ON A CLASS OF ANALYTIC FUNCTIONS RELATED TO CONIC DOMAINS AND ASSOCIATED WITH CARLSON-SHAFFER OPERATOR

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Abstract Making use of the Carlson–Shaffer convolution operator, we introduce and study a new class of analytic functions related to conic domains. The main object of this paper is to investigate inclusion relations, coefficient bound for this class. We also show that this class is closed under convolution with a convex function. Some applications are also discussed.

Key words Hadamard product (or convolution); Differential subordination; Carlson–Shaffer operator; \(k\)-starlike functions; \(k\)-uniformly convex function

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1 Introduction

Let \(A\) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disc \(U = \{ z \in \mathbb{C} : |z| < 1 \}\). Let \(S\) denote the subclass of \(A\) consisting of all univalent functions. We say that a function \(f\) is convex when \(f(U)\) is a convex set. Also, we say that a function \(f\) is starlike with respect to the origin when \(f(U)\) is a starlike set with respect to 0. By \(K \) or \(S^*\) we denote the subclasses of \(A\) consisting of all functions which are convex or starlike respectively, while by \(S^*(\delta)\) we denote the class of starlike functions of order \(\delta, \delta \in [0,1)\).

In 1991, Goodman [7] introduced the class \(UCV\) of uniformly convex functions. A function \(f \in CV\) is in the class \(UCV\) if for every circular arc \(\xi \subset U\), with center in \(U\), the arc \(f(\xi)\) is
convex. A more useful characterization of class $UCV$ was given by Ma and Minda [12] (see also [18]) as:

$$f \in UCV \iff f \in A \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad (z \in U).$$

In 1999, Kanas and Wiśniowska [9] (see also [10]) introduced the class of $k$-uniformly convex functions, $k \geq 0$, denoted by $k-UCV$ and the class $k-ST$ related to $k-UCV$ by Alexandar type relation, i.e.,

$$f \in k-UCV \iff zf' \in k-ST \iff f \in A \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad (z \in U).$$

In [9] and [10], respectively, there their geometric definitions and connections with the conic domains were also considered. For a fixed $k \geq 0$, the class $k-UCV$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $U$ with any disk centered at $\zeta$, $|\zeta| \leq k$, onto a convex domain. The notion of $k$-uniform convexity is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center $\zeta$ is the origin and the class $k-UCV$ reduces to the class $CV$. Moreover for $k = 1$ it coincides with the class of uniformly convex functions $UCV$ introduced by Goodman [7] and studied extensively by Rønning [18] and independently by Ma and Minda [12]. The class $k-UCV$ started much earlier in papers [4, 16] with some additional conditions but without the geometric interpretation.

We say that a function $f \in A$ is in the class $S_{k,\gamma}^\ast$, $k \geq 0$, $\gamma \in \mathbb{C} \setminus \{0\}$, if and only if

$$\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (z \in U).$$

A lot of authors investigated the properties of the class $S_{k,\gamma}^\ast$ and their generalizations in several directions, e.g., see, [2, 3, 6, 10, 17, 18, 21]. An analytic function $f$ is said to be subordinate to an analytic function $g$ (written as $f \prec g$) if and only if there exists an analytic function $\omega$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for } z \in U,$

such that

$$f(z) = g(\omega(z)) \text{ for } z \in U.$$

In particular, if $g$ is univalent in $U$, we have the following equivalence

$$f \prec g \iff [f(0) = g(0) \text{ and } f(U) \subset g(U)].$$

The convolution or Hadamard product of two functions of class $A$ is denoted and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f$ has the form (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in U).$$
Now, we recall the incomplete beta function \( \varphi(a, c; z) \) defined by

\[
\varphi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n, \quad (c \neq 0, -1, -2, \ldots; z \in U),
\]

where \((x)_n\) is the Pochhammer symbol:

\[
(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1, & \text{for } n = 0, x \neq 0, \\ x(x + 1) \cdots (x + n - 1), & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\}. \end{cases}
\]

Almost a couple of decades ago, using the technique of convolution, Carlson and Shaffer [5] defined a linear operator \( L(a, c) : \mathcal{A} \to \mathcal{A} \) as

\[
L(a, c)f(z) = \varphi(a, c; z) * f(z), \quad f \in \mathcal{A}.
\]  

(1.2)

Clearly, the operator (1.2) maps \( \mathcal{A} \) onto itself, and \( L(c, a) \) is an inverse of \( L(a, c) \) provided that \( a \neq 0, -1, -2, \cdots \). From (1.2) the following identity can easily derived

\[
z[L(a, c)f(z)]' = aL(a + 1, c)f(z) - (a - 1)L(a, c)f(z).
\]  

(1.3)

Throughout this paper, we assume that \( z \in U, k \geq 0, 0 \leq \alpha \leq 1, \ a, c \in \mathbb{C} \) with \( c \neq 0, -1, -2, \cdots \) and \( \gamma \in \mathbb{C} \setminus \{0\} \) unless otherwise stated.

In the present paper, using the operator \( L(a, c) \) we extend the works of Kanas and Wiśniowska [9, 10] and introduce the following new class of analytic functions related to conic domains. A closely related paper is also [8].

**Definition 1** A function \( f \in \mathcal{A} \) is in the class \( \mathcal{Q}(k, a, c, \alpha, \gamma) \) if and only if

\[
\Re \left[ 1 + \frac{1}{\gamma} (J(a, c, \alpha; z) - 1) \right] > k \left| \frac{1}{\gamma} [J(a, c, \alpha; z) - 1] \right| (z \in U),
\]  

(1.4)

where

\[
J(a, c, \alpha; z) = \frac{z [L(a, c)f(z)]' + \alpha z^2 [L(a, c)f(z)]''}{(1 - \alpha)L(a, c)f(z) + \alpha z [L(a, c)f(z)]'},
\]

(1.5)

with \( g(z) = \alpha z f'(z) + (1 - \alpha) f(z) \).

**Special Cases**

(i) For \( \alpha = 0 \) and \( f \in \mathcal{Q}(k, a, c, \alpha, \gamma) \), we have \( L(a, c)f(z) \in \mathcal{S}_{k,\gamma}^* \), while for \( \alpha = \gamma = 1 \), \( L(a, c)f(z) \) is in the class of \( k \)-uniformly convex functions \( k-\mathcal{UCV} \).

(ii) Moreover, \( \mathcal{Q}(k, a, a, 0, 1) \equiv k-\mathcal{ST} \) and \( \mathcal{Q}(k, a, a, 1, 1) \equiv k-\mathcal{UCV} \), see [9, 10].

**Geometric Interpretation of the inequality (1.4)**

A function \( f \in \mathcal{A} \) is in the class \( \mathcal{Q}(k, a, c, \alpha, \gamma) \) if and only if \( J(a, c, \alpha; z) \) takes all the values in conic domain \( \Omega_{k,\gamma} \) such that

\[
\Omega_{k,\gamma} = \gamma \Omega_k + (1 - \gamma),
\]  

(1.6)

where

\[
\Omega_k = \left\{ u + iv : u > k \sqrt{(u - 1)^2 + v^2} \right\}.
\]

The boundary \( \partial \Omega_k \) of the above set becomes the imaginary axis when \( k = 0 \), a hyperbola when \( 0 < k < 1 \), a parabola when \( k = 1 \), and an ellipse when \( k > 1 \). All of these curves have the
vertex at the point $k/(k+1)$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$; ever symmetric with respect to the real axis.

The functions which play the role of extremal functions for these conic regions are given as

$$p_{k,\gamma}(z) = \begin{cases} 
1 + \frac{1 - 2\gamma}{1 - z}, & k = 0; \\
1 + \frac{2\gamma}{1 - k^2} \sinh^2 \left[ \left( \frac{2}{\pi} \arccos k \right) \text{arctanh} \sqrt{z} \right], & 0 < k < 1; \\
1 + \frac{\gamma}{k^2 - 1} \sin \left( \frac{\pi}{2R(t)} \int_0^{\frac{z}{R(t)}} \frac{1}{\sqrt{1 - x^2}} \sqrt{1 - (tx)^2} dx \right) + \frac{\gamma}{1 - k^2}, & k > 1;
\end{cases}$$

where $u(z) = \frac{\sqrt{1 - k^2}}{1 - z}$, $t \in (0, 1)$, $z \in U$ and $z$ is chosen such that $k = \cosh \left( \frac{R'(t)}{R(t)} \right)$, $R(t)$ is the Legendre’s complete elliptic integral of the first kind and $R'(t) = \sqrt{1 - t^2}$ is complementary integral of $R(t)$. Moreover, $p_{k,\gamma}(U) = \Omega_{k,\gamma}$ and $p_{k,\gamma}(U)$ is convex univalent in $U$, see [9, 10]. Because $p_{k,\gamma}$ is a convex univalent function so we can write Definition 1 in subordination form

$$f \in Q(k, a, c, \alpha, \gamma) \Leftrightarrow f \in A \text{ and } J(a, c, \alpha; z) \prec p_{k,\gamma}, \quad (z \in U). \quad (1.7)$$

In [22] it was considered a related class $P_{a,c}(h, \gamma) = \{f \in A : J(a, c, \alpha; z) \prec h(z)\}$, where $h$ is a convex univalent function analytic in $U$ with $\Re \{h(z)\} > 0$, $h(0) = 1$. Therefore $Q(k, a, c, \alpha, \gamma) \subset P_{a,c}(h, \gamma)$ when $\Re \{p_{k,\gamma}(z)\} > 0$. For $c = 1$ the class $P_{a,c}(h, \gamma)$ becomes the class $P_{a,1}(h, \gamma)$ defined in [1].

2 Preliminary Results

**Lemma 2.1** [15] Let $h$ be a convex univalent function in $U$ with $\Re \{\lambda h(z) + \mu\} > 0$, where $\mu \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$, $z \in U$. If $p$ is analytic in $U$ with $p(0) = h(0)$, then

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec h(z) \quad \text{implies} \quad p(z) \prec h(z).$$

**Lemma 2.2** [14] Let $d$ be a complex number with $\Re d > 0$. Suppose that $\Psi : \mathbb{C}^2 \to \mathbb{C}$ is continuous and satisfies the conditions $\Re \Psi(1x, y) \leq 0$, when $x$ is real and

$$y \leq (-|d - ix|^2)/(2\Re d).$$

If $p$ is analytic in $U$ with $p(0) = d$ and $\Re \{\Psi(p(z), zp'(z))\} > 0$ for $z \in U$, then $\Re \{p(z)\} > 0$ in $U$.

**Lemma 2.3** [19] Let $f$ and $g$ be in the class $CV$ and $S^*$ respectively. Then, for every function $F$ analytic in $U$, we have

$$\frac{f(z) \ast g(z) F(z)}{f(z) \ast g(z)} \in \overline{co}[F(U)], \quad z \in U,$$

where $\overline{co}[F(U)]$ denotes the closed convex hull of the set $F(U)$. 
3 Main Results

In the following Theorem 3.1, we give a necessary and sufficient condition for a function \( f \in \mathcal{A} \) to be in the class \( \mathcal{Q}(k, a, c, \alpha, \gamma) \) and in Theorem 3.3, following essentially the same method given in [23], we generalize the result from [11] and the results from [23].

**Theorem 3.1** Let

\[
F(a, c, f(z)) = (1 - \alpha)L(a, c)f(z) + \alpha z [L(a, c)f(z)]'.
\]  

(3.1)

Then \( f \in \mathcal{Q}(k, a, c, \alpha, \gamma) \) if and only if \( F(a, c, f(z)) \in \mathcal{S}^*_k,\gamma \).

**Proof** Let \( F(a, c, f(z)) \in \mathcal{S}^*_k,\gamma \), then

\[
\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{zF'(a, c, f(z))(z)}{F(a, c, f(z))} - 1 \right) \right] \geq k \left| \frac{1}{\gamma} \left( \frac{zF'(a, c, f(z))(z)}{F(a, c, f(z))} - 1 \right) \right| .
\]

(3.2)

Thus (3.1) together with (3.2) implies

\[
\Re \left[ 1 + \frac{1}{\gamma} \left| J(a, c, \alpha; z) - 1 \right| \right] \geq k \left| \frac{1}{\gamma} \left| J(a, c, \alpha; z) - 1 \right| \right|,
\]

where \( J(a, c, \alpha; z) \) is given by (1.5). Therefore \( f(z) \in \mathcal{Q}(k, a, c, \alpha, \gamma) \). Converse is immediate. \( \square \)

**Theorem 3.2** If \( \Re[p_{k,\gamma}(z)] > \Re(1 - a) \), then

\( \mathcal{Q}(k, a + 1, c, \alpha, \gamma) \subset \mathcal{Q}(k, a, c, \alpha, \gamma) \).

**Proof** Let \( f \in \mathcal{Q}(k, a + 1, c, \alpha, \gamma) \) and

\[
\frac{z[L(a, c)f(z)]' + \alpha z^2[L(a, c)f(z)]''}{(1 - \alpha)L(a, c)f(z) + \alpha z[L(a, c)f(z)]'} = p(z),
\]

(3.3)

where \( p \) is analytic in \( U \) and \( p(0) = 1 \).

From (1.3) and (3.3) and after some simplification, we obtain

\[
\frac{\alpha az[L(a + 1, c)f(z)]' + (1 - \alpha)aL(a + 1, c)f(z)}{(1 - \alpha)L(a + 1, c)f(z) + \alpha z[L(a, c)f(z)]'} = p(z) + a - 1.
\]

(3.4)

By logarithmic differentiation of (3.4), we have

\[
\frac{z[L(a + 1, c)f(z)]' + \alpha z^2[L(a + 1, c)f(z)]''}{(1 - \alpha)L(a + 1, c)f(z) + \alpha z[L(a + 1, c)f(z)]'} = p(z) + \frac{zp'(z)}{p(z) + a - 1}.
\]

(3.5)

Since \( f \in \mathcal{Q}(k, a + 1, c, \alpha, \gamma) \), so

\[
p(z) + \frac{zp'(z)}{p(z) + a - 1} < p_{k,\gamma}(z).
\]

Thus, by using Lemma 2.1, \( p \prec p_{k,\gamma} \) and hence \( f \in \mathcal{Q}(k, a, c, \alpha, \gamma) \). \( \square \)

**Theorem 3.3** If \( f \) belongs to the class \( \mathcal{S}^*(\delta) \) of starlike functions of order \( \delta, \delta \in [0, 1) \), then

\[
\Re \sqrt{f(z)/z} > \frac{1}{2 - \delta} \quad (z \in U),
\]

where the determination of the square root is the principal one.
Proof Let \( f \in \mathcal{S}^*(\delta) \) and set

\[
\sqrt{\frac{f(z)}{z}} = (1 - \beta)p(z) + \beta,
\]

where \( p \) is analytic in \( U \) with \( p(0) = 1 \). We want to show that \( \Re p(z) > 0 \) when \( \beta = 1/(2 - \delta) \). Logarithmic differentiating the above equality, we obtain

\[
\frac{zf'(z)}{f(z)} = 1 + \frac{2zp'(z)}{p(z) + \frac{\beta}{1 - \beta}}.
\]

Let

\[
\Psi(u,v) = 1 - \delta + \frac{2v}{u + \frac{\beta}{1 - \beta}} \quad (u,v \in \mathbb{C}).
\]

We shall prove that the functions \( p \) and \( \Psi \) satisfy the assumptions of Lemma 2.2 to obtain \( \Re p(z) > 0 \). For real \( x, y \) such that \( y \leq -\frac{(1 + x^2)}{2} \), we have

\[
\Re \Psi(ix, y) = 1 - \delta + \frac{2y\frac{\beta}{1 - \beta}}{x^2 + \left(\frac{\beta}{1 - \beta}\right)^2}
\]

\[
\leq 1 - \delta - \frac{(1 + x^2)\frac{\beta}{1 - \beta}}{x^2 + \left(\frac{\beta}{1 - \beta}\right)^2}
\]

\[
= \frac{x^2 \left(1 - \delta - \frac{\beta}{1 - \beta}\right) + \frac{\beta}{1 - \beta} \left((1 - \delta)\left(\frac{\beta}{1 - \beta}\right) - 1\right)}{x^2 + \left(\frac{\beta}{1 - \beta}\right)^2}
\]

\[
\leq 0
\]

because the quantities in the numerator become, for \( \beta = \frac{1}{2 - \delta} \),

\[
1 - \delta - \frac{\beta}{1 - \beta} \leq 0 \quad \text{and} \quad (1 - \delta)\left(\frac{\beta}{1 - \beta}\right) - 1 = 0.
\]

Therefore, by Lemma 2.2 we get \( \Re p(z) > 0 \) and finally

\[
\Re \sqrt{\frac{f(z)}{z}} = \Re [(1 - \beta)p(z) + \beta] > \beta = \frac{1}{2 - \delta}. \quad \Box
\]

**Theorem 3.4** If \( f \in \mathcal{Q}(k,a,c,0,\gamma) \) for some \( k \geq 0 \) and \( 0 < \gamma \leq 1 \), then

\[
\Re \sqrt{\frac{L(a,c)f(z)}{z}} > \frac{k + 1}{\gamma + k + 1} \quad (z \in U),
\]

where the determination of the square root is the principal one.

**Proof** If \( k \geq 0 \) and \( 0 < \gamma \leq 1 \), then the domain \( \Omega_k \) (see (1.6)) becomes a conic domain with the vertex \( k/(k + 1) \), thus \( \min\{\Re z : z \in \mathfrak{t}\} = t/(t + 1) \). Let \( f \in \mathcal{Q}(k,a,c,0,\gamma) \). Then using (1.4)–(1.6), we can get

\[
\Re z[\frac{L(a,c)f(z)}{L(a,c)f(z)}] > \frac{k}{k + 1} \gamma + 1 - \gamma,
\]
so \( L(a,c)f \in \mathcal{S}^* (\delta) \), where \( \delta = \frac{k+1-\gamma}{k+1} \). Therefore, by using Theorem 3.3, it can be easily verified that
\[
\Re \sqrt{L(a,c)f(z)} > \frac{k+1}{\gamma k+1}.
\]

Taking \( \alpha = 1 \) in (1.5) and repeating the above considerations for \( z[L(a,c)f]' \) instead of \( L(a,c)f \) we can obtain that \( z[L(a,c)f]' \in \mathcal{S}^* (\delta) \), where \( \delta = \frac{k+1-\gamma}{k+1} \). Thus we get the following corollary.

**Corollary 3.5** If \( f \in Q(k,a,c,1,\gamma) \) for some \( k \geq 0, 0 < \gamma \leq 1 \), then
\[
\Re \sqrt{|L(a,c)f(z)|} > \frac{k+1}{\gamma k+1} (z \in U).
\]

**Special Cases**

(i) For \( \gamma = 1, a = c \), Theorem 3.4 and Corollary 3.5 become the main results of the paper [23].

(ii) For special values of \( k, a, c \) and \( \gamma \), we obtain the results given in [11, 13].

Let us consider the Bernardi integral operator \( F_\mu \) given by
\[
F_\mu f(z) = \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt.
\]  \( (3.6) \)

For \( \mu \) with \( \Re \mu > -1 \) the operator (3.6) has the property \( F_\mu : A \rightarrow A \) (see, for instance, \([11, p.11]\)). Next, we show that the class \( Q(k,a,c,\alpha,\gamma) \) is closed under Bernardi integral operator.

**Theorem 3.6** Let \( \mu > -1 \) and \( f \in Q(k,a,c,\alpha,\gamma) \), then \( F_\mu f \in Q(k,a,c,\alpha,\gamma) \).

Proof Let \( f \in Q(k,a,c,\alpha,\gamma) \) and set
\[
\frac{z [L(a,c)f(z)]'}{(1-\alpha)L(a,c)f(z) + \alpha z [L(a,c)f(z)']} = p(z),
\]
then the function \( p \) is analytic in \( U \) with \( p(0) = 1 \). Now working in a similar way as in the proof of Theorem 3.2, we obtain the required result.

**Theorem 3.7** Let \( f \in Q(k,a,c,\alpha,\gamma) \) and \( \psi \in CV \). If \( 0 < \gamma \leq 1 \), then \( \psi * f \in Q(k,a,c,\alpha,\gamma) \).

Proof Let \( F = \psi * f \). If \( f \in Q(k,a,c,\alpha,\gamma) \), then the condition (3.3) is satisfied with \( p \prec p_{k,\gamma} \). Using the usual convolution properties and (3.3), we get
\[
\frac{z [L(a,c)f(z)]'}{(1-\alpha)L(a,c)f(z) + \alpha z [L(a,c)f(z)']} = \frac{\psi(z) * \{z[L(a,c)f(z)]'}{(1-\alpha)L(a,c)f(z) + \alpha z [L(a,c)f(z)']} \n\]
\[
eq \psi(z) * \{ (1-\alpha)L(a,c)f(z) + \alpha z [L(a,c)f(z)'] \}.
\]

By Theorem 3.1, the function \( F(z) = (1-\alpha)L(a,c)f(z) + \alpha z [L(a,c)f(z)'] \in \mathcal{S}_{k,\gamma}^* \). For \( 0 < \gamma \leq 1 \), the set \( \Omega_{k,\gamma} \) lies in the right half-plane \( \Re w > 0 \), so in this case \( \mathcal{S}_{k,\gamma}^* \subset \mathcal{S}^* \) and the function \( F \) is starlike. Hence, by Lemma 2.3, we have
\[
\frac{\psi(z) * p(z) F(z)}{\psi(z) * F(z)} \in \overline{\mathbb{C}}[p(U)] \subset p_{k,\gamma}(U)
\]
because $p_{k,\gamma}$ is convex univalent and $p \prec p_{k,\gamma}$. From the geometric interpretation (1.6) we get $F = \psi \ast f \in Q(k, a, c, \alpha, \gamma)$.

It is easy to see that, if $f$ has the form (1.1), then the Bernardi integral operator $F_{\mu}$ (see (3.6)) becomes

$$F_{\mu}f(z) = \sum_{n=1}^{\infty} \frac{1 + \mu}{n + \mu} a_n z^n = f(z) \ast \sum_{n=1}^{\infty} \frac{1 + \mu}{n + \mu} z^n = f(z) \ast h(z).$$

Therefore, Theorem 3.7 shows that if $h \in CV$ and $0 < \gamma \leq 1$, then the class $Q(k, a, c, \alpha, \gamma)$ is closed under the Bernardi integral operator $F_{\mu}$. It also was proved in Theorem 3.6 under weaker assumptions. The problem of convexity of the function $h$ was partially solved in [20] and it is known that $h \in CV$ whenever $\Re \mu > 0$ or $\mu = 0$. Moreover, we have

$$\varphi(a, c) = \varphi(a + 1, c) \ast \sum_{n=1}^{\infty} \frac{1 + a}{n + a} z^n.$$ 

Thus, Theorem 3.7 with $\Re a > 0$ or $a = 0$ and convex function

$$\psi(z) = \sum_{n=1}^{\infty} \frac{1 + a}{n + a} z^n.$$ 

becomes also a special case of Theorem 3.2.

**Theorem 3.8** Let $f \in Q(k, a, c, \alpha, \gamma)$ given by (1.1), then

$$|a_n| \leq \frac{|\delta_{k,\gamma}| |(c)_n|}{(1 + \alpha(n - 1)) |(a)_n|},$$

where

$$\delta_{k,\gamma} = \begin{cases} 
\frac{8 \gamma (\cos^{-1} k)^2}{\pi^2 (1 - k^2)}, & 0 \leq k < 1, \\
\frac{8 \gamma}{\pi^2}, & k = 1, \\
\frac{\pi^2 \gamma}{4 \sqrt{7 (k^2 - 1) R^2 (1 + t)}}, & k > 1.
\end{cases}$$

**Proof** Let $f \in Q(k, a, c, \alpha, \gamma)$, then by Theorem 3.1,

$$F(z) = \{(1 - \alpha) L(a, c) f(z) + \alpha z [L(a, c) f(z)]'\} \in S_{k,\gamma}^* = z + \sum_{n=2}^{\infty} b_n z^n.$$  

Comparing the coefficients of (3.9), we obtain

$$\frac{(a)_n}{(c)_n} a_n [1 + \alpha(n - 1)] = b_n.$$  

Using (3.10) together with (3.8) and the result $|b_n| \leq \delta_{k,\gamma}$ given in [17], we directly obtain (3.7).

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