A backstepping approach to control a nonholonomic mobile robot

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Abstract—In this paper a trajectory tracking control problem for a nonholonomic mobile robot by making use of a kinematic oscillator has been solved. Firstly - time varying oscillator is examined to control nonholonomic mobile robot based only its kinematics. Secondly - backstepping procedure is proposed to include robot dynamics and servo loop. It is shown that overall multilevel controller is asymptotically globally stable to a small error different from zero. A wide range of simulation results are presented which illustrate behaviour of the controller with respect to tuning its parameters. Some preliminary experimental results are presented too.

Keywords—mobile robot, time varying controller, Lyapunov function.

I. INTRODUCTION

There is a wide literature concerning control of nonholonomic mobile robots published in last ten years. The main difficulty in set point stabilization of mobile robots is due to the fact that these systems cannot be stabilized by continuous feedback which depends on the state. This is due to the Brockett’s theorem [2]. Therefore many authors proposed different control schemes to overcome this fundamental difficulty. Among many others propositions there are three main approaches which are proposed in a robotics literature. The first one is concentrated on using time-varying control scheme; see for example reference [8]. The second one is based on applying discontinuous control scheme, see for example [1]. Finally, the third approach, named hybrid control scheme, is a kind of combination of the above two approaches. In this paper we refer to works published recently [3], [4], [5]. Authors of these papers present a general control scheme which uses an idea of oscillator for the purpose of control a nonholonomic mobile robot. This proposition is general in a sense that it allows to solve both set point stabilization and trajectory tracking control. We found this approach very attractive due to the fact that it is straightforward to implement and leads to smooth control signals. As a consequence a backstepping procedure can be easily implemented. Based on the kinematic oscillator we propose to consider dynamics of the mobile robot. It is proven in the paper that there is no any initial knowledge concerning dynamic parameters and uncertainties required and the proposed algorithm is robust. Finally, the last backstepping step includes the servo loop.

It is shown that the whole system is globally asymptotically stable to a small nonzero error. We believe that this kind of algorithm was not published in robotics literature and is new. The paper is organized as follows. In section two kinematical consideration are outlined. The third section is devoted to dynamical and servo loop considerations. Section four contains the main algorithm and its proof. Simulation results are presented in the next section. Some preliminary experimental results are presented too. Concluding remarks end the paper.

II. KINEMATICS

Consider a 2-wheeled mobile robot shown in Fig. 1. Both wheels have the same radius denoted by r. Width of the vehicle is equal to 2R. As it is shown in Fig. 1 XY denotes a base coordinate system. Centre of the mass of the mobile robot is located in point C, which is an origin of a coordinate system X, Y, z assigned to the vehicle. Point P is located in intersection of straight line passing through the middle of the vehicle and a section which is an axis of two wheels. Distance between points P and C is denoted by d. Orientation angle is described by θ. Linear velocity of the vehicle at point C is denoted by v while angular velocity around axis passing through point C is ω. It is assumed that wheels roll on the plane XY without longitudinal and transversal slippage.

Formally, last statement in mathematical terms can be written as follows

$$A(q)q = 0,$$  \hspace{1cm} (1)

where

$$A(q) = \begin{bmatrix} -\sin \theta & \cos \theta & -d & 0 & 0 \\ -\cos \theta & -\sin \theta & -R & r & 0 \\ 0 & 0 & 2R & r & -r \end{bmatrix},$$  \hspace{1cm} (2)

where \(\dot{q} = [\dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}_r, \dot{\varphi}_l]^T\) is a vector of velocities; \(\dot{x}, \dot{y}\) are the velocities of the centre of mass C expressed in the base coordinate system, \(\dot{\theta}\) denotes an angular velocity of the vehicle and \(\dot{\varphi}_r, \dot{\varphi}_l\) are angular velocities of the right and left wheel, respectively. Constraints given by Eq. (1) imply that there exists matrix \(S(q)\), which is full rank and consists of linearly independent vector fields which are spanned on the null space of matrix \(A(q)\), namely

$$S(q) = \begin{bmatrix} c(R \cos \theta - d \sin \theta) & c(R \cos \theta + d \sin \theta) \\ c(R \sin \theta + d \cos \theta) & c(R \sin \theta - d \cos \theta) \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$  \hspace{1cm} (3)

where \(c = \frac{R}{2R}\). It is straightforward to verify that \(A(q)S(q) = 0\). Since the constrained velocity is always in the null space of \(A(q)\), it is possible to define \((n - m)\) velocities \(\nu = [\nu_1, \ldots, \nu_{n-m}]^T\) such that (these velocities need not be integrable)

$$\dot{q} = S(q)\nu,$$  \hspace{1cm} (4)

where \(n\) is the dimension of vector \(q\) and \(m\) is a total number of holonomic and nonholonomic constraints imposed on the system and \(\nu = [\dot{\varphi}_r, \dot{\varphi}_l]^T\). In our case \(n = 5\) and \(m = 3\) (two nonholonomic and one holonomic). It is easy to verify (from geometrical considerations concerning appropriate velocities) that there
exists the following relationship between linear and angular velocities of the vehicle and its wheels’ velocities
\[ \nu = \begin{bmatrix} \dot{\psi}_r \\ \dot{\psi}_l \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 & \frac{1}{2r} \\ -\frac{1}{2r} & \frac{1}{r} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}. \] (5)

A square matrix which appears in the last equation is nonsingular, therefore it is not of importance which velocities we use to describe the kinematics of the vehicle considered in this paper. Substitution of Eq. (5) into Eq. (4) results in
\[ \dot{q} = \begin{bmatrix} \cos \theta & -d \sin \theta \\ \sin \theta & d \cos \theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = S(q_1)\nu, \] (6)

Now we skip last two equations from Eq. (6) and rewrite it as follows
\[ \dot{q}_1 = \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -d \sin \theta \\ \sin \theta & d \cos \theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = S_1(q_1)\nu, \] (7)

where \(\dot{q}_1 = [\dot{x}_c, \dot{y}_c, \dot{\theta}]^T\). Notice that two set of equations (5) and (7) constitute Eq. (6). Matrix \(A_1(q_1)\) which satisfies equation \(A_1(q_1)S_1(q_1) = 0\) has the following form
\[ A_1(q_1) = \begin{bmatrix} -\sin \theta & \cos \theta & -d \\ \sin \theta & \cos \theta & 0 \end{bmatrix} \] (8)

and certainly \(A_1(q_1)\dot{q}_1 = 0\) results in
\[ -d_x \sin \theta + d_y \cos \theta - d \dot{\theta} = 0 \] (9)

which is a nonholonomic constraint stating that the vehicle cannot move in direction transversal to the axis of symmetry of the vehicle. Note that in case of Eq. (6) we still have the same number of constraints, one holonomic and two nonholonomic, while in case of Eq. (7) we are dealing with only one nonholonomic constraint.

III. DYNAMICS

In this section we describe dynamics of the vehicle presented in Fig. 1. In general the dynamical model of a mobile robot is given by
\[ \dot{M}(q)\ddot{q} + V_m(q, \ddot{q})\dot{q} + F(q) + G(q) + \tau_r = \frac{1}{\tau_d}(B(q)\tau - A(q)\lambda), \] (10)

where \(M(q)\) is a symmetric, positive definite matrix, \(V_m\) is a centripetal and Coriolis matrix, \(F\) is a friction vector, \(G\) is a gravity vector, \(\tau_d\) is a vector of disturbances including unmodeled dynamics, \(B\) is an input transformation matrix, \(\tau\) is a control input vector, \(A\) is a matrix associated with the constraints, \(\lambda\) is a vector of constraint forces, and \(\dot{q}\) and \(\ddot{q}\) denote velocity and acceleration vectors, respectively. The dynamics of the driving and steering motors should be included in the robot dynamics, along with the gearing. Since in our case robot moves on a plane vector \(G = 0\). It would be more suitable to express the dynamic equations of motion in terms of internal velocities \(v\) and \(\omega\). In order to that we differentiate equation (4) with respect to time. One can get
\[ \ddot{q} = S(q)\nu + S(q)\nu. \] (11)

Next we substitute Eqs. (4) and (11) into equation (10) and multiply the resulting equation on the left hand side by matrix \(S^T(q)\). After some not complicated calculations we get
\[ \ddot{M}(q)\nu + \dot{V}_m(q, q)\nu + \ddot{F}(\nu) + \ddot{\tau}_e = \ddot{\tau}, \] (12)

where \(\ddot{M} = S^TMS, \dot{V}_m = S^T[M\dot{S} + \dot{V}_mS], \ddot{F} = S^T\ddot{F}, \ddot{\tau}_e = S^T\tau, \) and \(\ddot{B} = S^TB\). As before \(\tau\) is a set of two moments acting at the wheels, namely \([\tau_r, \tau_s]^T\). Matrix \(\ddot{M}(q)\) which appears in Eq. (12) is positive definite and matrix \(\ddot{B} = \ddot{B}(q)\tau\) is a skew symmetric matrix. Straightforward calculations show matrices \(\ddot{M}\) and \(\dot{V}_m\) have the following form (assuming that matrices \(A(q)\) and \(S(q)\) are of the form of Eqs. (5) and (7), for simplicity)
\[ \ddot{M} = \begin{bmatrix} m + 2C_n \frac{1}{\tau_r} & 0 & 0 \\ & & \end{bmatrix} J_1, \] (13)

where \(J_1 = J + 2C_n\frac{1}{\tau_r} + 2m_0R^2 + 2A, \) where \(m_0\) denotes mass of the wheel, \(m_0\) denotes mass of the vehicle, \(A\) is the moment of inertia of the wheel about a diameter, \(C_n\) is the axial moment of inertia including the actuator inertia, and \(J\) is a moment of the inertia around axis passing through point \(P\). Matrix \(\ddot{B}\) is of the form \(\ddot{B} = \begin{bmatrix} \frac{1}{\tau_r} & 0 \\ 0 & \frac{1}{\tau_s} \end{bmatrix} \). Note that matrix \(\ddot{B}\) is nonsingular.

Now we are in position to take into account electrical part of the actuator. It is assumed that both motors are DC motors. We denote by indices \(r\) and \(l\) right and left motor, respectively. Equations governing the right actuator can be written as follows
\[ \tau_{rr} = k_{i_r}i_r, \] (14)

\[ \ddot{U}_r = \frac{L_r}{R_r}\frac{di_r}{dt} + R_{i_r}i_r + k_{e_r}\dot{\psi}_r, \] (15)

where \(\tau_{rr}\) is the torque generated by the right motor. In Eq. (14) it is assumed that the current is proportional to the torque and constant coefficient which plays this role is denoted by \(k_{i_r}\). Equation (15) characterizes voltage equation of the armature which is described by inductance \(L_r\) and resistance \(R_{i_r}\), while \(k_{e_r}\) denote counter electromotive force coefficient. Torques for both motors can be written in more compact form as
\[ \tau_r = \begin{bmatrix} \tau_{rr} \\ \tau_{rl} \end{bmatrix} = \begin{bmatrix} k_{i_r} & 0 \\ 0 & k_{i_l} \end{bmatrix} \begin{bmatrix} i_r \\ i_l \end{bmatrix} = k_i\nu, \] (16)

and current-voltage equation has the following form
\[ \begin{bmatrix} \dot{U}_r \\ \dot{U}_l \end{bmatrix} = \begin{bmatrix} \frac{L_r}{R_r} & 0 \\ 0 & \frac{L_l}{R_l} \end{bmatrix} \begin{bmatrix} \frac{di_r}{dt} \\ \frac{di_l}{dt} \end{bmatrix} + \begin{bmatrix} R_{i_r} & 0 \\ 0 & R_{i_l} \end{bmatrix} \begin{bmatrix} i_r \\ i_l \end{bmatrix} + \begin{bmatrix} k_{e_r} & 0 \\ 0 & k_{e_l} \end{bmatrix} \begin{bmatrix} \dot{\psi}_{rr} \\ \dot{\psi}_{rl} \end{bmatrix}, \] (17)

or in a matrix vector form
\[ U = L\frac{di}{dt} + R_i i + k_e\dot{\psi}. \] (18)

Relationship between angular wheels’ velocities and internal velocities \(\nu\) is given by Eq. (5) which we rewrite as follows
\[ \begin{bmatrix} \dot{\psi}_r \\ \dot{\psi}_l \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & -\frac{1}{2r} \\ -\frac{1}{2r} & \frac{1}{r} \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = k_i\nu. \] (19)
Actuators are usually equipped with gears. Taking it into account one can write \( \tau_i = N\tau_s \) and \([\phi_{\tau^x}, \dot{\phi}_{\tau^x}]^T = N^{-1}\phi_s \). Then Eqs. (16) and (17) can be rewritten as follows
\[
\tau_i = NK_i \dot{i}
\] (20)
\[
U = L\frac{di}{dt} + R_i i + k_Nk_Nk_i\nu,
\] (21)
where \( N \) is a diagonal matrix \( N = \text{diag}(n_\theta, n_\phi) \), where gear ratios \( n_\theta \) and \( n_\phi \) are greater than 1. For subsequent analysis we assume that \( \tau = \tau_i \).

IV. KINEMATIC CONTROL

In this section we briefly outline on the idea of kinematic oscillator which was proposed by Dixon and coauthors [3], [4], [5]. First they introduced the following transformation for position and orientation tracking errors in the base coordinate system (they considered the case where \( d = 0 \))
\[
\begin{bmatrix}
w \\
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
-\delta \cos \theta + 2 \sin \phi \\
0 \\
\cos \theta 
\end{bmatrix} \begin{bmatrix}
x \\
y \\
\theta
\end{bmatrix},
\]
(22)
or equivalently
\[
w = Z\hat{x}.
\] (23)
In the last equation \( \hat{x}(t), \hat{y}(t), \hat{\theta}(t) \) denote difference between actual position and orientation \([x_c, y_c, \theta_c]\) of the center of mass of the vehicle and \([x_{rc}, y_{rc}, \theta_{rc}]\) denotes desired position and orientation both expressed of the base coordinate system
\[
\begin{align*}
\hat{x} & = q_i - q_{rc}, \\
\hat{y} & = y_i - y_{rc}, \\
\hat{\theta} & = \theta_i - \theta_{rc}.
\end{align*}
\] (24)
Differentiating both sides of Eq. (22) and using (7) one can write the following differential equation for auxiliary signals \( w, z_1 \) and \( z_2 \)
\[
\dot{w} = u^TJ^Tz + f \\
\hat{z} = \dot{z} + w(\hat{z}),
\] (25)
where \( J \in \mathbb{R}^{2 \times 2} \) is a skew symmetric matrix
\[
J = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix},
\] (26)
and function \( f \in \mathbb{R} \) has the form
\[
f = 2(v_{z2}z_2 - v_{x1}z_1 + dv_{z2}z_1 + dv_{z2}),
\] (27)
where \( v_{x1} \) and \( v_{z2} \) are desired linear and angular velocities of the vehicle. Now we define the auxiliary variable \( u(t) = [u_1(t) u_2(t)]^T \in \mathbb{R}^2 \) in terms of the position and orientation, velocities and desired trajectory as follows
\[
u = T^{-1}u = \begin{bmatrix} v_{x2} \\
v_{x1} \cos \theta + 2v_{z2} \sin \theta \end{bmatrix}.
\] (28)
Last equation in merely a rearrangement of the second equation of Eq. (25). Based on Eq. (28) one can calculate a vector of actual velocities of the vehicle
\[
\nu = Tu + \begin{bmatrix} v_{x1} \cos \theta + v_{z2} \sin \theta \\
v_{x1} \sin \theta + v_{z2} \cos \theta \end{bmatrix}.
\] (29)
In Eqs. (28) and (29) matrix \( T \in \mathbb{R}^{2 \times 2} \) has the following form
\[
T = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}.
\] (30)
Now we are in position to define an auxiliary error signal \( \hat{z}(t) \in \mathbb{R}^2 \) as the difference between the subsequently desired auxiliary signal \( x_d(t) \in \mathbb{R}^2 \) and the transformed variable \( z(t) \), defined in (23), as follows
\[
\hat{z} = z_d - z.
\] (31)
Based on the kinematic equation given by Eq. (23), Dixon and coauthors designed the auxiliary signal \( u(t) \) as follows
\[
u = u_i - k_2z,
\] (32)
where the auxiliary control term \( u_i(t) \in \mathbb{R}^2 \) is defined as
\[
u_i = \left( \begin{array}{c}
k_1w_f + f \\
\delta_d
\end{array} \right) Jz_d + \Omega_1z_d.
\] (33)
The auxiliary signal \( z_d(t) \) is defined by the following oscillator-like relationship
\[
\hat{z}_d = \delta_d z_d + \left( \frac{k_1w_f + f}{\delta_d^2} + w \Omega_1 \right) Jz_d,
\] (34)
with initial condition
\[
z_d(0)z_d(0) = \delta_d^2(0),
\] (35)
and \( \Omega_1 \in \mathbb{R} \) is defined as
\[
\Omega_1 = k_2 + \frac{\delta_d \Omega_1}{\delta_d^2} + w \left( \frac{k_1w_f + f}{\delta_d^2} \right),
\] (36)
where
\[
\delta_d = \alpha e^{-\nu_{z1}^T} + \varepsilon_1.
\] (37)
All constants \( k_1, k_2, \alpha_0, \alpha_1 \) and \( \varepsilon_1 \in \mathbb{R} \) are positive control gains. Dixon and coauthors proved in a systematic manner the following theorem [3]. Provided the desired trajectory is selected to be bounded for all time \( t \geq 0 \), the kinematic control law given by Eqs. (23)-(37) ensures the position and orientation tracking errors defined in Eq. (22) are globally uniformly bounded in the sense that
\[
|\hat{x}(t)|, |\hat{y}(t)|, |\hat{\theta}(t)| \leq \beta_0e^{-\gamma_0 t} + \beta_1 \varepsilon_1,
\] (38)
where \( \varepsilon_1 \) was defined in Eq. (34), and \( \beta_0, \beta_1, \gamma_0 \in \mathbb{R} \) are some positive constants. Proof of the theorem in based on Lyapunov stability analysis and can be found in reference [3]. The stability analysis is carried out in terms of variables \( w(t), z_1(t), z_2(t) \), however inverse transformation to one given in Eq. (22) leads to variables \( \hat{x}, \hat{y}, \hat{\theta} \) according to the following equation
\[
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{\theta}
\end{bmatrix} = \begin{bmatrix}
\frac{\dot{\theta}}{\cos \theta} & \frac{d}{\sin \theta} & \frac{\dot{\theta}}{\cos \theta} \\
\frac{\dot{\theta}}{\cos \theta} & \frac{d}{\sin \theta} & \frac{\dot{\theta}}{\cos \theta} \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix} w \\
z_1 \\
z_2
\end{bmatrix}.
\] (39)
Now we incorporate dynamic equations of motion of the vehicle in the form of Eq. (12). In order to that we rewrite Eq. (29) as follows
\[
\nu = Tu + \Pi_1,
\] (40)
where vector $\mathbf{P}_1$ is defined clearly in Eq. (29). Differentiating Eq. (40) with respect to time gives

$$\dot{v} = \dot{T}u + Tu + \mathbf{P},$$

and $u_v$ is defined in Eq. (33). Following the ideas presented by Dixon and coauthors [3], [4] it is easy to derive the differential equations governing errors $w$ and $\dot{z}$

$$\dot{w} = -k_v w + u_v^T J \dot{z} + \eta^T J z,$$  \hspace{1cm} (52)

and

$$\dot{\dot{z}} = -k_z \ddot{z} + w J u_v + \eta.$$  \hspace{1cm} (53)

Now we propose the following Lyapunov function candidate

$$V = \frac{1}{2} w^T + \frac{1}{2} \dot{z}^T \ddot{z} + \frac{1}{2} \eta^T \tilde{M} \eta + \frac{1}{2} \alpha^T \tilde{M} \eta + \frac{1}{2} \alpha^T \tilde{M} \alpha.$$  \hspace{1cm} (54)

First two components represent energy of the transformed errors, the third component is a kinetic energy of the vehicle described in terms of input error, and finally the last term describes a square of certain function which depends on time. Time derivative of the Lyapunov function has the following form

$$\dot{V} = w \dot{w}^T + \dot{\dot{z}}^T \ddot{z} + \eta^T \tilde{M} \dot{\eta} + \frac{1}{2} \eta^T \frac{d}{dt} [\tilde{M}] \eta + a \alpha.$$  \hspace{1cm} (55)

Substitution of Eqs. (52), (53) and (50) into Eq. (55) gives

$$\dot{V} = \frac{w}{2} [-k_v w + u_v^T J \dot{z} + \eta^T J z] + \frac{1}{2} \eta^T [\tilde{M}] \eta + a \alpha + \frac{1}{2} \eta^T \tilde{M} \tilde{M} \eta + \left[-R - T^T \tilde{M} \eta + \eta^T \tilde{M} \tilde{M} \eta + \left[-R - T^T \tilde{B} \eta + \alpha \alpha \right].$$  \hspace{1cm} (56)

It is easy to prove that matrix $\frac{1}{2} \frac{d}{dt} \tilde{M} - T^T \tilde{M} \tilde{T} - T^T \tilde{V}_m T$ is a skew symmetric matrix. Therefore one can write $V$ as follows

$$\dot{V} = -w k_v w + w \eta^T J \dot{z} - \dot{z}^T k_z \dot{z} + \dot{z}^T \eta + \eta^T \tilde{B} \eta + \alpha \alpha.$$  \hspace{1cm} (57)

In order to reduce furthermore the last expression one can use the following relationship $R_l = R - J \omega - w \omega$ which gives

$$\dot{V} = -w k_v w - \dot{z}^T k_z \dot{z} + \eta^T \left[-R_l \right] + a \alpha - T^T \tilde{B} \dot{\eta}.$$  \hspace{1cm} (58)

Now we propose the following control law at the dynamic level

$$\tau = \left[T^T \tilde{B} \dot{\eta}\right]^T \eta.$$  \hspace{1cm} (59)

In the case considered here matrix which appears in the last equation is nonsingular. It is easy to calculate that

$$T^T \tilde{B} \dot{\eta} = \frac{1}{T} \begin{bmatrix} \dot{x} \sin \theta - \dot{y} \cos \theta + R & \dot{x} \sin \theta - \dot{y} \cos \theta - R \end{bmatrix}.$$  \hspace{1cm} (60)

Note that det $[T^T \tilde{B} \dot{\eta}] = 2R_1$ therefore the last matrix is nonsingular. Substituting control law given by Eq. (59) in Eq. (58) gives

$$\dot{V} = -w k_v w - \dot{z}^T k_z \dot{z} + \eta^T \eta + \eta^T \left[-R_l \right] + a \alpha.$$  \hspace{1cm} (61)

In order to reduce the last equation we assume that

$$a \alpha = \eta^T R_l,$$  \hspace{1cm} (62)

which allows to rewrite the time derivative of the Lyapunov function as follows

$$V = -w k_v w - \dot{z}^T k_z \dot{z} - \eta^T \eta.$$  \hspace{1cm} (63)
An idea of introducing function $a(t)$ but in different manner was introduced by Galic̆ki [6]. Conditions under which Eq. (62) is satisfied are given in reference [7]. Here we state that it is true for passive mechanical systems. Now we include the torques acting at the motor level, which is the last level of control. Intuitively we define the following Lyapunov function candidate

$$V_1 = \frac{1}{2} w^2 + \frac{1}{2} \tilde{z}^T \tilde{z} + \frac{1}{2} \eta^T M\eta + \frac{1}{2} \tau^T \epsilon_0 \tau + \frac{1}{2} (\tau_d - \tau)^T (\tau_d - \tau).$$

(64)

where we have made use of Eq. (54), $\tau_d$ denotes a vector of desired torques and $\tau$ denotes a vector of actual torques. The time derivative of the Lyapunov function candidate, $V_1$, assuming that the differential equation (62) is satisfied has the following form

$$\dot{V}_1 = -k_1 w^2 - \tilde{z}^T k_2 \tilde{z} - \eta^T T^T \hat{B}(q) \tau + (\tau_d - \tau)^T k_c (\tau_d - \tau).$$

(65)

Now we consider the last term which appears in Eq. (65). Time differentiation of Eq. (20) gives

$$\frac{d\tau_d}{dt} = Nk_c \frac{di}{dt}$$

(66)

Recall that we have assumed that $\tau = \tau_d$. Substituting Eq. (66) in Eq. (20) leads to

$$L[Nk_c]^{-1} \frac{d\tau_d}{dt} = U - H(i, \nu),$$

(67)

where

$$H(i, \nu) = R_i i + k_c Nk_c \nu.$$

(68)

Notice that $\nu$ is a vector of linear and angular velocities of the centre of mass of the vehicle. Now we propose the following control law at the voltage level

$$U = L[Nk_c]^{-1} \frac{d\tau_d}{dt} + H(i, \nu) + L[Nk_c]^{-1} \frac{d\tau_d}{dt} + L[Nk_c]^{-1} \eta^T T^T \hat{B}(q)^T,$$

(69)

where $k_c$ is a positive definite diagonal matrix. Substituting Eq. (69) in Eq. (67) gives the following result

$$\frac{d\tau_d}{dt} - \frac{d\tau_d}{dt} = -k_c (\tau_d - \tau) - \left[ \eta^T T^T \hat{B}(q) \right]^T.$$

(70)

Making use of Eq. (70) the time derivative $\dot{V}_1$ takes the following form

$$\dot{V}_1 = -k_1 w^2 - \tilde{z}^T k_2 \tilde{z} - \eta^T T^T \hat{B}(q) \tau + (\tau_d - \tau)^T \left[ -k_c (\tau_d - \tau) - \left[ \eta^T T^T \hat{B}(q) \right]^T \right].$$

(71)

We rewrite the above equation as follows

$$\dot{V}_1 = -k_1 w^2 - \tilde{z}^T k_2 \tilde{z} - \eta^T T^T \hat{B} \tau - (\tau_d - \tau)^T k_c (\tau_d - \tau) + -\tau^T \left[ \eta^T T^T \hat{B} \right]^T + \tau^T \left[ \eta^T T^T \hat{B} \right]^T.$$

(72)

Now we propose the following desired torque vector

$$\tau_d = [T^T \hat{B}]^{-1} \eta.$$  

(73)

This construction is consistent with the backstepping procedure. Recall that matrix $T^T \hat{B}(q)$ is defined correctly since it is nonsingular. Calculating transpose of Eq. (73) gives

$$\tau_d^T = \eta^T [T^T \hat{B}]^{-T}.$$

(74)

Substituting the last equation in Eq. (72) results

$$\dot{V}_1 = -k_1 w^2 - \tilde{z}^T k_2 \tilde{z} - \eta^T T^T \hat{B}(q) \tau - (\tau_d - \tau)^T k_c (\tau_d - \tau) + -\eta^T \left[ (T^T \hat{B})^{-1} T^T \right] \eta + \tau^T \left( \eta^T T^T \hat{B} \right)^T.$$ 

(75)

Now taking into account the following identity $\tau^T \left( \eta^T T^T \hat{B} \right) = \left[ \eta^T T^T \hat{B} \right]^T \eta$, finally we rewrite the time derivative of the Lyapunov function as follows

$$\dot{V}_1 = -k_1 w^2 - \tilde{z}^T k_2 \tilde{z} - \eta^T \eta - (\tau_d - \tau)^T k_c (\tau_d - \tau).$$  

(76)

Note that Eq. (70) describes the differential equation governing torque error. In order to implement control law Eq. (69) one has to substitute time derivative of the desired torques given by Eq. (74). For subsequent analysis we denote $\tau_m = \tau_d - \tau$. Note that $V_1$ is negative for all signals $w \neq 0, \tilde{z} \neq 0, \eta \neq 0$ and $\tau_d \neq 0$, and equal zero otherwise. Signals $w, \tilde{z}$ and $\eta$ are continuous and bounded. Signal $\tau_d$ is bounded due to physical constraints of the motors. Now making use of the La Salle’s-Yoshizawa invariance principle one can deduce that $w \to 0, \tilde{z} \to 0, \eta \to 0$ and $\tau_d \to 0$. Consequently by making use of Eqs. (31)-(37) we obtain that $\tilde{x}, \tilde{y}$ and $\dot{\theta}$ tend to a small nonzero error. Summarizing the above considerations we can formulate the following theorem.

Theorem 1. Assuming that the desired trajectory $x_{des}, y_{des}, \theta_{des}$ and their derivatives are bounded signals for all $t \geq 0$ the dynamic control law given by Eqs. (31)-(37), (59), and (69) ensures global asymptotic stability of the mobile vehicle described by kinematical, dynamical and drive system equations. 

V. SIMULATION AND EXPERIMENTAL RESULTS

In this section we present some simulation results and preliminary experiments. Simulation results were carried out using Matlab® and Simulink® tools. In general we did many different simulations; namely using only kinematic oscillator, kinematic oscillator and dynamic model and finally using three levels of the proposed control scheme. Here we present only representative results, while a wide analysis of them can be found in reference [7]. Kinematic oscillator has been used for both set point and trajectory control. For the purpose of simulation we assumed the following desired velocities $\omega_x = 0.2 \frac{m}{s}$ and $0.5 \frac{m}{s}$. Robot has to follow trajectory which is a circle of a radius 0.4[m] located at the base coordinate system with its middle point $[x_0 y_0] = [0 0.4]$. The starting point is defined as $[0 0 0]$. An auxiliary signal $x_{aux}(0) = [0.10610.1061]$. The coefficients which characterize the oscillator are assumed to be $k_1 = 100$, $k_2 = 100, \alpha_0 = 0.1, \alpha_1 = 0.5$ and $\epsilon_1 = 0.05$. Desired (dotted line) and actual trajectories (continuous line) are presented in Fig. 2. The steady state errors for this simulation are as follows $[\epsilon] = 0.3 \cdot 10^{-3}[mm], [\dot{\epsilon}] = 4.8 \cdot 10^{-3}[mm], [\ddot{\epsilon}] = 2.8 \cdot 10^{-2}[\epsilon]$. In the experiments it was observed that when $\epsilon_1 > 0.5$ oscillator has a high dumping and the steady state errors have bigger numbers, while when $\epsilon_1 < 0.5$ the tracking errors have smaller values. In the above experiment $d = 0$. Now the simulation experiments were carried out with dynamic model of the robot itself. Here we assumed the following parameters of the robot $m_0 = 0.35[kg], m_1 = 0.05[kg], A_1 = 0.23 \cdot 10^{-5}[kgm^2], C_\theta = 0.1 \cdot 10^{-3}[kgm^2], J = 0.4 \cdot 10^{-3}[kgm^2], \theta = 0.075[mm], r = 0.0265[mm]$. Kinematic oscillator parameters are as follows.
In this paper we proposed a three level control algorithm which includes three levels of control: kinematic, dynamic and servo loop. In addition to the parameters listed above we introduce other parameters $L_a = 24[mH], R_a = 11.3[\Omega]$, and $T_{max} = 1.36[mNm]$. Here it was assumed that robot follows a sine trajectory and the desired velocities have the following values $v_{r1} = 0.15\frac{\pi}{2} [\frac{mm}{s}]$ and $v_{r2} = 0.2 \sin(0.5t) [\frac{mm}{s}]$. The parameters of the kinematic oscillator are $k_1 = 30, k_2 = 50, \alpha_0 = 0.1, \alpha_1 = 1.0$ and $\varepsilon_1 = 0.05$. The three levels algorithm was tested on several examples which show that the position and orientation errors are within the following bounds $|\vec{x}| < 6[mm]$, $|\vec{v}| < 6[mm/s]$ and $|\theta| < 0.7[^\circ]$. In order to test the proposed algorithm experimentally we built a prototype of a mobile robot named Mini Tracker2β presented in Fig. 3. The control system of this robot consists of the two microprocessor systems. Obviously the main program resides on a PC compatible computer. Communication between the host computer and robot computer is realized via fast RS232 IrDA. Robot weights 0.45[kg] and can move fast with speed around 1[m/s], and acceleration around 0.5[m/s²]. Some preliminary results are shown in Fig. 4. Here robot is supposed to follow a sine trajectory with the desired velocities $v_{r1} = 0.2\frac{\pi}{2} [\frac{mm}{s}]$ and $v_{r2} = 0.2 \sin(0.5t) [\frac{mm}{s}]$. At the moment the experimental results presented here are realized in open loop due to some difficulties of the communication between Simulink® and on-board computer via RS232C which works only in one direction, namely from the host computer to the on-board computer. Position and orientation in cartesian space were measured by the vision system which is installed above the plane on which mobile robot moves.

VI. CONCLUDING REMARKS

In this paper we proposed a three level control algorithm which is a backstepping type for the trajectory tracking and set point control. The proposed control algorithm consists of kinematic, dynamic and servo loop and uses an idea of kinematic oscillator. It was proven that the closed loop system is asymptotically stable to certain bounds of the tracking position and orientation errors. Simulation results illustrate the theoretical considerations. Some experimental work is presented too. In the near future more experimental work will be done.

REFERENCES