THE SOLUTION TO THE CONJECTURE ON PROPERNESS OF WEAKLY RELAXED DELAYED CONTROLS

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Abstract. In 1986 Warga proposed a “weak” relaxation procedure applicable to fully nonlinear problems with delays in the control variables and showed that the resulting relaxed problem has a solution. However, in the event of commensurate delays, several examples were found for which weakly relaxed controls cannot be approximated with original controls, so that this extension fails to be “proper.” Although the case of commensurate delays was solved by the introduction of a “strong” model, the question of how to properly relax noncommensurately delayed controls has remained open, and a natural candidate has been precisely that of weakly relaxed controls. In this paper, a general counterexample is constructed which rules out the weak relaxation as a proper relaxation when there are two or more delays. It is hoped that this result will provide some insight into the problem of finding a general representation of properly relaxed controls.

Key words. optimal control problems, systems with time delays, proper relaxation procedures

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1. Introduction. This paper concerns the problem of finding a proper relaxation procedure for optimal control problems with nonadditively coupled delays (or, more generally, shifts) in the control variables. Usually in relaxation theory the aim is to find a relaxation procedure for a given original problem which leads us to its proper extension. This means that the set of controls for the original problem is dense in the space of controls for the relaxed problem. Such procedures are well studied for delay free problems. For problems with delays the situation is more difficult and less understood. For example, there are natural relaxation procedures which give a proper extension while there are other natural relaxation procedures which do not (see [2, 8]).

Research in this area starts in [11]. There are two basic problems:

1. the existence of a proper relaxation, and
2. the representation of the set of relaxed controls when it exists.

Problem 1 has been affirmatively answered in [11] when there is only one constant delay, in [8] when the constant delays are commensurate, in [13] for arbitrary constant delays, and in [12] for certain variable delays.

This paper deals with problem 2. Specifically, a general counterexample is constructed which rules out the weak relaxation proposed in [11] as a proper relaxation when there are two or more delays. This is a significant step forward in this direction which finally clarifies questions raised in [1, 2, 3, 4, 5, 6, 7, 8, 11]. In order to understand this contribution, let us briefly state the problem we shall be concerned with together with some basic notation and previous results.

For \( T \subset \mathbb{R} \) compact and \( R \) a compact metric space, denote by \( \mathcal{M}(T, R) \) the space of measurable functions mapping \( T \) to \( rpm(R) \) with the weak star topology of \( L^1(T, C(R))^* \), where \( rpm(R) \) is the space of Radon probability measures on \( R \) with

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the weak star topology of $C(R)^*$. Let $U(T, R)$ be the space of measurable functions mapping $T$ to $R$, embedded in $M(T, R)$ by identifying each $u \in U(T, R)$ with the function $t \mapsto \delta_{u(t)}$, where $\delta_u$ (also written as $\delta_a$) is the Dirac measure at $a$.

It is well known that $M(T, R)$ coincides with $\text{cl} \ U(T, R)$, the weak star closure of $U(T, R)$. For optimal control problems where $U(T, R)$ is the space of ordinary controls, under the usual assumptions on the data of the problem, existence of minimizers in the space $M(T, R)$ of relaxed controls can thus be assured, and they can be approximated with ordinary controls. This fact is summarized by saying that $M(T, R)$ provides a “proper” relaxation procedure for $U(T, R)$. For a full account of these ideas, together with a thorough study of relaxation and its importance in optimal control theory, we refer to Warga’s book [10].

For optimal control problems involving delays in the controls, several attempts have been made to find proper relaxation procedures. The space of ordinary delayed controls we shall consider (also studied in [1, 2, 3, 4, 5, 6, 7, 8]), which illustrates the main difficulties encountered when addressing relaxation questions, is given by

$$U(\theta_1, \ldots, \theta_k) = \{(u_0, u_1, \ldots, u_k) \in U(T, \Omega^{k+1}) \mid u_i(t) = u_{i-1}(t - \Delta_i)$$

almost everywhere (a.e.) in $T_i$ $(i = 1, \ldots, k)$,

where $T = [0, 1]$, $0 < \theta_1 < \cdots < \theta_k < 1$ are given real numbers, $\Omega$ is a given compact metric space, and $\theta_0 = 0$, $\Delta_i = \theta_i - \theta_{i-1}$, $T_i = [\Delta_i, 1]$ $(i = 1, \ldots, k)$.

Warga [11] proposed (in a more general setting) a natural extension of $U(\theta_1, \ldots, \theta_k)$, which we call the weak relaxation procedure, given by

$$M_w(\theta_1, \ldots, \theta_k) = \{\mu \in M(T, \Omega^{k+1}) \mid P_i \mu(t) = P_{i-1} \mu(t - \Delta_i) \text{ a.e. in } T_i \ (i = 1, \ldots, k)\},$$

where if, say, $\mu \in M(T, \Omega^n)$ for some $n \in N$ and $S \subset \{0, 1, \ldots, n-1\}$, then $P_S \mu(t)$ denotes the projection onto the $S$ coordinates of $\mu(t)$. Equivalently, $\mu \in M(T, \Omega^{k+1})$ is a weakly relaxed control if and only if

$$\int_{T_i} dt \int \varphi(t, r_i) \mu(t)(dr) = \int_{T_i} dt \int \varphi(t, r_{i-1}) \mu(t - \Delta_i)(dr)$$

for all $\varphi$ in $L^1(T, C(\Omega))$ and $i = 1, \ldots, k$, where $r = (r_0, \ldots, r_k)$.

One readily verifies that the set of weakly relaxed controls contains the set of ordinary controls and, regarding it as a subspace of $M(T, \Omega^{k+1})$ with the weak star topology, it is compact. In [11] the question of properness of this model (that is, if the equality $M_w(\theta_1, \ldots, \theta_k) = \text{cl} \ U(\theta_1, \ldots, \theta_k)$ holds) was posed but could not be proved. For the one delay case, it is shown in [3, 8] that this model is indeed a proper relaxation procedure.

In [8] Rosenblueth and Vinter introduced an abstract relaxation procedure, which we call the $D$-model, and properness of this procedure was established by Warga and Zhu [13]. However, as we point out in [9], determining the set of $D$-relaxed controls for specific problems is a very difficult and perhaps even a hopeless task, so there is a need to find more concrete characterizations of the closure of the space of ordinary delayed controls.

Now, the conjecture mentioned in our title, considered in [1, 2, 3, 4, 5, 6, 7, 8, 11], is that

for any $\Omega \subset R^m$ compact and $0 < \theta_1 < \theta_2 < 1$, $M_w(\theta_1, \theta_2) = \text{cl} \ U(\theta_1, \theta_2)$.

This statement is false. In [8], Rosenblueth and Vinter exhibit an element of $M_w(\theta_1, \theta_2)$ lying outside $\text{cl} \ U(\theta_1, \theta_2)$ for the case when $\theta_2 = 2\theta_1$ and $\Omega = [0, 1]$.353
Rosenblueth [4] extends the inequality $\mathcal{M}_w(\theta_1, \theta_2) \neq \text{cl} \mathcal{U}(\theta_1, \theta_2)$ to pairs different than $(\theta_1, 2\theta_1)$ and whose quotient is a rational number (this settles a question raised by Andrews [1]). In the event of commensurate delays, a “strong” relaxation procedure was introduced in [8] and shown to be proper (see also [4, 9]). For the noncommensurate case, the problem of how to characterize $D$-relaxed controls has remained unsolved, but a natural candidate has been precisely the space of weakly relaxed controls (see [1, 5, 6]). In particular, it is shown in [6] that, if $\Omega = \{0, 1\}$ and $\theta_1/\theta_2$ is irrational, then any constant element of $\mathcal{M}_w(\theta_1, \theta_2)$ can be approximated with elements of $\mathcal{U}(\theta_1, \theta_2)$. The present paper finally solves this question. We show that inequality also holds for noncommensurate $\theta_1, \theta_2$:

For any $0 < \theta_1 < \theta_2 < 1$, there exists $\mu \in \mathcal{M}_w(\theta_1, \theta_2)$ such that $\mu \notin \text{cl} \mathcal{U}(\theta_1, \theta_2)$.

It is hoped that the counterexample used to solve this question will provide some insight into the problem of finding a general representation of properly relaxed controls.

2. The solution to the conjecture. In the following theorem we assume that $\Omega = [0, 1]$. Essentially the same arguments apply if $\Omega \subset \mathbb{R}$ is any compact set containing at least two points.

Theorem 2.1. For any $0 < \theta_1 < \theta_2 < 1$ there exists $\mu \in \mathcal{M}_w(\theta_1, \theta_2)$ such that $\mu \notin \text{cl} \mathcal{U}(\theta_1, \theta_2)$.

Proof. Let $0 < \theta_1 < \theta_2 < 1$ and set $\alpha := \theta_2 - \theta_1$. For all $(u, v, w) \in \Omega^3$ let

$$h(u, v, w) := \min\{(u, v - 1, w - 1), (u - 1, v, w)\},$$

$$g(u, v, w) := \min\{(u, v - 1, w - 1), (u - 1, v - 1, w)\},$$

and, for any $t \in T, x_0, x_1 \in \mathbb{R}$, and $(u, v, w) \in \Omega^3$, let

$$f(t, x_0, x_1, u, v, w) := \begin{cases} (x_0 - t/2)^2 + h(u, v, w) & \text{if } t \in [0, \theta_2), \\ (x_0 - t/2)^2 + g(u, v, w) & \text{if } t \in [\theta_2, 1]. \end{cases}$$

Consider the problem (P) of minimizing $x_1(1)$ subject to

$$\begin{cases} (\dot{x}_0(t), \dot{x}_1(t)) = (u(t), f(t, x_0(t), x_1(t), u(t), v(t), w(t))) \text{ a.e. in } T, \\ (x_0(0), x_1(0)) = (0, 0), \\ (u, v, w) \in \mathcal{U}(\theta_1, \theta_2). \end{cases}$$

Let $\mu \in \mathcal{M}(T, \Omega^3)$ be given by

$$\mu(t) = \begin{cases} \frac{1}{2}\delta(0, 1, 1) + \frac{1}{2}\delta(1, 0, 0) & \text{if } t \in [0, \theta_2), \\ \frac{1}{2}\delta(0, 0, 1) + \frac{1}{2}\delta(1, 1, 0) & \text{if } t \in [\theta_2, 1]. \end{cases}$$

Since

$$\mathcal{P}_i \mu(t) = \frac{1}{2}\delta_i + \frac{1}{2}\delta_i' \text{ for all } t \in T \text{ and } i = 0, 1, 2,$$

we have $\mu \in \mathcal{M}_w(\theta_1, \theta_2)$. Note that its corresponding cost is zero and, since the cost cannot be negative, the minimum of the problem posed on $\mathcal{M}_w(\theta_1, \theta_2)$ is zero.

Let $0 < a < \min\{\alpha, \theta_1, 1 - \theta_2\}$ so that the intervals $I := [0, a)$, $I + \alpha$, and $I + \theta_2$ are disjoint and belong to $[0, 1)$. Let $(x_0, x_1, u, v, w)$ be any admissible original process for (P), so that $(u, v, w) \in \mathcal{U}(T, \Omega^3)$ and

\begin{align*}
&(2.1) \quad v(t) = u(t - \theta_1) \text{ a.e. in } [\theta_1, 1], \\
&(2.2) \quad w(t) = v(t - \alpha) \text{ a.e. in } [\alpha, 1],
\end{align*}
and observe that the following relations hold a.e. in $I$:

$$w(t + \alpha) = v(t), \quad v(t + \theta_2) = u(t + \alpha), \quad w(t + \theta_2) = u(t).$$

Indeed, by (2), $w(t + \alpha) = v(t)$ a.e. in $[0, 1 - \alpha]$ and, since $\alpha < \theta_1 < 1 - \theta_2 + \theta_1 = 1 - \alpha$, we have $I \subset [0, 1 - \alpha]$. By (1), $v(t + \theta_2) = u(t + \alpha)$ a.e. in $[-\alpha, 1 - \theta_2] \supset I$. Finally, by (2), $w(t + \theta_2) = v(t + \theta_1)$ a.e. in $[-\theta_1, 1 - \theta_2]$ and, by (1), $v(t + \theta_1) = u(t)$ a.e. in $[0, 1 - \theta_1]$. Hence $w(t + \theta_2) = u(t)$ a.e. in $[0, 1 - \theta_2] \supset I$ and the three relations hold a.e. in $I$.

Therefore,

$$x_1(1) = \int_0^{\theta_2} \{(x_0(s) - s/2)^2 + h(u(t), v(t), w(t))\} dt$$

$$+ \int_0^1 \{(x_0(s) - s/2)^2 + g(u(t), v(t), w(t))\} dt$$

$$\geq \int_0^{\alpha} h(u(t), v(t), w(t)) dt + \int_0^{\alpha + \theta_2} h(u(t), v(t), w(t)) dt$$

$$+ \int_0^{\theta_2 + \alpha} g(u(t), v(t), w(t)) dt$$

$$= \int_0^{\alpha} \{h(u(t), v(t), w(t)) + h(u(t + \alpha), v(t + \alpha), v(t)) + g(u(t + \theta_2), u(t + \alpha), u(t))\} dt.$$

Fix $t \in I$ and let $r_0 = u(t)$, $r_1 = v(t)$, $r_2 = u(t + \alpha)$, $s_0 = w(t)$, $s_1 = v(t + \alpha)$, and $s_2 = u(t + \theta_2)$. Define

$$\varphi(t) := h(r_0, r_1, s_0) + h(r_2, s_1, r_1) + g(s_2, r_2, r_0),$$

and let

$$m_0 := \begin{cases} 1 & \text{if } h(r_0, r_1, s_0) = |(r_0, r_1 - 1, s_0 - 1)|, \\ 0 & \text{if } h(r_0, r_1, s_0) = |(r_0 - 1, r_1, s_0)|, \end{cases}$$

$$m_1 := \begin{cases} 1 & \text{if } h(r_2, s_1, r_1) = |(r_2, s_1 - 1, r_1 - 1)|, \\ 0 & \text{if } h(r_2, s_1, r_1) = |(r_2 - 1, s_1, r_1)|, \end{cases}$$

$$m_2 := \begin{cases} 1 & \text{if } g(s_2, r_2, r_0) = |(s_2, r_2, r_0 - 1)|, \\ 0 & \text{if } g(s_2, r_2, r_0) = |(s_2 - 1, r_2 - 1, r_0)|. \end{cases}$$

Observe now that

$$m_0 \neq m_1 \Rightarrow \varphi(t) \geq |1 - r_1| + |r_1|,$$

$$m_1 \neq m_2 \Rightarrow \varphi(t) \geq |1 - r_2| + |r_2|,$$

$$m_0 = m_2 \Rightarrow \varphi(t) \geq |1 - r_0| + |r_0|.$$

On the other hand, if $m_0 = m_1$ and $m_1 = m_2$, then $m_0 = m_2$ and so, in all cases, $\varphi(t) \geq 1$. It follows that

$$x_1(1) \geq \int_0^{\alpha} \varphi(t) dt \geq a > 0$$

and so the infimum of (P) posed over the original admissible processes is positive. This implies that $\mu$ cannot be approximated with elements of $\mathcal{U}(\theta_1, \theta_2)$. \qed
3. Extensions to other delay-relaxation problems. The term “weak relaxation procedure” was first introduced in [8] referring to the model proposed by Warga in [11]. To be exact, the latter is slightly different from the one studied in [8] and mentioned in section 1. The subtle difference, which we shall explain below, leads to the study of another delay-relaxation problem for which a proof similar to the one of Theorem 2.1 can be applied.

Consider the following optimal control problem involving constant delays in the control variables. Let \( T := [0, 1] \) and suppose we are given real numbers \( 0 < \theta_1 < \cdots < \theta_k < 1 \), a point \( \xi \in \mathbb{R}^n \), a compact set \( \Omega \subset \mathbb{R}^m \), and functions \( g \) mapping \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( f \) mapping \( T \times \mathbb{R}^n \times \mathbb{R}^{m(k+1)} \) to \( \mathbb{R}^n \). Let \( \hat{T} := [-\theta_k, 1] \) and consider the problem (P) of minimizing \( g(x(1)) \) subject to

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t), u(t - \theta_1), \ldots, u(t - \theta_k)) \text{ a.e. in } T, \\
x(0) &= \xi, \\
u(t) &\in \Omega \text{ a.e. in } \hat{T},
\end{aligned}
\]

where \( u \) is any measurable function mapping \( \hat{T} \) to \( \mathbb{R}^m \).

In [11] Warga reformulated this “original control problem” (P) by treating the control functions as independent variables which satisfy certain compatibility conditions in terms of the delays. The model of relaxation proposed by Warga was obtained by generalizing these conditions in the corresponding space of relaxed controls. To be specific, let

\[
\mathcal{W}(\theta_1, \ldots, \theta_k) := \{(u_0, u_1, \ldots, u_k) \in U(\hat{T}, \Omega^{k+1}) | u_i(t) = u_0(t - \theta_i) \text{ a.e. in } T (i = 1, \ldots, k)\}
\]

and consider the problem (W) of minimizing \( g(x(1)) \) subject to

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)) \text{ a.e. in } T, \\
x(0) &= \xi, \\
u &\in \mathcal{W}(\theta_1, \ldots, \theta_k).
\end{aligned}
\]

As Warga mentions in [11], it is a simple fact to show that (P) and (W) are equivalent. The “weak” extension of \( \mathcal{W}(\theta_1, \ldots, \theta_k) \) proposed by Warga is given by

\[
\mathcal{S}_w(\theta_1, \ldots, \theta_k) := \{\mu \in M(\hat{T}, \Omega^{k+1}) | P_i \mu(t) = P_{i-1} \mu(t - \Delta_i) \text{ a.e. in } T (i = 1, \ldots, k)\}.
\]

In [8] Rosenblueth and Vinter considered the problem (see the notation of section 1), which we label (RV), of minimizing \( g(x(1)) \) subject to

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)) \text{ a.e. in } T, \\
x(0) &= \xi, \\
u &\in U(\theta_1, \ldots, \theta_k).
\end{aligned}
\]

It should be noted that, in this reformulation of the problem, ordinary controls are measurable functions defined on the interval \( T = [0, 1] \) and not on \( \hat{T} = [-\theta_k, 1] \) as in the reformulation (W) of (P) given in [11]. As before, (P) and (RV) are equivalent (see [7] for details). The notion of “weakly relaxed controls” applied to (RV) yields the set

\[
\mathcal{M}_w(\theta_1, \ldots, \theta_k) = \{\mu \in \mathcal{M}(T, \Omega^{k+1}) | P_i \mu(t) = P_{i-1} \mu(t - \Delta_i) \text{ a.e. in } T (i = 1, \ldots, k)\}
\]
and, since the three problems are equivalent, so is the question of properness of the two models of relaxed controls.

Now, in [5, 6] we studied a similar but larger space of ordinary controls to which the notion of weakly relaxed controls can also be applied. A different class of optimal control problems is derived from these spaces, and the question of properness of the weak model with respect to the larger class of ordinary controls can be posed. Consider the space of original controls

\[ M = \{(u_0, u_1, \ldots, u_k) \in U(T, \Omega^{k+1}) \mid u_i(t) = u_0(t - \theta_i) \text{ a.e. in } [\theta_i, 1] \ (i = 1, \ldots, k)\}. \]

The notion of weakly relaxed controls applied to problem (R) of minimizing \( g(x(1)) \) subject to

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)) \text{ a.e. in } T, \\
x(0) &= \xi, \\
u &\in U'(\theta_1, \ldots, \theta_k)
\end{align*}
\]

corresponds to

\[ M'_w(\theta_1, \ldots, \theta_k) := \{\mu \in M(T, \Omega^{k+1}) \mid P_i \mu(t) = P_0 \mu(t - \theta_i) \text{ a.e. in } [\theta_i, 1] \ (i = 1, \ldots, k)\}, \]

which we shall call the space of \( \mathcal{R} \)-weakly relaxed controls, and the open question has been, again, if the relation \( M'_w(\theta_1, \ldots, \theta_k) = \text{cl } U'(\theta_1, \ldots, \theta_k) \) holds.

Note that (R) is similar to (P) but not equivalent. The definition of \( U'(\theta_1, \ldots, \theta_k) \) as the space of ordinary delayed controls does not correspond to a reformulation of (P) and, as one can easily show,

\[ U'(\theta_1, \ldots, \theta_k) = \{(u_0, u_1, \ldots, u_k) \in U(T, \Omega^{k+1}) \mid u_i(t) = u_{i-1}(t - \Delta_i) \text{ a.e. in } [\theta_i, 1] \ (i = 1, \ldots, k)\}. \]

Comparing with the definition of the set \( U(\theta_1, \ldots, \theta_k) \), it is clear that \( U(\theta_1, \ldots, \theta_k) \subset U'(\theta_1, \ldots, \theta_k) \), but the two sets may not coincide.

In [7] we proved an important consequence of this fact by exhibiting an element of both \( M'_w(\theta_1, \theta_2) \) and \( M'_w(\theta_1, \theta_2) \) which belongs to the weak star closure of \( U'(\theta_1, \theta_2) \) but not to that of \( U(\theta_1, \theta_2) \). Also in [7] we showed that, for certain delays, the space of \( \mathcal{R} \)-weakly relaxed controls does provide a proper relaxation procedure. The result proved in [7] states that, given \( 0 < \theta_1 < \theta_2 < 1 \) and \( \Omega \subset \mathbb{R}^m \) compact, if \( \theta_1 \geq 1/2 \), then \( M'_w(\theta_1, \theta_2) = \text{cl } U'(\theta_1, \theta_2) \) and, if \( \theta_1 < 1/2 \) and \( \theta_1 \) and \( \theta_2 \) are commensurate, then \( \text{cl } U'(\theta_1, \theta_2) \) may be strictly contained in \( M'_w(\theta_1, \theta_2) \).

A new result for this model can now be obtained with a proof similar to the one of Theorem 2.1. The foregoing arguments can be applied to the conjecture stated in terms of \( M'_w(\theta_1, \theta_2) \) and \( U'(\theta_1, \theta_2) \). If \( \theta_1 + \theta_2 < 1 \), with a problem similar to the previous one, it is not difficult to see that

\[
\mu(t) = \begin{cases} 
\frac{1}{2} \delta(0, 1, 1) + \frac{1}{2} \delta(1, 0, 0) & \text{if } t \in [0, \theta_1 + \theta_2), \\
\frac{1}{2} \delta(0, 0, 1) + \frac{1}{2} \delta(1, 1, 0) & \text{if } t \in [\theta_1 + \theta_2, 1]
\end{cases}
\]

belongs to \( M'_w(\theta_1, \theta_2) \) but not to \( \text{cl } U'(\theta_1, \theta_2) \). We state this result.

**Theorem 3.1.** For any \( 0 < \theta_1 < \theta_2 < 1 \) with \( \theta_1 + \theta_2 < 1 \) there exists \( \mu \in M'_w(\theta_1, \theta_2) \) such that \( \mu \notin \text{cl } U'(\theta_1, \theta_2) \).
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