Linear Functional Observers for Systems with Delays in State Variables: A Multirate Output Sampling Approach

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Abstract—This paper presents an approach for the design of a non-dynamical functional observers for linear time-invariant systems with time-delay in state variables. The proposed method is based on multirate sampling of plant output (fast output sampling). The main feature of the proposed technique is that the functional will be estimated with minimal number of output samples and the condition of full state observability is not necessary. The design procedure, simplicity and merits are illustrated through a numerical example.

Index Terms—Functional observer, time-delay, linear systems, multirate sampling.

I. INTRODUCTION

In the mathematical modeling of a real system, it is often assumed that the future behavior of the system depends only on current state. However, such an assumption is not always true due to the existence of time-delay elements such as material or information transfer. If the delay is neglected, sometimes the model cannot reflect the system sufficiently well, leading to poor performance. Great deal of interest has been devoted towards the stability, the stabilization, the controller and the observer design for time-delay systems. State feedback control has many distinct advantages over other forms of control. When entire state is not measurable directly, for the implementation of state feedback law for pole-placement or quadratic performance index, it is sufficient to estimate to estimate a subset of state vector or the value of a linear function of the state from the past and present inputs and outputs. Such an estimator termed as functional observer. Because the estimation of a function of states does not necessarily require the estimation of all the states, the order of a functional observer can be significantly less than that of state observer. Linear functional observers estimate linear functions of the state vector.

One of the main focus of research in design of functional observers for systems with delay [8]-[12] is reduction of the order of observer. Darouach in [8], [9] presented necessary and sufficient conditions for existence of pth-order functional observer for estimation of p-functional for both continuous and discrete-time systems. A new observer for the estimation of function of the state vector for linear time-delay systems is presented in [11]. But, the problem with the method proposed in [11] is that the functional observer may not exist unless the number of available outputs is greater than half of the number of states. To overcome this drawback, an alternative procedure is proposed in [10]. Trinh et.al., in [12] provided a comprehensive treatment on the design of functional observers which has an attractive features of being low-order and delay free, thereby cost effective and easy to implement.

Despite all contributions related to the problem of designing a minimum-order functional observers, we can mention two major disadvantages of observer based design are increase in the order of composite system and the robustness. Hence, need to search for non observer-based designs has been an ongoing process. Much research was reported which intended to solve the above problem by using multirate digital techniques as specified in [1]-[7]. All these techniques have a common feature of periodically time-varying elements in its controllers. In a multirate digital system the input and output signals of the system are sampled at different rates. For convenience of explanation, we introduce the sampling period of discrete-time system as $\tau$ sec. In the first approach, the input is changed N times more often than the output is sampled for every $\tau$ sec. The second approach is to sample the output of the continuous plant N times more often than the input to the plant. We will refer the first technique is multirate input sampling and later termed as multirate output sampling.

Based on multirate output sampling, Janardhanan and Bandyopadhyay [13], proposed an algorithm to estimate a full state vector for uncertain time-delay systems. The proposed algorithm works under the assumption that the system is observable and the number of output samples must be greater than observability index of the plant. But, the design of non-dynamical functional observer for time-delay systems not yet reported. In this paper, the authors propose for construction of linear functional of state vector for time delay systems. The proposed method is simple, less complex thereby easy to implement. In contrast to observer based design in which accuracy of estimation of states improves after long time, exact computation of states in just one sampling period is feasible if multirate output sampling is employed.

The structure of the paper is as follows: Section II provides a statement of the problem. Section III describes discretization of continuous-time system and derivations of multirate output model. Section IV presents necessary and sufficient conditions for existence of functional observers. Design procedure and illustrative example for validating the proposed method, discussion on results will appear in section V. Concluding remarks will appear in section VI followed by references.
II. Problem Formulation

Consider a continuous-time system having delay in state is represented by
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_1x(t - \tau_x) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]  
where the delay \( \tau_x \geq 0 \) and \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^r, y(t) \in \mathbb{R}^m \) are state, input, and output vectors respectively. \( A \in \mathbb{R}^{n \times n}, A_1 \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, C \in \mathbb{R}^{m \times n} \) are known matrices. The aim of this paper is to design a multirate output sampling based functional observer of the form
\[
z(k\tau) = Lx(k\tau); k > 1
\]  
where \( L \in \mathbb{R}^{m \times n}, p < n \). Without loss of generality, it is assumed that \( \text{rank}(C) = m, \text{rank}(L) = p \) and \( \text{rank} \begin{bmatrix} C \\ L \end{bmatrix} = m + p \). The objective of designing multirate output sampling based functional observer (2) is two-fold:
- First, establish a mathematical model of lifted system for uniformly sampled fast output \( y(k\tau + \mu\Delta) \) and step invariant input \( u(k\tau) \) for given sampling interval \( [k\tau, k\tau + \tau] \), where \( \mu = 0, 1, 2, \ldots, (N - 1), k = 0, 1, 2, \ldots \) and \( \Delta = \frac{2}{N}, N \) is integer.
- Second, based on lifted system, necessary and sufficient conditions is derived for existence and a constructive procedure is to be presented for estimation of functional observer of the form (2).

III. Discretization of Continuous Time Delay System

Consider the solution of (1) to calculate the values of the state \( x(t) \) given by
\[
x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\alpha)}A_1x(\alpha - \tau_x) \, d\alpha + \int_{t_0}^{t} e^{A(t-\alpha)}Bu(\alpha) \, d\alpha
\]  
where \( t_0 \) is initial time and \( t \) is time at which system is to be computed. A continuous-time system of structure represented in (1) is discretised at a sampling period \( \tau \) seconds. It is assumed that system (1) is driven by an input that is piecewise constant over the sampling interval, i.e. the zero-order hold (ZOH) assumption holds true:
\[
u(\alpha) = u(k\tau); \ k\tau \leq \alpha < k\tau + \tau
\]  
The delay-time \( \tau_x = k_x\tau \), where \( k_x \) is whole number, yield the following best-approximated discrete-time model
\[
\begin{align*}
x((k+1)\tau) &= \Phi(\tau)x(k\tau) + \Phi_d(\tau)x((k-k_x)\tau) + \Gamma(\tau)u(k\tau) \\
y(k\tau) &= Cx(k\tau)
\end{align*}
\]  
where
\[
\begin{align*}
\Phi(\tau) &= e^{A\tau} \\
\Phi_d(\tau) &= \int_{0}^{\tau} e^{A\alpha}A_1 \, d\alpha \\
\Gamma(\tau) &= \int_{0}^{\tau} e^{A\alpha}B \, d\alpha
\end{align*}
\]  
Remark 1: It is to be understood that exact discretisation of a time-delay system is impossible to be represented in a closed form [16]. This is a result of the fact that the actual contribution of the delayed state in the \( \tau \) system is
\[
\int_{k\tau}^{(k+1)\tau} e^{A((k+1)\tau-\alpha)}A_1x(\alpha - \tau_x) \, d\alpha
\]  
and it is approximated to
\[
\left( \int_{0}^{\tau} e^{A\alpha}A_1 \, d\alpha \right) x(k) = \Phi_d(\tau)x(k)\]
for the purpose of obtaining a closed-form representation.

A. Multirate Output Model Derivations

Consider the case in which sampling period \( \tau \) sec, is divided into equal sub-interval of \( \Delta = \frac{\tau}{N} \), where \( N \) is integer. The sampled output at these sub-intervals will be
\[
y(k\tau + \mu\Delta) = Cx(k\tau + \mu\Delta)
\]  
where \( \mu = 0, 1, \ldots, N - 1 \). From (3), the state computed at time instants \( t = k\tau + (\mu + 1)\Delta \) will be
\[
\begin{align*}
x(k\tau + \Delta) &= \Phi(\Delta)x(k\tau) + \Phi_d(\Delta)x((k-k_x)\tau) + \Gamma(\Delta)u(k\tau) \\
x(k\tau + 2\Delta) &= \Phi(2\Delta)x(k\tau) + \Phi_d(2\Delta)x((k-k_x)\tau) + \Gamma(2\Delta)u(k\tau)
\end{align*}
\]  
proceeding on similar lines
\[
\begin{align*}
x((k+1)\tau) &= \Phi(N\Delta)x(k\tau) + \Phi_d(N\Delta)x((k-k_x)\tau) + \Gamma(N\Delta)u(k\tau)
\end{align*}
\]  
Substitute (11), (12) and (13) in (10) results into
\[
y_{k+1} = C_0x(k) + C_dx((k-k_x)\tau) + D_0u(k)
\]  
Hence, a lifted system been constructed by updating input at every \( \tau \) sec and output samples sampled at \( \Delta \) sec, will be given by (5) and (14). The known constant matrices \( C_0 \in \mathbb{R}^{Nn \times n}, C_d \in \mathbb{R}^{Nn \times n}, D_0 \in \mathbb{R}^{Nm \times r} \), and output vector during sampling period \( y_{k+1} \in \mathbb{R}^{Nm \times 1} \) are given by
\[
C_0 = \begin{bmatrix} C \\ C\Phi(\Delta) \\ C\Phi(2\Delta) \\ \vdots \\ C\Phi(\tau - \Delta) \end{bmatrix}, C_d = \begin{bmatrix} 0 \\ C\Phi_d(\Delta) \\ C\Phi_d(2\Delta) \\ \vdots \\ C\Phi_d(\tau - \Delta) \end{bmatrix}
\]
\[
D_0 = \begin{bmatrix}
0 \\
CT(\Delta) \\
CT(2\Delta) \\
\vdots \\
CT(\tau - \Delta)
\end{bmatrix}
\]

and

\[
y_{k+1} = \begin{bmatrix}
y(k\tau) \\
y(k\tau + \Delta) \\
y(k\tau + 2\Delta) \\
\vdots \\
y((k+1)\tau - \Delta)
\end{bmatrix}
\]

The detailed derivation of \(y_{k+1}\) and coefficients in (14) were given by Janardhanan and Bandyopadhyay [13] for general plant. The restriction on \(N\) for computing full state in [13], is that it must be greater than or equal to the observability index \(\nu\), where \(\nu\) is the smallest positive integer such that

\[
\text{rank} \begin{bmatrix}
C \\
C\Phi(\Delta) \\
\vdots \\
C\Phi^{\nu-1}(\Delta)
\end{bmatrix} = \text{rank} \begin{bmatrix}
C \\
C\Phi(\Delta) \\
\vdots \\
C\Phi^\nu(\Delta)
\end{bmatrix}
\]

By choosing \(N \geq \nu\), an estimate of full state at \(k\)th sampling period is computed from linear combination fast output during \(k\)th period \(y_k\) and past input \(u(k-1)\).

Recently, Janardhanan and Satyanarayana [14] proposed a new method for computation of linear function of state vector for systems without delay. In [14], an algorithm is proposed with number of output samples less than observability index. The algorithm in [14] has been extended for the design of sliding mode controller based on functional observation and multirate output sampling to the systems with uncertainty in [15].

IV. EXISTENCE OF FUNCTIONAL OBSERVER CONDITION

Consider the lifted discrete-system, whose output is sampled at every \(\Delta\) sec, and input updated once in a sampling period \(\tau\) sec, given by (5) and (14). Let us define \(N\) being the smallest integer such that

\[
\text{rank} \Omega = \text{rank} \begin{bmatrix}
L\Xi \\
\Omega
\end{bmatrix}
\]

with

\[
\Xi = \begin{bmatrix}
\Phi(\tau) \\
\Phi_d(\tau)
\end{bmatrix}
\]

\[
\Omega = \begin{bmatrix}
C_0 \\
C_d
\end{bmatrix}
\]

From (16) there exist \(\Lambda \in \mathbb{R}^{p \times mN}\) such that

\[
L\Xi = \Lambda \Omega
\]

Multiplying (19) with \(\pi(k)\) from right side and using (5) and (14) results into

\[
Lx(k) = \Lambda y_k + (LT_\tau - \Lambda D_0)u(k-1)
\]

where

\[
\pi(k) = \begin{bmatrix}
x(k-1) \\
x(k-k_x - 1)
\end{bmatrix}
\]

Hence required function (2) realized as linear combination of fast sampling output and previous input.

Remark 2: The feasibility of (19) will not be dependent on the condition of number of output samples \(N\) should be greater than observability index of plant. Therefore, the complete observability of system is relaxed for estimation of functional. This is the one of the unique feature of proposed technique.

Theorem 3: Eqn. (20) is an estimate of \(Lx(k\tau)\), if and only if there exist \(N\), being the smallest integer such that (16) is satisfied. Then (20) is multirate output sampling based functional observer for (2).

Proof: Noticing that \(\Phi_\tau = e^{A\tau}\) is always invertible, it is possible to derive \(x(k)\) from (5) and substitute its value into (14), thus obtaining

\[
y_{k+1} = \tilde{C}_0x(k+1) + \tilde{C}_d x(k-k_x) + \tilde{D}_0 u(k)
\]

where, \(
\tilde{C}_0 = C_0 \Phi^{-1}(\tau)\) and

\[
\tilde{C}_d = C_d - \tilde{C}_0 \Phi_d(\tau)
\]

\[
\tilde{D}_0 = D_0 - \tilde{C}_0 \Gamma(\tau)
\]

Equation (22) is a basic formula of the multirate sampling of plant output. Rewriting (22) results

\[
\tilde{C}_0 x(k+1) = y_{k+1} - \tilde{C}_d x(k-k_x) - \tilde{D}_0 u(k)
\]

Multiplying (25) with \(\Lambda\) from left side, simplifying subject to satisfaction of (19) we get

\[
\Lambda \tilde{C}_0 = L, \quad \Lambda \tilde{C}_d = 0
\]

then it is easy to see that

\[
Lx(k+1) = \Lambda y_k + (LT_\tau - \Lambda D_0)u(k)
\]

i.e., the desired functional \(Lx(k)\) is equivalently realized by the concept of multirate sampling of plant output for all \(k \geq 1\). The condition (26) is equivalent to (16).

V. DESIGN PROCEDURE AND NUMERICAL EXAMPLE

In this section we develop the procedure to design functional observer when the conditions of existence are satisfied. Later a numerical example is given. Through the example, the design procedure and advantages of the proposed functional observer is illustrated.
A. Functional design procedure

Consider the lifted discretized representation (5), (14) of continuous-time system (1) with input changed once in a every sampling period $\tau$ sec and output is sampled at every $\Delta = \frac{\tau}{N}$ sec. With a minimum value of multirate output samples $N$ there exist a matrix $\Lambda$ such that (19) is satisfied. Then, the required functionals $Lx(k)$ can be expressed as linear combination of fast output samples and previous input as

$$Lx(k) = \Lambda y_k + Qu(k - 1) \quad (28)$$

where

$$Q = LT\Gamma - \Lambda D_0 \quad (29)$$

**Algorithm 1** Design Algorithm

1. consider the design of linear functional of state vector $Lx(k)$.
2. Choose a suitable sampling period $\tau$ seconds, such that $\tau_x = k_x \tau$.
3. Choose the number of output samples $N < \nu$.
4. Construct lifted multirate model of the plant according (5) and (14).
5. Check for the condition (16) with a minimum number of output samples $N$.
6. Determine matrix $\Lambda$ so that (19) is satisfied.
7. Derive matrix $Q$ from (29).
8. The functional $Lx(k)$ will be estimated according to (28).

B. Numerical example

In this section a numerical example is given. Through the example, the procedure and advantages of the proposed functional observer design is illustrated.

Let us consider the following system matrices of (1):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1.4 & 3 \\ -1 & 0.5 & -0.75 \\ 3.24 & -4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1.3 \\ -8.9 \\ -2.5 \end{bmatrix},$$

and $\tau_x = 0.2$ seconds. Using multirate sampling of plant output, we proposed to design a linear functional (2) with

$$L = \begin{bmatrix} -2 & -9 & -0.9 \end{bmatrix}$$

Here the selection functional gain need not be explained in detail. The (nominal) parameters assumed for functional observer design are $\tau = 0.2$ sec. In the first step, discretizing this continuous-time system, we get state dynamics as

$$x(k + 1) = \Phi(\tau)x(k) + \Phi_d(\tau)x(k - 1) + \Gamma(\tau)u(k)$$

where,

$$\Phi(\tau) = \begin{bmatrix} 0.9974 & 0.1986 & 0.0193 \\ -0.0386 & 0.9781 & 0.1889 \\ -0.3779 & -0.2275 & 0.8837 \end{bmatrix}$$

$$\Phi_d(\tau) = \begin{bmatrix} -0.3840 & 0.2862 & 0.5858 \\ -0.1412 & 0.0185 & -0.1392 \\ 0.5581 & -0.8210 & 0.0898 \end{bmatrix}$$

$$\Gamma(\tau) = \begin{bmatrix} 0.0026 \\ 0.0386 \\ 0.3779 \end{bmatrix}$$

With output samples $N = 1$, the $rank[\Omega] = 1$ and the condition (16) is satisfied. Hence, there exist $\Lambda$ such that the functional $Lx(k)$ is estimated using (28)

$$Lx(k) = y_k - 0.6923u(k - 1) \quad (30)$$

Figure. 1 shows the error between functional $Lx(k)$ and its estimate. From Figure. 1, it is clear that proposed method will exactly estimate predefined functional after one sampling period.

C. Discussions on results

From system matrices, the observability index is 3. To estimate functional based on earlier multirate output sampling technique [13], requires $N$ should be at least 3 against $N = 1$ in this method. The proposed method is simple and straight forward as it computes the required (predefined) functional in terms of only input and output and there is no dynamics are involved. The estimated functional will exactly equal to required functional in one sampling period.
VI. CONCLUSIONS

In this paper a new technique for computation of functional observer for LTI system with delays in state variables is presented using the concept of multirate output sampling. It has been demonstrated that the number of output samples required to exactly compute a linear function of state vector is less than the observability index, which in turn minimum value for computation of full state. The implementation of functional observer is simple as it does not have any dynamics and uses only multirate sampling output and past input, hence it can be termed as open-loop functional observer. Through numerical example the design procedure and its advantages are illustrated.

REFERENCES