Cubic graphs with large circumference deficit

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Abstract

The circumference \( c(G) \) of a graph \( G \) is the length of a longest cycle. Exploiting our recent results on resistance of snarks, we provide upper bounds on the circumference ratio \( c(G)/|V(G)| \) for classes of cyclically 4-, 5- and 6-edge-connected cubic graphs. In addition, we construct snarks with large girth and large circumference, solving Problem 1 proposed in [J. Hägglund and K. Markström, On stable cycles and cycle double covers of graphs with large circumference, Disc. Math. 312 (2012), 2540–2544].

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Classification: 05C15, 05C38

1 Introduction

A cycle is one of the most basic structures in a graph, so it comes as no surprise that cycles have been analysed from the very beginnings of graph theory. This article focuses on longest cycles in cubic graphs. The circumference \( c(G) \) of a graph \( G \) is the length of a longest cycle. The circumference ratio is the ratio of circumference to order. The circumference deficit is the difference between order and circumference.

A lot of attention has been given to Hamiltonian graphs, that is, graphs with zero circumference deficit. Compared to the vast tomes written on hamiltonicity, non-Hamiltonian graphs appear rather neglected despite there is plenty of investigation to be done. The problem of determining the circumference of a given graph is NP-hard and even approximation is a very tough problem [5], so no simple characterisations are expected.

It transpired in many areas that the most interesting cubic graphs are those with circular chromatic index four. Such graphs are called snarks; we will additionally require snarks to have girth at least five and cyclic edge-connectivity at least four. We will encounter them in Section 3 where we prove the existence of a snark of girth at least \( g \) and circumference deficit at least \( g \) for every integer \( g \). Section 2 is devoted to upper bounds on circumference ratio; we provide linear bounds for cyclically 4-, 5-, and 6-edge-connected cubic graphs.
Each snark can be transformed into a 3-edge-colourable graph by removing sufficiently many edges. The least number of edges that need to be removed is the resistance of the snark. There are snarks with arbitrarily large resistance (see e.g. [14, 12]). Most of our constructions are based on building blocks created from snarks with large resistance; usefulness of such blocks is demonstrated in Lemma 1.

2 Circumference ratio of cubic graphs

Circumference ratio of cubic graphs strongly depends on connectivity. Since each vertex of a cubic graph is separated by three edges from the rest of the graph, the classical notions of vertex-connectivity and edge-connectivity are of limited use. The most important parameter here is cyclic edge-connectivity. A graph is cyclically $k$-edge-connected if at least $k$ edges must be removed to disconnect it into components among which there are at least two containing a cycle. For cubic graphs, the notion of cyclic $k$-edge-connectivity coincides with $k$-vertex-connectivity and $k$-edge-connectivity for $k \in \{1, 2, 3\}$ [13].

If we allow bridges in our graphs, there are infinitely many trivial cubic graphs with circumference 5. What is more interesting, Bondy and Entringer [2] proved that every 2-edge-connected cubic graph $G$ contains a cycle of length at least $4\log |V(G)| - 4\log \log |V(G)| - 20$. This bound is essentially best possible, as shown by Lang and Walther [11]. Bondy and Simonovits [3] conjectured the existence of a constant $c$ such that every 3-connected cubic graph $G$ has circumference at least $|V(G)|^c$ and showed that $c \leq \log_9 8 \approx 0.946$. The conjecture was verified by Jackson [9] for $c = \log(1 + \sqrt{5}) - 1 \approx 0.694$; the constant $c$ has been improved to 0.753 recently [1].

Bondy also conjectured the existence of a constant $c$ such that every cyclically $4$-edge-connected cubic graph $G$ has circumference at least $c|V(G)|$ (see [9, Conjecture 1]). The dominating cycle conjecture [6] implies that $c \geq 0.75$ since a dominating cycle in a cubic graph $G$ has length at least $0.75|V(G)|$. In addition, Thomassen [15] conjectured that there exists an integer $k$ such that every cyclically $k$-edge-connected cubic graph is Hamiltonian. This would mean $c = 1$ for sufficiently connected cubic graphs.

We summarize the currently known results together with our contributions in Table 1. The column LB displays a lower bound which holds for all graphs; the column UB shows an upper bound on circumference of an infinite class of graphs. All the bounds are asymptotic.

The crucial observation used in our construction of graphs with large circumference deficit is captured in the following lemma.

<table>
<thead>
<tr>
<th>connectivity</th>
<th>LB</th>
<th>conjectured LB</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\log n$</td>
<td></td>
<td>$\log n$</td>
</tr>
<tr>
<td>3</td>
<td>$n^{0.753}$</td>
<td></td>
<td>$n^{0.946}$</td>
</tr>
<tr>
<td>4</td>
<td>$n^{0.753}$</td>
<td>0.75$n$</td>
<td>0.875$n$ *</td>
</tr>
<tr>
<td>5</td>
<td>$n^{0.753}$</td>
<td>0.75$n$</td>
<td>0.960$n$ *</td>
</tr>
<tr>
<td>6</td>
<td>$n^{0.753}$</td>
<td>0.75$n$</td>
<td>0.990$n$ *</td>
</tr>
<tr>
<td>7+</td>
<td>$n^{0.753}$</td>
<td>0.75$n$ and $n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 1: Summary of results on circumference (our contribution marked by *).
Lemma 1. Let $H$ be a subgraph of a bridgeless cubic graph $G$. If $H$ has resistance $k$, then any cycle in $G$ not contained in $H$ misses at least $k$ vertices of $H$.

Proof. If a cycle $C$ of $G$ is not contained in $H$, then its intersection with $H$ is a union of vertex-disjoint paths. Take each of those paths in turn and colour the edges along the path alternately by colours 1 and 2. Colour all the remaining edges by 3. There are two possibilities for a vertex $v$ of $H$: either its incident edges are coloured by 1, 2, 3, or all of its incident edges are coloured by 3 (we call such a vertex bad). After removing all the bad vertices, we obtain a 3-edge-colourable graph from $H$. However, $H$ has resistance $k$, and thus there are at least $k$ bad vertices. Obviously, no bad vertex belongs to $C$, hence $C$ misses at least $k$ vertices of $H$.

In order to construct infinite classes of graphs with circumference ratio promised in Table 1, we employ cubic construction blocks described in [12]. We will give neither a detailed description nor a proof of all their properties here because we are only interested in certain parameters: order, resistance, and connectivity properties.

The cubic graph $N_2$ has order 26, resistance 2, and two pairs of dangling edges (see Figure 1 and [12, Section 7]). We create a graph $G$ by arranging $m \geq 2$ copies of $N_2$ along a circle and for each copy $K$, we connect one pair of dangling edges of $K$ to a pair of dangling edges of the following copy and the other pair of dangling edges to a pair in the previous copy. (The exact way of how we do it does not matter because we only need to preserve cyclic 4-edge-connectivity.) According to Lemma 1, any cycle of $G$ misses at least 2 vertices in each of the $m$ copies of $N_2$ (or has length at most 26), hence $G$ has circumference deficit at least $2m$ and circumference ratio at most $24m/26m = 12/13 \approx 0.92$. A different idea used in Theorem 2 leads to an even better upper bound for cyclically 4-edge-connected graphs.

The construction described in the previous paragraph can be repeated with the cyclically 5-edge-connected building block $Z$ with order 25 and resistance 1 (see Section 8 and Figure 5 in [12] for a description of $Z$; this block was also used by Steffen under the name $T$ [14, Theorem 2.3]). The graph $Z$ has seven dangling edges naturally split into two triples and one single dangling edge. We repeat the circular construction with $m$ copies of $Z$ in place of $N_2$; the role of the pairs of dangling edges is now played by the triples. The single dangling edges are joined to a cycle of length $m$. The resulting graph has circumference ratio $24/25 = 0.96$ and is cyclically 5-edge-connected.

The same construction can also be used to construct cyclically 6-edge-connected graphs; however, the details are more complicated. Section 9 of [12] describes a cyclically 6-edge-connected graph $M_r$ of order $99r$ with resistance at least $r$ for each even positive integer $r$, but there are no dangling edges in this graph, thus we cannot use it directly: we first have to cut a few suitable edges to obtain a block which would allow a construction of 6-edge-connected graphs.

The graph $M_r$ is obtained by symmetrically applying superposition to a graph $L_r$ composed of $r$ isomorphic copies of the Petersen graph with one vertex removed. Therefore, $M_r$ also contains $r$ isomorphic blocks $A_1, A_2, \ldots, A_r$ arranged along a circle. Each two
consecutive blocks of $M_r$ are joined by three edges. We cut all the edges between $A_1$ and $A_2$ to form a cubic graph $M'_r$ with two triples of dangling edges. The graph $M'_r$ has order $99r$ and resistance at least $r - 3$. (According to the definition of resistance, the removal of an edge can decrease resistance by at most 1. The resistance of $M'_r$ is actually $r$, but that would require a detailed proof; we will use the obvious lower bound of $r - 3$ here because it is sufficient for our purpose.)

Consequently, we can use $M'_r$ in place of $N_2$ in the above-described construction (with triples of dangling edges instead of pairs). The resulting cyclically 6-edge-connected graph $G$ has order $m \cdot 99r$ and circumference deficit at least $m(r - 3)$, thus its circumference ratio is at most $1 - (r - 3)/99r$. By taking a sufficiently large $r$ we can make this ratio to be arbitrarily close to $98/99 \approx 0.990$.

**Theorem 2.** For each integer $m$, there exists a cyclically 4-edge-connected cubic graph with order $8m$ and circumference $7m + 2$.

**Proof.** Let $u$ and $v$ be two adjacent vertices of the Petersen graph $P$. We remove the path $uv$, but keep the dangling edges incident to exactly one of its endvertices; the dangling edges incident to $u$ are input edges and the edges incident to $v$ are output edges. We say that a path passes through $B$ if it starts with a vertex incident to an input edge and ends in a vertex incident to an output edge. We say that a cycle passes through $B$ if a portion of this cycle (a path) passes through $B$. The resulting graph $B$ has two properties interesting to us.

First, if a path passes through $B$, it cannot pass through all the vertices of $B$: otherwise we would be able to extend this path by $u$ and $v$ to a Hamiltonian cycle of $P$, but it has no such cycle. Second, if we take two disjoint paths passing through $B$, there is at least one vertex of $B$ missed by both of these paths. Otherwise, we can extend the first path by $u$, extend the other path by $v$, and then concatenate them together by adding two edges to form a Hamiltonian cycle of $P$ which is a contradiction.

Let $G$ be the graph obtained from $m$ copies of $B$ arranged along a circle in such a way that the output edges of each copy are identified with the input edges of the following copy (see Figure 2). The graph $G$ is cyclically 4-edge-connected and has order $8m$.

Let $C$ be a cycle in $G$. Note that $C$ passes through each copy of $B$ at most twice. According to the two properties of $B$ proved above, no matter how many times $C$ passes through $B$, at least one vertex of $B$ is missed. The only possibility for $C$ to contain all vertices of $B$ is to enter by an input edge and then leave by the other input edge (of course, it can also both enter and leave by output edges). Consequently, the cycle $C$ misses at least one vertex in each of at least $m - 2$ copies of $B$, and thus the circumference of $G$ is at most $8m - (m - 2) = 7m + 2$. Since $G$ contains a cycle of length $7m + 2$, the derived upper bound on its circumference is tight.

We propose the following strengthening of both the conjecture of Bondy [9, Conjecture 1] and the dominating cycle conjecture.
Conjecture 1. Every cyclically 4-edge-connected cubic graph has circumference ratio at least $7/8$.

3 Large girth and large circumference deficit

The motivation for this section is provided by Cycle Double Cover Conjecture (CDCC). Huck [8] showed that the smallest possible counterexample to CDCC has girth at least 12. Brinkmann et al. [4, 7] proved that if a bridgeless cubic graph $G$ has a cycle of length at least $|V(G)| - 10$, then $G$ has a cycle double cover. Put together, smallest counterexamples to CDCC can only be found in the class of snarks with girth at least 12 and with circumference deficit at least 11. Since no such snark has been known before, the following problem is very relevant.

Problem (Hägglund and Markström [7]). Let $g$ be a constant. Construct a snark of girth at least $g$ and circumference deficit at least $g$.

We give a solution to this problem in Theorem 3. The construction used in the proof of Theorem 3 can be modified to produce snarks with arbitrarily large girth and linear circumference deficit.

Theorem 3. For every integer $g$ there exists a snark with girth at least $g$ and circumference deficit at least $g$.

Proof. We construct the desired snark for every integer $g \geq 5$ which is enough to prove the theorem.

Let $H_0$ be a snark of girth at least $g$; the existence of such snarks has been proved by Kochol [10]. Let $v_1$ and $v_2$ be two adjacent vertices of $H_0$ and let $e_i$ and $f_i$, for $i \in \{1, 2\}$, be the two edges incident to $v_i$ and not incident to $v_{3-i}$. Let $H_1$ be the cubic graph obtained from $H_0$ by removing $v_1$ and $v_2$ while keeping the dangling edges $e_1, e_2, f_1, f_2$. The well-known Parity Lemma assures that the edges $e_1$ and $f_1$ have the same colour in every 3-edge-colouring of $H_1$ (otherwise $H_0$ would be 3-edge-colourable, but it is a snark).

Take two copies of $H_1$ and join them as indicated in Fig. 3 (the edge $f_1$ of the first copy is identified with the edge $f_1$ of the second copy and the edges $e_1$ of both copies are attached to an additional vertex $v$). If $H$ was 3-edge-colourable, then the colour of $e_1$ of the first copy of $H_1$ would be the same as the colour of $f_1$ and, in turn, the same as the colour of $e_1$ of the second copy, and we would get a contradiction at $v$. Hence, $H$ is not colourable and has resistance at least 1. Moreover, there is no cycle of length less than $g$ in $H$, and any path both starting and ending at a vertex incident to a dangling edge of $H$ passes through at least $g - 1$ vertices of $H$.

Let $G$ be a cubic graph obtained from $g$ copies of $H$ arranged along a circuit in such a way that two dangling edges of a copy of $H$ are attached to the previous copy and two of them are attached to the next. The remaining $g$ edges can be joined to a cycle of length
in an arbitrary way (preserving maximum degree 3). The graph $G$ clearly has girth $g \geq 5$, is cyclically 4-edge-connected and is not 3-edge-colourable. Any cycle in $G$ misses at least one vertex in each copy of $H$ thanks to Lemma 1, and thus $G$ has circumference deficit at least $g$.

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References


