On even cycle decompositions of 4-regular line graphs

Edita Mácaiová¹, Ján Mazák²

{macajova, mazak}@dcs.fmph.uniba.sk

¹ Univerzita Komenského, Mlynská dolina, 842 48 Bratislava
² Trnavská univerzita, Priemyselná 4, 918 43 Trnava

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Abstract

We prove that the Petersen colouring conjecture implies a conjecture of Markström saying that the line graph of every bridgeless cubic graph is decomposable into cycles of even length. In addition, we describe two infinite families of 4-regular graphs: the first family consists of 3-connected graphs with no even cycle decomposition and the second one consists of 4-connected signed graphs with no even cycle decomposition.

1 Introduction

Decompositions of graphs into paths, cycles, trees or various other structures constitute a vast research area in graph theory—even entire books have been written on diverse individual problems from this area. One of the very old problems is the decomposition of an Eulerian graph into cycles (2-regular connected subgraphs), which is easily seen to be always possible, but becomes much more interesting if we impose additional restrictions on the cycles in the decomposition. A natural restriction that has recently caught attention is that all the cycles in the decomposition must have even length (such cycles are called even). These even cycle decompositions (ECDs for short) had been addressed in 1981 by Seymour, who proved in [11] that any planar Eulerian graph has an ECD, and then reappeared again and again. Zhang [13] improved Seymour’s result to $K_5$-minor-free Eulerian graphs and conjectured that $K_5$ is the only 3-connected Eulerian graph with no ECD. This conjecture was refuted by Jackson [4] and later by Rizzi [10] who constructed an infinite family of 4-connected Eulerian graphs with no ECD.

Let $P$ be the Petersen graph and let $L(G)$ denote the line graph of a simple graph $G$, that is, the vertices of $L(G)$ bijectively correspond to the edges of $G$ and two vertices are adjacent if and only if their corresponding edges are incident. We denote by $e(v)$ the edge of $G$ corresponding to a vertex $v$ of $L(G)$.

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There are three very recent articles [2, 3, 7] on ECDs; apart from certain theorems, they also offer interesting open problems. In the first research direction, Markström [7] considered ECDs of 4-regular graphs. He constructed an infinite family of 2-connected 4-regular graphs with no ECD and gave an example of a 3-connected such graph. Rizzi's construction [10] does not apply here since his graphs also contain vertices of degree 6. Our Theorem 1 asserts the existence of an infinite family of 3-connected 4-regular graphs with no ECD. Markström conjectured that every line graph of a bridgeless cubic graph has an ECD [7, Conjecture 3.1] and proved this conjecture for cubic graphs with oddness at most 2. The conjecture has been verified for graphs with at most 36 vertices by Brinkmann et al. in [1]. We show in Section 3 that Markström's conjecture is a consequence of the well-known Petersen colouring conjecture [5, 6].

Petersen colouring conjecture. If $G$ is a bridgeless cubic graph, then there is a mapping $p : E(G) \to E(P)$ such that the three edges incident with an arbitrary vertex of $G$ map bijectively onto the three edges incident with a vertex of $P$.

In the second direction, two interesting articles [2, 3] appeared that consider even cycle decompositions of signed graphs. A signed graph $(G, \Sigma)$ is an undirected simple graph $G$ together with a signature $\Sigma \subseteq E(G)$. The edges in $\Sigma$ are negative; all the other edges of $G$ are positive. A cycle of $G$ is balanced if it contains an even number of negative edges. An even cycle decomposition of $(G, \Sigma)$ is a decomposition of $G$ into balanced cycles. This is, of course, possible only if $\Sigma$ has an even number of elements. Our contribution consists in constructing an infinite class of 4-connected 4-regular signed graphs with no ECD (see Theorem 2). We have also considered signed 4-regular line graphs and pose the following question.

Question 1. Let $G$ be a bridgeless cubic graph. Is it true that $(L(G), \Sigma)$ has an ECD for any $\Sigma \subseteq E(L(G))$ such that $|\Sigma|$ is even?

The assumption of no bridges in $G$ is important because some cubic graphs containing a bridge have a signature not allowing any ECD; the smallest of them has 10 vertices. To put our question in context, Huynh et al. [2] proved that every simple Eulerian odd-$K_4$-minor-free signed graph $G$ with an even number of odd edges has an ECD and conjectured that the condition can be replaced by $G$ being odd-$K_5$-minor-free signed graph. (A signed graph is odd-$H$-minor-free if it contains no signed minor isomorphic to $H$ with all edges negative; we refer the reader to the article [2] for more details on minors in signed graphs.) Huynh et al. [3] also proved that all simple Eulerian complete $k$-partite graphs different from $K_5$ have an ECD. Obviously, line graphs of cubic graphs are almost never complete multipartite graphs and they can contain arbitrarily large complete minors, hence our question has not been answered yet.

With the help of a computer and the well-known generator genreg [8] we have verified that the answer to Question 1 is positive for all signed graphs arising from line graphs of bridgeless cubic graphs with at most 10 vertices.

2 Families with no ECDs

Theorem 1. There exists an infinite family of 3-connected 4-regular graphs with no ECD.

Proof. Let $C = v_0v_1 \ldots v_nv_0$ be a cycle of length $n + 1$ with each edge replaced by a pair of parallel edges (for $n = 1$, we take a pair of vertices joined by four parallel edges instead of $C$). For $i \in \{1, 2, \ldots, n\}$, we replace the vertex $v_i$ with a copy $H_i$ of $K_4$ and attach
the four edges incident with $v_i$ to different vertices of $K_4$ (the construction is depicted in Figure 1 left). The resulting graph $G_n$ is clearly 4-regular and 3-connected. We prove by induction that $G_n$ has no ECD.

The graph $G_1$ is isomorphic to $K_5$ and thus has no ECD. Assume that the graph $G_{n+1}$ has an ECD $D$ for some $n \geq 1$. At least one cycle of $D$ passes through both $v_0$ and $H_1$. If this cycle was a subgraph of $G[V(H_1) \cup \{v_0\}]$, then it would have length three or five or there would be a triangle in $D$ contained in $H_1$, and in both of these cases we have a contradiction with the properties of $D$. Hence all the cycles through $v_0$ that enter $H_1$ just pass through it and continue to $H_2$. If there are two such cycles, each of them contains two or four vertices of $H_1$; if there is just one such cycle, it has to pass $H_1$ two times and thus contains four vertices of $H_1$ (in both cases, there can be one 4-cycle of $D$ contained in $H_1$). In either case we can simply remove $H_1$ from $G_{n+1}$ and join the dangling edges of the resulting graph in the way indicated by the cycles of $D$ passing through $H_1$. The resulting graph is $G_n$ together with an ECD, but this contradicts the induction hypothesis.

**Theorem 2.** There exists an infinite family of 4-connected 4-regular signed graphs with no ECD.

**Proof.** First, we describe a useful operation. Let $G$ be a 4-regular signed graph. The blowup $G^v$ of $G$ at a vertex $v$ of $G$ is the signed graph obtained by replacing $v$ with a copy $H$ of $K_4$ with all edges positive; the neighbours of $v$ are joined to the vertices of $H$ bijectively while preserving the signs of the edges incident with $v$ in $G$. Note that if $G$ was 4-connected, then any blowup $G'$ of $G$ is 4-connected.

We define a family $\mathcal{F}$ of 4-connected 4-regular graphs as follows:

(i) $K_5$ with all edges negative belongs to $\mathcal{F}$;

(ii) if $(G, \Sigma) \in \mathcal{F}$, then any blowup of $(G, \Sigma)$ belongs to $\mathcal{F}$.

Finally, we prove that no graph $G$ from $\mathcal{F}$ has an ECD. We do this by induction on the number of blowups that has been performed to obtain $G$ from $K_5$. The graph $K_5$ has all edges negative, hence it has no ECD. Assume that $G^v$, a blowup of $G$, has an ECD $D'$. We can easily construct an ECD $D$ of $G$ from $D'$: the only thing we need to do is to replace all the vertices and edges of the inserted copy of $K_4$ in each cycle of $D'$ by $v$. This contradicts the induction hypothesis. \qed
3 Line graphs of cubic graphs

**Theorem 3.** If Petersen colouring conjecture is true, then the line graph of any bridgeless cubic graph has an ECD.

**Proof.** Let $G$ be a bridgeless cubic graph. If the Petersen colouring conjecture is true, then there is a mapping $p : E(G) \rightarrow E(P)$ such that for every vertex $v$ of $G$, the three edges incident with $v \in V(G)$ are mapped by $p$ into three distinct edges incident with a vertex $v'$ of $P$ (this naturally gives rise to a mapping $p_v : V(G) \rightarrow V(P)$ which maps $v$ to $v'$). The mapping $p$ can also be viewed as a homomorphism $h : L(G) \rightarrow L(P)$. Indeed, if two vertices $u$ and $v$ of $L(G)$ are adjacent, then their corresponding edges $e(u)$ and $e(v)$ are incident in $G$, and thus $p(e(u))$ and $p(e(v))$ are two incident edges of $P$, hence $h(u)$ and $h(v)$ are adjacent.

Let $D_P$ be the decomposition of $L(P)$ into even cycles depicted in Figure 1 right; the Petersen graph is embedded in the projective plane and its line graph is decomposed into four cycles with lengths 10, 8, 6, 6. Our aim is to define a decomposition $D_G$ of $L(G)$ into even cycles. We first decompose $L(G)$ into closed trails and then decompose these trails into even cycles. In order to describe the desired trails, it is sufficient to form two pairs from the four edges incident with each vertex of $L(G)$; each pair will belong to a trail in $L(G)$. Consider a vertex $v$ of $L(G)$; let $w$ and $w'$ be the endvertices of $e(v)$. There are two edges different from $e(v)$ incident with $w$; let $u_1$ and $u_2$ be the vertices of $L(G)$ corresponding to these two edges. We define $u_1'$ and $u_2'$ analogously: they correspond to the two edges of $G$ incident with $w'$. There are two possibilities: either $p_v$ maps both $w$ and $w'$ into the same vertex of $P$, or not. If $p_v(w) = p_v(w')$, we pair $u_1$ with $u_1'$ so that $h(u_1) = h(u_1')$. If $p_v(w) \neq p_v(w')$, we pair $u_1$ with $u_1'$ so that $h(u_1)$ and $h(u_1')$ are in the same cycle of the decomposition $D_P$.

The final step is to prove that all the trails in the constructed decomposition $D$ can be decomposed into even cycles. Let $T$ be the subgraph of $L(G)$ induced by a trail of $D$. According to the pairing described above, all the vertices of $T$ are mapped by $h$ into the same cycle $C$ of $D_P$. Therefore, the graph $h(T)$ is a subgraph of $C$, and since $C$ is even and $h$ is a homomorphism, $T$ must be bipartite. Consequently, $T$ can be decomposed into even cycles. The desired decomposition $D_G$ is obtained by putting all cycles arising from decompositions of all the trails together. \qed

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**References**


