (i) of $R(x)$ holds. Setting, for example,

$$u(t) = \begin{cases} \sin 2t \cos t & t \in [0, \pi/2] \\ 0 & t \in [\pi/2, \pi]. \end{cases}$$

it follows that $\gamma \neq 0$ and therefore $\pi \in R(x)$. Hence, if $a \geq \pi$, the problem has no solution.

It should be noted that this conclusion follows in an even simpler way by the use of $S(x)$. The arguments given at the end of Section 4 show that the choice $y(t) = \sin t$ is not an idle one. Equation (B) corresponds to $\dot{y}(t) + y(t) = 0$ whose solution vanishing at $t = 0$ is precisely $y(t) = a \sin t$. Since $y$ does not satisfy (A) given by $t\ddot{y}(t) + \dot{y}(t) + ty(t) = 0$, we conclude that $\pi \in S(x)$.

On the other hand, the use of any of these simple functions yields a contradiction in the definitions of $C_1(x)$ and $C_2(x)$ and, in Berlanga & Rosenblueth (2002), we left open the question of their emptiness in the open interval $(0, \pi)$. We shall now give an answer to this question.

Observe first that Jacobi’s equation corresponds to $t\ddot{y}(t) + \dot{y}(t) + ty(t) = 0$, Bessel’s equation of order zero. If $J_\nu$ and $Y_\nu$ denote, respectively, the Bessel functions of the first and second kind of order $\nu$, the general solution (see, for example, Bowman, 1958; Watson, 1939) is given by $y(t) = c_1 J_\nu(t) + c_2 Y_\nu(t)$, which is valid for all finite $t$ excepting $Y_\nu(t)$ as $t \to 0$. This function then ceases to be a solution, and we are left with $c_1 J_\nu(t)$. Since $J_\nu(0) = 1$, the only $(C^2)$ solution vanishing at $t = 0$ is $y_0 \equiv 0$.

This fact may look disappointing for our problem. Since there are no non-trivial solutions vanishing at the initial endpoint it implies, apparently, that we cannot make use of Jacobi’s equation to find out if $H$ is empty or not. However, we can certainly consider non-zero piecewise $C^1$ solutions to the equation.

With this in mind, let us proceed as follows. Let $\eta \approx 2.4048$ be the first zero of $J_0(t)$ ($t > 0$) and let $\zeta_0 \approx 0.8935$ and $\zeta_1 \approx 3.9576$ be the first two zeros of $Y_0(t)$ ($t > 0$). Suppose first that $a > \eta$. Fix a point $\eta < s < \min\{a, \zeta_1\}$ and define

$$F(t) := J_0(s)Y_0(t) - J_0(t)Y_0(s) \quad \text{for all } t \in (0, \zeta_0).$$
Note that \( J_0(\xi_0) > 0 \) and, since \( s \in (\eta, \xi_1) \), \( J_0(s) < 0 < Y_0(s) \). Thus
\[
F(\xi_0) = -J_0(\xi_0)Y_0(s) < 0 \quad \text{and} \quad \lim_{t \to 0} F(t) = +\infty.
\]
Since \( F \) is continuous, there exists \( \epsilon \in (0, \xi_0) \) such that \( F(\epsilon) = 0 \). Let
\[
c := \frac{J_0(\epsilon)}{Y_0(\epsilon)} = \frac{J_0(s)}{Y_0(s)} \quad \text{and} \quad y(t) := J_0(t) - cY_0(t) \quad (t \in [\epsilon, s])
\]
and observe that \( y(\epsilon) = y(s) = 0 \) and \( y \) satisfies Jacobi’s equation. Extending \( y \) so that \( y(t) := 0 \) for all \( t \in [0, \epsilon] \), it follows immediately that \( s \in S(x) \) and, consequently, \( (\eta, a) \subset S(x) \) for any \( x \in X \).

The piecewise \( C^1 \) function \( y \) defined above can also be used to show that \( s \in C_1(x) \). To prove it, note first that \( v(t) = t\dot{y}(t) \) and \( w(t) = -t\dot{y}(t) \). Let \( q(t) := v(t) \) for all \( t \in T \). Since \( y \in Y \), \( y \) satisfies Jacobi’s equation \( v(t) = w(t) \), and \( q(s) \neq 0 \), the first three conditions defining membership of \( C_1(x) \) are satisfied. If \( a \) is any function belonging to \( Y \) for which \( u(s)q(s) < 0 \), condition 3.1(b) holds and the result follows. This implies that \( (\eta, a) \subset C_1(x) \) (the final endpoint is not included, contrary to what happens with \( S(x) \), since \( x \) would require \( u \in Y \) and \( u(a)q(a) \neq 0 \)). By Theorems 3.4 and 3.6, and observing that the non-emptiness of \( C_2(x) \) and \( R(x) \) implies that \( a \) belongs to them, we have \( (\eta, a) \subset C_2(x) \subset R(x) \).

Suppose now that \( a \leq \eta \). Let \( \epsilon \in (0, \xi_0) \) and consider the problem of minimizing
\[
I_\epsilon(x) = \frac{1}{2} \int_\epsilon^a t[\dot{x}^2(t) - x^2(t)]dt
\]
subject to \( x(\epsilon) = x(a) = 0 \). With arguments similar to those used above, one can show that the first (usual) conjugate point to \( \epsilon \), with respect to any trajectory \( x \), is greater than \( \eta \).

Hence, there are no conjugate points to \( \epsilon \in (\epsilon, a] \) and the non-singularity of the extremal \( x_0 \) (of class \( C^1 \)) implies that \( I''_\epsilon(x_0; y) > 0 \) for any non-zero variation \( y : [\epsilon, a] \to \mathbb{R} \) satisfying \( y(\epsilon) = y(a) = 0 \). By continuity considerations, \( I''(x_0; y) \geq 0 \) for any \( y \in Y \), that is, \( x_0 \in H \), which is equivalent to \( S(x_0) = \emptyset \). Since \( S(x) = S(x_0) \) for any \( x \in X \), the three sets \( C_1(x) \), \( C_2(x) \) and \( R(x) \) are also empty.

Summarizing, we have shown that \( (\eta, a) \subset C_1(x) \subset (\eta, a] \) and, moreover, \( C_2(x) \), \( R(x) \) and \( S(x) \) coincide and are equal to \( (\eta, a] \). Finally, with respect to the original problem, we conclude that, if \( a \leq \eta \), then \( x_0 \equiv 0 \) solves (P) and, if \( a > \eta \), the problem has no solution.

6. Examples

In this section we shall provide several examples for which determining the sets \( C_1(x) \) or \( C_2(x) \) may be extremely difficult, in some cases perhaps even a hopeless task, but one can trivially detect non-optimality of \( x \) by showing that \( S(x) \neq \emptyset \). All these examples are one-dimensional and the problem (P) of Section 2 is posed for an interval \( T = [t_0, t_1] \), \( A = T \times \mathbb{R} \times \mathbb{R} \), \( \xi_0 = \xi_1 = 0 \), and
\[
L(t, x, \dot{x}) = \frac{1}{2}[p(t)x^2 - r(t)x^2] \quad \text{for all} \ (t, x, \dot{x}) \in T \times \mathbb{R} \times \mathbb{R}.
\]
Note that, since $L_{x i}(t, x, \dot{x}) = p(t)$, $L_{x j}(t, x, \dot{x}) = -r(t)$ and $L_{x x}(t, x, \dot{x}) = 0$ for any $(t, x, \dot{x}) \in A$, the second variation $I''(x, \cdot)$ is independent of $x$. In particular, if $S(x) \neq \emptyset$ for some $x \in X, (A)$, the problem has no solution.

Recall that, if $y \in Y_\varepsilon \cap C^2$ satisfies either of the equations (A) or (B) of Section 4, then $K(y) = 0$. From the definitions of $C_1(x)$ and $C_2(x)$ it is clear that, except in one case, there are no criteria for finding a function $y \in Y_\varepsilon$ which can be used to show that these sets contain a point $s \in (t_0, t_1]$. The only exception is when $y \in Y_\varepsilon$, with $y \neq 0$, satisfies Jacobi’s equation, that is, equation (A). This fact is illustrated with great detail in the example given in Section 5, where the general solution to (A) is well-known. There are, however, simple examples for which finding a variation satisfying (A) may be extremely cumbersome, but not for (B). Some of the examples that follow are based on this fact.

**Example 6.1** Let $p(t) = r(t) = at + b$, where $a \neq 0$, $b \in R$ and $t_0 = -b/a$. In this case, equation (A) corresponds to

$$(at + b)\ddot{y}(t) + a\dot{y}(t) + (at + b)y(t) = 0$$

while (B) is $\dot{y}(t) + y(t) = 0$. Setting $y(t) = \sin(t + b/a)$ ($t \geq t_0$), we have $y \in Y \varepsilon$ with $s = \pi - b/a$, and $y$ satisfies (B). Since $y$ does not satisfy (A), $s \in S(x)$. Hence, if $t_1 \geq \pi - b/a$, the problem has no solution.

We have thus shown in a trivial way, by applying the set introduced in this paper, that the problem has no solution for certain upper bound on the time interval. We do not claim that $s = \pi - b/a$ is the best bound (that is, the infimum of $S(x)$). In fact, if $a = 1$ and $b = 0$, we know from Section 5 that, if $\eta \approx 2.4048$ is the first zero of $J_0(t)$ ($t > 0$) and $t_1 > \eta$, then the problem has no solution. However, the usefulness and simplicity of $S(x)$, for the more general case we are considering, are evident.

On the other hand, to prove directly that $s = \pi - b/a$ belongs to $C_1(x)$ or $C_2(x)$, one could of course try by testing simple functions like the one defined above, or functions with a piecewise (non-zero) constant derivative. However, as one readily verifies, they yield a contradiction in the definitions of the two sets. One might be successful with other functions, but the only objective criterion seems to be that of solving (A). For this example, however, its general solution is rather complicated. To give some idea, referring to Polyanin & Zaitsev (1995, equation 2.1.2.103), the solution to (A) is given by

$$y(t) = e^{\alpha t}J(1/2, 1; -2i(at + b)/a)$$

where $J(\alpha, \beta; t)$ is an arbitrary solution of the degenerate hypergeometric equation

$$t\ddot{y}(t) + (\beta - t)\dot{y}(t) - a\dot{y}(t) = 0$$

(see Polyanin & Zaitsev, 1995, equation 2.1.2.65 for its solution with $\alpha = 1/2$ and $\beta = 1$).

**Example 6.2** Let $p(t) = t$, $r(t) = at e^{\lambda t}$, where $\lambda \neq 0$, $a > 0$ and $t_0 = 0$. In this case,

$$t\ddot{y}(t) + \dot{y}(t) + a\lambda t e^{\lambda t}y(t) = 0 \quad \text{(A)}$$

$$\dot{y}(t) + a\lambda e^{\lambda t}y(t) = 0. \quad \text{(B)}$$
Equation (A) is not included in Polyanin & Zaitsev (1995), and it does not seem an easy task to find its general solution. However, the solution to (B) (see Polyanin & Zaitsev, 1995, equation 2.1.3.1) is given by

\[ y(t) = c_1 J_0(2\sqrt{a\lambda}^{-1}e^{s/2}) + c_2 Y_0(2\sqrt{a\lambda}^{-1}e^{s/2}). \]

One can then proceed as in Section 5 to determine a bound on the time interval for which \( S(x) \neq \emptyset \).

**Example 6.3** Let \( p(t) = t, r(t) = t(ae^t - b), \) where \( a > 0, b \geq 0 \) and \( t_0 = 0 \). In this case,

\[
\begin{align*}
t\ddot{y}(t) + \dot{y}(t) + t(ae^t - b)y(t) &= 0 \\
\dot{y}(t) + (ae^t - b)y(t) &= 0.
\end{align*}
\]

As before, equation (A) is not included in Polyanin & Zaitsev (1995), but (B) has the solution (see Polyanin & Zaitsev, 1995, equation 2.1.3.2)

\[ y(t) = c_1 J_{2\sqrt{a\lambda}}(2\sqrt{ae^{s/2}}) + c_2 Y_{2\sqrt{a\lambda}}(2\sqrt{ae^{s/2}}). \]

**Example 6.4** Let \( p(t) = t^n, r(t) = t^n + n(n-1)t^{n-2}/2, \) where \( n \) is a positive integer and \( t_0 = 0 \). In this case, (A) is given by

\[ t^n \ddot{y}(t) + nt^{n-1}\dot{y}(t) + (t^n + n(n-1)t^{n-2}/2)y(t) = 0 \]

whose solution is not included in Polyanin & Zaitsev (1995). On the other hand, (B) corresponds to \( \dot{y}(t) + y(t) = 0 \) and so, if \( t_1 \geq \pi \), then \( S(x) \neq \emptyset \).

**Example 6.5** Let \( p(t) = t^n, r(t) = b, \) where \( n \) is a positive integer, \( b > 0 \) and \( t_0 = 0 \). Let \( s > 0 \) and set

\[ y(t) := \begin{cases} 
\int_0^t & t \in [0, s/2] \\
\int_{s/2}^t & t \in [s/2, s].
\end{cases} \]

We have

\[ K(y) = \int_0^{s/2} (t^n - bt^2)dt + \int_{s/2}^s (t^n - b(s - t)^2)dt \]

\[ = \int_0^{s/2} (t^n - bt^2)dt + 2bs \int_{s/2}^s tdt - bs^2 \int_{s/2}^s dt \]

\[ = \frac{s^{n+1}}{n+1} - \frac{bs^3}{3} + \frac{bs^3}{4} \]

\[ = \frac{12s^{n+1} - (n+1)bs^3}{12(n+1)} \]

and therefore \( K(y) < 0 \iff b > 12s^{n-2}/(n+1) \). In particular, if \( n = 2 \), the inequality corresponds to \( b > 4 \) (independently of \( s \)) and, if \( n \geq 3 \) and \( t_1 > t_0 \) is arbitrary, one can
always find $0 < s < t_1$ for which the integral is negative, implying that $s \in S(x)$. On the other hand, equation (A) corresponds to

$$t^n \ddot{y}(t) + nt^{n-1} \dot{y}(t) + by(t) = 0$$

whose general solution, for $n$ arbitrary, may be extremely cumbersome to find (it is not included in Polyanin & Zaitsev, 1995).

**EXAMPLE 6.6** Let $p(t) = \sin^2 t$, $r(t) = \cos^2 t$ and $t_0 = 0$. We have

$$\ddot{y}(t) + 2\dot{y}(t) \cot t + y(t) \cot^2 t = 0$$

(A)

whose solution, given in Polyanin & Zaitsev (1995, equation 2.1.6.55), is expressed in terms of the Gauss hypergeometric equation, and we prefer not even to copy it. Equation (B) is $\ddot{y}(t) + y(t) = 0$ and so, setting $y(t) = \sin t$, it follows that $\pi \in S(x)$ if $t_1 \geq \pi$.

Just as in the previous examples, the function $y \in Y$, used to show that $S(x) \neq \emptyset$, cannot be used in trying to prove non-emptiness of $\mathcal{C}_1(x)$ or $\mathcal{C}_2(x)$. We have omitted the proof before but, to illustrate this fact, let us prove it in this particular case. Note first that

$$v(t) = p(t)\dot{y}(t) = \sin^2 t \cos t \quad \text{and} \quad w(t) = -r(t)y(t) = -\cos^2 t \sin t.$$  

We are interested in seeing if there exists $c \in \mathbb{R}$ such that $\dot{y}(t)(q(t) - v(t)) \geq 0$ for all $t \in [0, \pi]$, where

$$q(t) = c + \int_0^t w(\tau)d\tau = c - \int_0^t \cos^2 \tau \sin \tau d\tau = c - \frac{1 - \cos^3 t}{3}.$$  

The above inequality holds if and only if $(\cos t)(c - z(t)) \geq 0$ for all $t \in [0, \pi]$, where

$$z(t) = \sin^2 t \cos t + \frac{1 - \cos^3 t}{3}.$$  

Thus, necessarily, $c \geq z(t)$ for all $t \in [0, \pi/2]$ and $c \leq z(t)$ for all $t \in (\pi/2, \pi]$. But, as one readily verifies, no such constant exists (simply note that, if $t = \pi/4$, then $c > 1/3$ and, if $t = 3\pi/4$, then $c < 1/3$).

**EXAMPLE 6.7** Let $t_0 = 0$, $t_1 = 1$, $n \geq 3$ an integer, $r(t) = t^{n-2}$ and

$$p(t) := \begin{cases} t^n & t \in [0, 1/2] \\ t^{n-2}(1-t)^2 & t \in [1/2, 1] \end{cases}$$

Observe that, for this example, $p(t) = L_{\dot{x}}(t, \dot{x})$ is continuous on the whole interval $[0, 1]$, but its derivative is discontinuous at $t = 1/2$. Let $m \in \mathbb{R}$ ($m \neq 0$) and set

$$y(t) := \begin{cases} mt & t \in [0, 1/2] \\ m(1-t) & t \in [1/2, 1] \end{cases}$$

Clearly, $I''(x; y) = 0$. Since Jacobi’s equation is given, in the interval $[0, 1/2)$, by

$$t^n \ddot{y}(t) + nt^{n-1} \dot{y}(t) + t^{n-2}y(t) = 0,$$

the above function shows that $t_1 = 1$ belongs to $S(x)$. As before, one readily verifies that the use of this function yields a contradiction in the definitions of $\mathcal{C}_1(x)$ and $\mathcal{C}_2(x)$.
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