A Pricing Model of Fuzzy Rainbow Options

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Abstract
The studies of financial derivatives such as options are emerging prosperously. The rainbow options which link to two or more underlying assets get more concerns increasingly because of more flexibility to support diversification and more available investment strategies than one-asset options. Nevertheless, imprecise evaluation of input parameters usually results in misestimation of option value. In order to handle vague and imprecise problems, This paper extend Lee et al.’s [6] fuzzy Black-Scholes option pricing model in place of one-asset with multi-asset to develop a fuzzy multi-asset rainbow option pricing model with analytical approach which integrates fuzzy set theory and Bayesian theorem.

1. Introduction
Options are financial derivatives which represent the rights to buy or sell a financial value. Many researchers devote to develop the option pricing model (OPM) for improving valuation of options. Black and Scholes [1] introduced European-style call options pricing model in continuous time with analytical model. Cox, Ross, and Rubinstein [13] proposed European- and American-style option pricing model in discrete time with numerical analysis approach.

The Black-Scholes (B-S) model is based on mathematical modeling of a stochastic system, and it works successfully under the estimation of uncertain factors and dynamic movement in certain environment. Hence, in case of fuzziness occur in the process of options pricing, general pricing models are restricted by their nature and unsuitable for dealing with the imprecise problems mentioned above.

Moreover, the options price would be affected by parameters which derived from option pricing methods including value of stocks, risk-free interest rates, volatilities and others. It is difficult to assign appropriate value to these parameters. For example, there are many risk-free interest rates in financial market, and which one would be more appropriate is hard to decide. Particularly under uncertain environment, input parameters need investors to set their value, and the performance of option pricing models depend on the assignment of these parameters.

The fuzzy set theory proposed by Zadeh [23] is a useful tool to solve imprecise problems. Ribero et al. [12] presented financial applications by applying fuzzy set theory.

There are some papers establishing option pricing models under uncertain environment, such as European one-asset option pricing model in continuous and discrete time, and American one-asset option pricing model in continuous time. For example, Yoshida [21] proposed fuzzy Black-Scholes European option pricing model that discusses the stochastic process with uncertainty from the viewpoint of fuzzy expectation by taking account of sellers’/buyers’ subjective judgments to provide acceptable range of rational expected prices to investors. Lee et al. [6] applied fuzzy set theory to the Cox, Ross, and Rubinstein (CRR) model to set up the fuzzy binomial European option pricing model which provides reasonable range of option price based on the volatilities of stock prices from greatest to smallest.

It emerges many kinds of options in financial market which are different from European-style option and called exotic options (Jarrow [4]). One of exotic options named a rainbow option which links to two or more assets elicits researchers’ interest. Stulz [18] proposed the maximum and minimum of two-asset European option pricing model with the analytical approach. And Johnson [5] developed multi-asset European option pricing model that extended Stulz’s research from two-asset to many assets. Rubinstein [13] proposed a two-asset rainbow option pricing model with the numerical analysis approach.

However, these studies take no account of imprecise problems within rainbow option pricing models. In this paper, European fuzzy multi-asset rainbow option pricing model in continuous time with analytical model is developed to improve imprecise problems under
uncertain environment. Besides, this paper also takes the situation of dividend payment into account to raise the completeness of established option pricing model.

2. Research Methodology

A fuzzy European multi-asset option pricing model would be introduced by extending the fuzzy B-S option pricing model (Lee et al. [6]) to multi-assets.

2.1. Model inference

The expected value of \( R_v \), \( \sigma_v \), \( \rho_{ij} \) and the fuzzy multi-asset rainbow call option pricing are derived in the following.

2.1.1. Expected value of standard deviation \( E_{\Delta v}(\bar{\sigma}_v) \).

In multi-asset rainbow option pricing model, the value of standard deviation of each underlying asset is one of the important variables. The value of volatility of each underlying asset is depend on trade-volume whether large or small, it also be viewed as an investment decision. Each underlying asset owns its volatility set

\[
\sigma_v = \{\sigma_{v1}, \sigma_{v2}, ..., \sigma_{vn}\}, \quad \sigma_{vi}, \quad Y_v = \{1, 2, ..., Y_v\} \text{ stands for a possible state or actual condition of the volatility under the underlying asset.}
\]

In real world, it is hard to assign a precise volatility value to a decision action, so experts’ experiences are introduced to support decision-making, and then the fuzzy decision set \( \tilde{A}_{(v,n)} = \{\tilde{A}_{v1}, \tilde{A}_{v2}, ..., \tilde{A}_{vn}\} \) is made, where \( \tilde{A}_{(v,n)} \) denotes a fuzzy decision set of asset \( v \).

Lee et al. [6] presented the idea of a “probability of a fuzzy action”, i.e., the probability of \( \tilde{A}_{(v,n)} \), as follows.

\[
P_{\Delta v}(\bar{\sigma}_v) = \sum_{i=1}^{Y_v} \mu_{\sigma_{vi}}(\bar{\sigma}_{vi})P(\sigma_{vi})
\]

Therefore, the expected value of \( \sigma_v \) in \( \tilde{F}_i \) state,

\[
E_{\Delta v}(\bar{\sigma}_v) = \sum_{i=1}^{Y_v} \sigma_{vi} \mu_{\sigma_{vi}}(\bar{\sigma}_{vi})P(\sigma_{vi})
\]

(1)

Where \( y_v = I, 2, ..., Y_v \) stands for different volatility in \( \sigma_{vi} \).

2.1.2. Expected value of correlation coefficient \( E_{L(a,b)}(\tilde{\rho}_{(a,b)}) \).

In multi-asset rainbow option pricing models, the relationship between assets must be considered. The value of correlation coefficient between underlying asset \( a \) and \( b \) \( \rho_{(a,b)} \) also be viewed as a relation degree \( L_{(a,b),w} \), where \( a \neq b \) denotes underlying asset \( a \) and \( b \), \( w \) denotes a possible state of correlation degree.

Correlation coefficient sets between underlying asset \( a \) and \( b \), \( \rho_{(a,b)} = \{\rho_{(a,b),1}, \rho_{(a,b),2}, ..., \rho_{(a,b),H}\} \).

\[
\rho_{(a,b),a} = \{\tilde{L}_{(a,b),1}, \tilde{L}_{(a,b),2}, ..., \tilde{L}_{(a,b),M}\}
\]

stand for an possible state of the correlation coefficient between two underlying assets. A fuzzy relation degree set \( \tilde{L}_{(a,b),m} = \{\tilde{L}_{(a,b),1}, \tilde{L}_{(a,b),2}, ..., \tilde{L}_{(a,b),M}\} \), where \( a, b \) denotes underlying asset \( a \) and \( b \); \( m \) denotes a possible state of fuzzy relation degree. The “probability of a fuzzy relation decision”, i.e., the probability of \( \tilde{L}_{(a,b),m} \), is defined as

\[
P_{L_{(a,b)}}(\tilde{\rho}_{(a,b)}) = \sum_{h=1}^{H} \mu_{\tilde{L}_{(a,b),h}}(\tilde{\rho}_{(a,b),h})P(\tilde{\rho}_{(a,b),h})
\]

(2)

Where \( h = 1, 2, ..., H \) stands for different correlation coefficient in \( \rho_{(a,b),h} \).

2.1.3. Expected value of covariance between asset \( a \) and asset \( b \) \( \sigma_{(a,b)} \) is one of important variable in multi-asset rainbow option pricing model.

The expected value of \( \sigma_{(a,b)} \) in \( \tilde{F}_i \) state,

\[
E_{\Delta a}(\tilde{\sigma}_{(a,b)}) = \tilde{E}_{\sigma_{(a,b)}} = \tilde{E}_{\sigma_{(a,b)}}^2 - 2\tilde{E}_{\tilde{\rho}_{(a,b)}} \tilde{E}_{\tilde{\sigma}_{(a,b)}} + \tilde{E}_{\tilde{\sigma}_{(a,b)}}^2
\]

(3)

Where \( a, b = 1, 2, ..., V \) stands for different underlying asset, \( a \neq b \).

2.1.4. Expected value of triple indexed correlation coefficient \( E_{L_{(a,b)}}(\tilde{\rho}_{(a,b)}) \) and \( \rho_{(a,b)} \) are important variables in multi-asset rainbow option pricing models, the expected value of \( \tilde{\rho}_{(a,b)} \) and \( \rho_{(a,b)} \) in \( \tilde{F}_i \) state,

\[
E_{\tilde{\rho}_{(a,b)}}(\tilde{\rho}_{(a,b)}) = \frac{E_{\tilde{\rho}_{(a,b)}}(\tilde{\sigma}_{a})^2 - E_{\tilde{\rho}_{(a,b)}}(\tilde{\rho}_{(a,b)})E_{\tilde{\sigma}_{a}}(\tilde{\sigma}_{a})}{E_{\tilde{\rho}_{(a,b)}}(\tilde{\sigma}_{a})}
\]

(4)

Where \( a, b = 1, 2, ..., V \) stands for different underlying asset, \( a \neq b \), and \( E_{\tilde{\rho}_{(a,b)}}(\tilde{\rho}_{(a,b)}) \) can be defined as

\[
E_{\tilde{\rho}_{(a,b)}}(\tilde{\rho}_{(a,b)}) = \sum_{i=1}^{Y_v} \mu_{\tilde{\rho}_{(a,b),i}}(\tilde{\rho}_{(a,b),i})P(\tilde{\rho}_{(a,b),i})
\]
2.1.5. Expected value of risk-less interest $R_t$. Experts will assign an appropriate membership value to each possible state of the risk-free interest rates state set by his/her experience. In addition, experts can also discover hidden information or hidden fluctuation by their own information such as economy, politics, and so on. Hence, the sample information will affect the gap between reality (future) and initial rate (now).

The gap between reality and initial rate doesn’t exist in the efficient market because all information is revealed. However, if an information gap exists, it will lead to an inappropriate decision-making. If the probability of information about risk-free rate rising in the future $P(R_t | H_t)$ could be measured, then $R_t(H_t)$ would be computed by

$$R_t(H_t) = R_t + \sum_{i=1}^{R} P(x_i | H_t) H_t X_t$$

A fuzzy risk-free rate is set by experts’ experiences. $R_t(F_t)$ could be calculated using expected concept, and the expected value of $R_t$ in state $\tilde{F}_t$, $R_t(\tilde{F}_t)$, can be defined as

$$R_t(\tilde{F}_t) = \sum_{i=1}^{R} ((H_t + \sum_{j=1}^{R} P(x_j | H_t) H_t X_t)) u_{\tilde{F}_t}(H_t) P(H_t)$$

2.1.6. Expected value of dividend-paid rate. Experts will assign an appropriate membership value to each possible state of the dividend-paid rate state set by his/her experience. If the probability of sample information about dividend-paid rate rising in the future $P(Y_t | T_q)$ could be measured, then $\delta(T_q)$ would be calculated by

$$\delta(T_q) = T_q + \sum_{i=1}^{Z} P(Y_t | T_q) Y_t T_q$$

Where $T_q$ denotes a possible dividend-paid rate state, $Y_t$ denotes sample information space about dividend-paid rate rising in the future.

A fuzzy dividend-paid rate is set by experts’ experiences. $\delta(T_q)$ could be calculated using expected concept, and denoted by

$$\delta(T_q) = \sum_{i=1}^{Z} (T_q + \sum_{j=1}^{Z} P(Y_t | T_q) Y_t T_q) u_{\tilde{F}_t}(T_q) P(T_q)$$

2.1.7. The fuzzy multi-asset rainbow option pricing model. Important variables in fuzzy rainbow option pricing model are obtained by six previous items, so as to calculate the value

$$C(\tilde{A}_{1}, \tilde{A}_{2}, ..., \tilde{A}_{N}), (L_{1}, L_{2}, ..., L_{N}), M_j) = \sum_{i=1}^{E} \sum_{j=1}^{R} \mu_{\tilde{F}_i}(H_i) M_j$$

According to the optimal value of

$$C(\tilde{A}_{1}, \tilde{A}_{2}, ..., \tilde{A}_{N}), (L_{1}, L_{2}, ..., L_{N}), M_j)$$

the minimal value is obtained as follows.

$$C(\tilde{A}_{1}, \tilde{A}_{2}, ..., \tilde{A}_{N}), (L_{1}, L_{2}, ..., L_{N}), M_j)$$

2.2. A fuzzy model returns to crisp model

According to the definition given by Zadeh [23], If a fuzzy set belongs to X, then $\mu_{\tilde{A}}: X \rightarrow [0, 1]$  , $\mu_{\tilde{A}}$ is a membership function of $\tilde{A}$, i.e., if $x \in X$, then $\mu_{\tilde{A}}(x) \in [0,1]$. When the value domain of membership function has become $\{0, 1\}$, then $\tilde{A}$ is converted into crisp set A, and $\mu_{\tilde{A}}$ is converted into crisp characteristic function $C_A$. The conversion means no fuzziness existing for input parameters in multi-asset OPMs, i.e., $\mu_{\tilde{A}}(x) = 1$ or 0. If $x \notin A$, then $\mu_{\tilde{A}}(x) = C_A(x) = 1$; If $x \in A$, then $C_A(x) = 0$. Therefore, the probability of a fuzzy event $\tilde{A}$ could be defined as

$$P(A) = \sum_{i=1}^{R} \mu_{\tilde{A}}(X_i) P(X_i) = \sum_{i=1}^{R} C_A(X_i) P(X_i) = P(A)$$

Analogously, each crisp parameter and call option price could be computed.

3. Conclusion and suggestion

In this research, the fuzzy problems in multi-asset rainbow option pricing have been solved by fuzzy Bayesian theory which is composed of fuzzy set theory and Bayesian theorem. To compare fuzzy multi-asset rainbow option pricing model with non-fuzzy one, some phenomena occur (by mathematic corollaries which are left out in this paper) if neglecting fuzzy problems.

1. The volatility of each underlying asset price would be overestimated.
2. If correlation coefficient is positive then it would be overestimated.
3. If correlation coefficient is negative then it would be underestimated.
4. The covariance would be overestimated if correlation coefficient is negative.
5. Interest rate would be overestimated.
6. In-the-money and on-the-money call option might be overestimated.

No matter how standard deviation and correlation coefficient are estimated, overestimated or underestimated, ρ_{ab} would be affected by those imprecise variables. Hence, it easily results in incorrect option price, and fuzzy multi-asset rainbow option pricing model could solve these problems to provide relatively precise option price.

By corollary and simulation results which are left out due to large length, it is found that some variables would be overestimated or underestimated. And it would lead to inappropriate decision-making and incorrect option valuation when fuzziness exists but be ignored.

Reference