Preserving continuity by Zadeh extension and invariance of separation axioms under surjections

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Abstract

This note provides conditions under which Zadeh extension of a map between \(L\)-topological spaces preserves semicontinuity and continuity of functions with values in the \(L\)-unit interval. Invariance of complete \(L\)-regularity under continuous open–closed surjections is established among others.

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1. Introduction

A property of \(L\)-topological spaces is said to be invariant under surjective maps of certain type if, whenever the source space has that property, the target space has that property too. Normality type axioms (i.e., \(L\)-normality, complete \(L\)-normality and perfect \(L\)-normality) are all preserved under closed surjections if \(L = 2\), while regularity type axioms (i.e., \(L\)-regularity and complete \(L\)-regularity) are invariant under open–closed surjections if \(L = 2\) (see [3]).

Rodabaugh [13] has proved that \(L\)-normality and perfect \(L\)-normality are invariant under continuous closed surjections for an arbitrary complete lattice \(L\) with an order reversing involution (recall that all the higher \(L\)-topological separation axioms require that involution via definition).

This note shows among others that complete \(L\)-regularity is preserved by continuous open–closed surjections. The technique we are using is similar to that of general topology. Namely, Zadeh extension not only preserves (semi)continuity of real-valued maps but also of \(L\)-real-valued maps (see Section 4 for details).

Standing assumption. In this paper, \(L\) is a complete lattice. We write \((L,')\) if \(L\) has an order reversing involution \((\cdot)'\).

2. A remark on Zadeh extension principle

The Zadeh extension principle (ZEP) is of fundamental importance in fuzzy set theory.

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Let $M$ be a complete lattice and let $X$ be a set (we use $M$ just because latter on we will use two complete lattices at the same time: $L$ and $I(L)$). Then $M^X$ is the complete lattice of all maps from $X$ to $M$ under pointwise ordering. Each map $f: X \to Y$ induces two maps

$$M^X \xrightarrow{f^+_M} M^Y \xrightarrow{f^-_M} M^X$$

defined by

$$f^+_M(a)(y) = \bigvee_{x \in f^{-1}(y)} a(x) \quad \text{for all } y \in Y$$

(ZEP)

and

$$f^-_M(b) = b \circ f$$

(the composite of $f$ and $b$). The two maps were introduced by Zadeh [15] with $M = \mathbb{I}$ (the real unit interval). They are also called fuzzy powerset operators (for a detailed study of them we refer to Rodabaugh [14]).

**Historical remark.** It may be of interest to point out that the idea of ZEP has earlier been used by Frolík [4, Lemma 3.4] and has become of importance in topology (cf. [9,12,1–3]). Namely, let $f: X \to Y$ and let $a: X \to \mathbb{R}$ be bounded on all fibres of $f$. Let $a^*: Y \to \mathbb{R}$ be defined by (in notation of [3])

$$a^*(y) = \sup_{x \in f^{-1}(y)} a(x) \quad \text{for all } y \in Y.$$  

Clearly, $f^+_1(a) = a^*$ if $a$ is $1$-valued.

### 3. Preliminary $L$-topological notions

Let us set up basic terminology that we shall need in what follows.

**$L$-topologies.** A set $\tau \subseteq L^X$ is an $L$-topology, its elements are open $L$-sets, and $(X, \tau)$ is an $L$-topological space (usually denoted just by $X$) if $\tau$ is closed under arbitrary sups and finite infs (formed in $L^X$). A map $f: (X, \tau) \to (Y, \sigma)$ is continuous if $f^-_L(u) \in \tau$ for all $u \in \sigma$. With $(L, \cdot^L)$ we also have closed $L$-sets: $k \in L^X$ is closed iff $k^L$ is open (where $k^L(x) = k(x)^L$).

**$L$-unit interval.** Let $H_L$ be the complete lattice of all order reversing maps from $1$ to $L$ (under pointwise ordering). Define

$$\cdot^+, \cdot^-: H_L \to H_L$$

by

$$\lambda^+(t) = \bigvee \lambda(t, 1) \quad \text{and} \quad \lambda^-(t) = \bigwedge \lambda[0, t].$$

Then $(\cdot)^+$ preserves arbitrary sups and $(\cdot)^-$ preserves arbitrary infs. Both are idempotent. The equivalence relation

$$\lambda \sim \mu \iff \lambda^+ = \mu^+$$

is such that $\lambda_j \sim \mu_j (j \in J)$ implies $\bigvee_{j \in J} \lambda_j \sim \bigvee_{j \in J} \mu_j$ and the same for infs. Thus, the set

$$\mathbb{L}(L) = H_L / \sim = \{ [\lambda] : \lambda \in H_L \}$$

ordered by

$$[\lambda] \leq [\mu] \iff \lambda^+ \leq \mu^+$$

(iff $\lambda^- \leq \mu^-$ iff $\lambda^+ \leq \mu^-[10, 1.3.1]$) is a complete lattice in which $\bigvee_{j \in J} [\lambda_j] = [\bigvee_{j \in J} \lambda_j]$ and similarly for infs. It is called the $L$-unit interval [7].
Given \( t \in 1 \), let \( R_t \in L^{\mathbb{1}(L)} \) be defined by \( R_t[\lambda] = \lambda^+(t) \). Then
\[
\mathcal{R}_L = \{ R_t : t \in 1 \} \cup \{ 0, 1 \}_{\mathbb{1}(L)}
\]
is an \( L \)-topology on \( \mathbb{1}(L) \). With \( (L, ') \), we have another \( L \)-topology on \( \mathbb{1}(L) \), viz.
\[
\mathcal{L}_L = \{ L_t : t \in 1 \} \cup \{ 0, 1 \}_{\mathbb{1}(L)},
\]
where \( L_t[\lambda] = \lambda^-(t) \). The usual \( L \)-topology \( \mathcal{U}_L \) of \( \mathbb{1}(L) \) is the smallest \( L \)-topology which contains \( \mathcal{R}_L \cup \mathcal{L}_L \).

**Semicontinuous maps.** Let \((X, \tau)\) be an \( L \)-topological space and let \( a : X \to \mathbb{1}(L) \). Then \( a \) is called lower semicontinuous (upper semicontinuous) if it is continuous as a map from \((X, \tau)\) to \((\mathbb{1}(L), \mathcal{R}_L)\) [to \((\mathbb{1}(L), \mathcal{L}_L)\)]. If it is continuous as a map from \((X, \tau)\) to \((\mathbb{1}(L), \mathcal{U}_L)\), then it is simply called continuous. It is important to recall that for any complete lattice \((L, ')\), \( a \) is continuous if it is both lower and upper semicontinuous (see \([11, \text{Section 2}]\) for details).

### 4. Preservation of semicontinuity and continuity by Zadeh extension

Since \((\cdot)^+\) preserves arbitrary sups, if \( f : X \to Y, a : X \to \mathbb{1}(L) \) and \( t \in 1 \), then
\[
R_t \circ (f_{\mathbb{1}(L)}^+(a)) = f_{L^+}^-(R_t \circ a).
\]

The map \((\cdot)^-\) fails to preserve arbitrary sups, but we have the following:

**4.1. Lemma.** Let \( A \subseteq H_L \) and \( t \in 1 \). Then:
\[
\left( \bigvee A \right)^-(t) = \bigwedge_{s \leq t} \lambda^-(s).
\]

**Proof.** For each \( s < t \) in \( 1 \) we have
\[
\left( \bigvee A \right)^-(t) \leq \left( \bigvee A \right)^+(s) = \bigvee_{\lambda \in A} \lambda^+(s) \leq \bigvee_{\lambda \in A} \lambda^-(s),
\]
so that
\[
\left( \bigvee A \right)^-(t) \leq \bigwedge_{s \leq t} \lambda^-(s).
\]
Since \((\cdot)^-\) is order preserving, we have \( \bigvee_{\lambda \in A} \lambda^- \leq (\bigvee A)^- \). Further,
\[
\bigwedge_{s \leq t} \lambda^-(s) \leq \bigwedge_{s \leq t} \left( \bigvee A \right)^-(s) = \left( \bigvee A \right)^- (t) = \left( \bigvee A \right)^-(t),
\]
which completes the proof. \(\square\)

**4.2. Fact.** Let \((L, ')\) be a complete lattice and let \( f : X \to Y \). For each \( a : X \to \mathbb{1}(L) \) and \( t \in 1 \) we have
\[
L_t' \circ (f_{\mathbb{1}(L)}^+(a)) = \bigwedge_{s \leq t} f_{L^+}^-(L_t' \circ a).
\]

**Proof.** Let \( y \in Y \). Let \( a_x \in H_L \) denote a representative of \( a(x) \), i.e., \( a(x) = [a_x] \). Then
\[
L_t'(f_{\mathbb{1}(L)}^+(a)(y)) = \left( \bigvee_{x \in f^{-1}(y)} a_x \right)^-(t)
\]
and
\[ \bigwedge_{s \leq t} f_L^{-1}(L'_s \circ a)(y) = \bigwedge_{s \leq t} \bigvee_{x \in f^{-1}[y]} a_x^{-}(s). \]

Now it is seen that (L) holds on account of Lemma 4.1. \(\square\)

Recall that a continuous \(f: X \to Y\) is called open (closed) if \(f^{-1}(w)\) is open (closed) in \(Y\) whenever \(w\) is open (closed) in \(X\). A map which is both open and closed is called open–closed. The following provides conditions under which \(f_{\|L\}^{-1}\) preserves (semi)continuities. We recall again that if \(f\) is open–closed and \(a\) is continuous, then \(f_{\|L\}^{-1}(a): Y \to \|L\) is a pointwise sup of continuous maps. By Proposition 5.1, \(f_{\|L\}^{-1}\) preserves arbitrary sups, we obtain

4.3. Proposition. Let \(X\) and \(Y\) be \(L\)-topological spaces, \(f: X \to Y\), and \(a: X \to \|L\). Then:

1. If \(f\) is open and \(a\) is lower semicontinuous, then \(f_{\|L\}^{-1}(a)\) is lower semicontinuous.

2. If \(f\) is closed and \(a\) is upper semicontinuous, then \(f_{\|L\}^{-1}(a)\) is upper semicontinuous.

3. If \(f\) is open–closed and \(a\) is continuous, then \(f_{\|L\}^{-1}(a)\) is continuous.

Proof. This is an immediate consequence of (R) and (L). \(\square\)

Note. When \(L = 2\), (1) and (2) of Proposition 4.3 are in Isiwata [9], while (3) is in Frolík [4].

5. Invariance of separation axioms under maps

The following characterization of complete \(L\)-regularity is proved in [10]:

5.1. Proposition. Let \((L',')\) be a complete lattice. An \(L\)-topological space \(X\) is completely \(L\)-regular if and only if each lower semicontinuous function from \(X\) to \(\|L\) is a pointwise sup of a family of continuous functions from \(X\) to \(\|L\).

The theorem which follows can be proved by involving the definition of complete \(L\)-regularity as formulated in [8] (cf. [1] where \(L = 2\)), but we prefer to use Proposition 5.1.

5.2. Theorem. Complete \(L\)-regularity is preserved under continuous open–closed surjections for each complete lattice \((L',')\).

Proof. Let \(f: X \to Y\) be a continuous open–closed surjection and let \(X\) be completely \(L\)-regular. Let \(a: Y \to \|L\) be lower semicontinuous. Then \(f_{\|L\}^{-1}(a) = a \circ f: X \to \|L\) is lower semicontinuous too. By Proposition 5.1, there is a family \(G\) of continuous maps from \(X\) to \(\|L\) such that \(f_{\|L}^{-1}(a) = \bigvee G\) (a pointwise sup). As \(f\) is onto and \(f_{\|L}^{-1}\) preserves arbitrary sups, we obtain

\[ a = f_{\|L}^{-1}(f_{\|L}^{-1}(a)) = \bigvee_{g \in G} f_{\|L}^{-1}(g). \]

By (3) of Proposition 4.3, \(a\) is a pointwise sup of continuous maps. By Proposition 5.1, \(Y\) is completely \(L\)-regular. \(\square\)

Recall that \((X, \tau)\) is \(L\)-regular (cf. [8]) iff each \(u \in \tau\) can be written as \(u = \bigvee \mathcal{V} = \bigvee \overline{\mathcal{V}}\) where \(\overline{\mathcal{V}} = \{v : v \in \mathcal{V}\}\) with \(\mathcal{V}\) a subset of \(\tau\) and \(\overline{\mathcal{V}}\) being the closure of \(v\) in \((X, \tau)\). One easily gets the following:

5.3. Proposition. \(L\)-regularity is preserved under continuous open–closed surjections for each complete lattice \((L',')\).

As has already been mentioned, \(L\)-normality and perfect \(L\)-normality are preserved by continuous closed surjections with \((L',')\) a complete lattice [13]. Recall that \(X\) is completely \(L\)-normal if, given \(a, b \in L^X\) with \(\overline{a} \leq b\) and \(a \leq \text{Int} b\),
there are open $u$ and closed $k$ such that $a \leq u \leq k \leq b$. Arguing exactly as in [13, proof of Theorem 4.2], one shows that:

5.4. Proposition. Complete $L$-normality is preserved under continuous open–closed surjections for each complete lattice $(L, \cdot)$.

None of the $L$-topological lower separation axioms of [10,5,6] is invariant under continuous open surjections as the case $L = 2$ shows (cf. [3, p. 70]). Clearly, the 2-topological $T_1$-axiom is preserved under closed surjections. Consequently, 2-Tychonoff spaces are preserved under continuous open–closed surjections. Unfortunately, we have not been able to prove the same for $L \neq 2$. Recall that $X$ is $L$-Tychonoff if it is completely $L$-regular and $L$-$T_0$ (meaning: open $L$-sets separate points). We note that $L$-$T_0$, $L$-$T_1$ of [10] and the two Hausdorff type axioms of [5,10] are all equivalent in the class of $L$-regular, hence $L$-completely regular spaces, for every complete $(L, \cdot)$ (cf. [10,5,6]).

Question. Prove or disprove that $L$-Tychonoff is invariant under continuous open–closed surjections.

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