Multi-Domain Fourier-Continuation/WENO Hybrid Solver for Conservation Laws

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Abstract

We introduce a multi-domain Fourier-Continuation/WENO hybrid method (FC-WENO) that enables high-order and non-oscillatory solution of systems of non-linear conservation laws, and which enjoys essentially dispersionless, spectral character away from discontinuities, as well as mild CFL constraints (comparable to those of finite difference methods). The hybrid scheme employs the expensive, shock-capturing WENO method in small regions containing discontinuities and the efficient FC method in the rest of the computational domain, yielding a highly effective overall scheme for applications with a mix of discontinuities and complex smooth structures. The smooth and discontinuous solution regions are distinguished using the multi-resolution procedure of Harten [J. Comput. Phys. 115 (1994) 319-338]. We consider WENO schemes of formal orders five and nine and a FC method of order five. The accuracy, stability and efficiency of the new hybrid method for conservation laws is investigated for problems with both smooth and non-smooth solutions. In the latter case, we solve the Euler equations for gas dynamics for the standard test case of a Mach three shock wave interacting with an entropy wave, as well as a shock wave (with Mach 1.25, three or six) interacting with a very small entropy wave and evaluate the efficiency of the hybrid FC-WENO method as compared to a purely WENO-based approach as well as alternative hybrid based techniques. We demonstrate considerable computational advantages of the new FC-based method, suggesting a potential of an order of magnitude acceleration over alternatives when extended to fully three-dimensional problems.

Keywords:
Fourier continuation Methods, high-order WENO methods, multi-resolution methods, conservation laws, shock waves

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1. Introduction

Solutions of high-speed flow problems are often characterized by a challenging combination of sharp gradients, discontinuities and regions of complex but smooth flow structures; examples of such situations are provided by the well known shock-induced multi-material flow instabilities, mixing, and acoustic noise generation by turbulent flows. To accurately capture all regimes in such complicated flow it is thus necessary to account accurately for both sharp nonlinear discontinuities as well as complex smooth flow structures. While it is well known that high-order accurate methods, the high-order finite difference methods or spectral methods, are well suited for the smooth elements of the solution, it is also well appreciated that such techniques introduce oscillatory behavior near discontinuities [20, 21].

A well-known alternative for the solution of such flows is based on a high-order weighted essentially non-oscillatory (WENO) finite difference method; in particular, high-order WENO algorithms have been used to produce successful simulations of the Rayleigh-Taylor instability [23] and Richtmyer-Meshkov instability [24, 25] in two and three space dimensions.

Unfortunately, the WENO finite difference methods are generally computationally very expensive and do not compare favorably with linear schemes such as finite difference or spectral methods. The reason for this is found in costly WENO operations: the characteristic decomposition of fluxes to control oscillations and the calculation of nonlinear weights for smooth and non-smooth flux contributions. Having in mind the need to model three-dimensional unsteady compressible flows, governed by the Navier-Stokes equations, it is essential to continue to strive for more efficient solution strategies.

We shall focus on problems in which the solution discontinuities are mainly local phenomena in both space and time. This suggests that a hybrid scheme could advantageously be used whereby the “expensive” WENO procedure is only used in parts of the space-time domain containing discontinuities, while a less costly, high-order numerical method is employed for the portion of the domain containing complex but smooth features. For problems with localized shocks and extended regions of mixing or acoustic noise propagation, we would expect such an approach to yield a considerably faster algorithm.

The core of such a hybrid approach is the use of an efficient numerical method for smooth regions together with an effective “smoothness-indicator” strategy for identification of areas of smoothness and discontinuity. For the former, a central difference (upwind-biased or unbiased) or a Chebyshev-based spectral methods are possible choices that have been explored previously, e.g., in [10, 12] a hybridization of spectral Chebyshev and WENO methods for conservation laws in one and two space dimensions is proposed and applied to a complex simulation of early-stage two-dimensional Richtmyer-Meshkov instability while a hybrid central difference-WENO scheme in one space dimension is presented in [11]. Both of these developments accurately identify discontinuities using the multi-resolution algorithm of Harten [9], based on differences of the point-values of a function and its high-order interpolated function on coarser grids [10, 12].
The advantage of the very high order Chebyshev spectral methods over the central difference methods lies in the higher order of accuracy near the non-periodic boundaries and, perhaps of most importance, solutions free of dispersion error (also known as pollution error). Absence of dispersion error is crucial for efficient simulation of problems with a wide range of spatial and temporal scales including transitional and turbulent flows [27].

As we shall discuss shortly, the hybrid scheme proposed here maintains these properties while overcoming some severe restrictions imposed by the scheme based on Chebyshev polynomials. A particular consequence of the use of the Chebyshev spectral method is the presence of grid points that cluster near the boundaries of each such interval or sub-domain, impacting the overall performance of the scheme in different ways. On one hand, the WENO finite difference method is defined on an equidistant grid. Thus, the two underlying grids in a Chebyshev-WENO hybrid do not conform in any natural manner which implies that extra interpolation operations are required for transfer of data at overlapping regions between adjacent domains with two different discretization schemes. This furthermore complicates the multi-resolution analysis. Furthermore, for explicit time integration strategies, a hybrid Chebyshev-WENO method requires a much smaller ($O(1/N^2)$) time step than a scheme based on an equidistant grid due to the nature of the approximation. This leads to a very stringent time-step restriction which is particularly problematic if long time integration and/or very high accuracy is required.

In this work we propose an attractive alternative in which we hybridize the WENO method with a recently proposed Fourier continuation (FC) method [1, 6, 7]. The FC approximation is based on a high-order periodic continuation of a (possibly non-periodic) function, yet being based on a Fourier method, the FC approximation has no dispersion (pollution) error and, utilizing an equispaced grid, provides a simple and efficient interface with WENO scheme and multi-resolution analysis while allowing for an efficient temporal integration through a much more relaxed restriction as compared to the Chebyshev-WENO scheme. At high resolution it furthermore allows for the use of the Fast Fourier Transform, enabling a further acceleration.

While we shall focus here on one dimensional problems, the work in [1, 6, 7] confirms that the extension to multiple dimensions is entirely possible and we expect the proposed FC-WENO approach to be applicable and, indeed, be highly competitive for systems of conservation laws in both two- and three-dimensional space.

The remainder of the paper is organized as follows. In Sec. 2 we describe the Fourier continuation (FC) method for the approximation of functions, and in Sec. 3 we discuss our multi-domain FC-WENO hybrid method for one-dimensional conservation laws. This sets the stage for Sec. 4 where we present a variety of numerical results for problems governed by the advection equation, the inviscid Burgers’ equation and the Euler equations. Among other things, we demonstrate the accuracy and stability of the FC method in the context of advection problems with smooth solutions in both single-domain and multi-domain formulations, and we assess the overall efficiency of our method in the
context of challenging computational problems—including nonlinear systems of
conservation laws with discontinuous solutions. In Sec. 5 we conclude with a
few remarks and suggested directions of future research.

2. The Fourier Continuation approximation

Given a (typically non-periodic) function \( f \) defined over the interval \([0, 1]\)
and with values of \( f \) given at an \( N \)-point equi-spaced grid \( \{x_k\}_{k=0}^{N-1} \subseteq [0, 1] \),
the Fourier Continuation (Extension) method seeks to produce, on the basis of
\( \{f(x_k)\}_{k=0}^{N-1} \), a periodic function \( \tilde{f} \) expressed by a finite number \( M \) of Fourier
modes,

\[
\tilde{f}(x) = \sum_{j \in g(M)} \tilde{a}_j e^{(\frac{2\pi i}{1+d}jx)}.
\]

and defined over a larger domain \([0, 1 + d]\) in such a way that it matches closely
the original function \( f \) throughout the original interval \([0, 1]\). Here \( g(M) = \{j \in \mathbb{N}| -M/2 + 1 \leq j \leq M/2\} \) for \( M \) even and \( g(M) = \{j \in \mathbb{N}| -(M - 1)/2 \leq j \leq (M - 1)/2\} \) for \( M \) odd. In general, \( N \) must be chosen larger than \( M \) due to the
intrinsic ill-conditioning of the resulting linear system; as shown in [2, 5], the
least-squares solution via singular value decomposition of the over-determined
system

\[
\sum_{j \in g(M)} \tilde{a}_j e^{(\frac{2\pi i}{1+d}jx)} = f(x_k) \quad \forall k = 0, \cdots, N - 1,
\]

for the coefficients \( \tilde{a}_j \) leads to an effective Fourier Continuation strategy.

We, however, observe that a smooth and accurate continuation of \( f(x) \) does
not necessarily require the value of the function over the entire interval \([0, 1]\)
and only the function values on small intervals near the boundaries (boundary
intervals) are sufficient. A continuation method based on this philosophy re-
quires finding a matching function \( \tilde{h} \) defined over \([1 - \Delta, 1 + 2d + \Delta]\) with
periodicity \( 2d + 2\Delta \) as

\[
\tilde{h}(x) = \sum_{j \in g(M)} \tilde{a}_j e^{(\frac{2\pi i}{1+\Delta}jx)},
\]

\( \tilde{h}(x) \) matches the original function \( f(x) \), in the least squares sense, as

\[
\tilde{h}(z_k + 1 - \Delta) = f(z_k + 1 - \Delta) \quad (4a)
\]
\[
\tilde{h}(1 + d + z_k) = f(z_k) \quad (4b)
\]

for a sufficiently large fine equi-spaced grid points \( z_k \) defined over \([0, \Delta]\). The
Fourier continuation of \( f(x) \), \( \tilde{f}(x) \), is then obtained as

\[
\tilde{f}(x) = f(x) \quad 0 \leq x \leq 1, \quad (5a)
\]
\[
\tilde{f}(x) = \tilde{h}(x) \quad 1 < x \leq 1 + d. \quad (5b)
\]

Fig. 1 illustrates the basic idea behind the continuation and the matching
function for a sample function.

As an alternative approach for decreasing the impact of the ill-conditioning without incurring the high \( \mathcal{O}(N^3) \) computational cost arising from the evaluation of the singular value decomposition \([8]\), \([6]\) proposes a different strategy based on the use of Gram polynomials \([29]\). Since this is a key element of our hybrid scheme, we shall discuss this in more detail.

Let the left and right boundary segments of equal size \( \Delta \) of the original interval \([0, 1]\) be denoted by \( \Delta^{\text{left}} \) and \( \Delta^{\text{right}} \). Moreover, let the corresponding orthogonal (Gram) polynomial basis of degree \( \leq p \) be based on a number \( \gamma \) of equi-spaced grid points in \( \Delta^{\text{right}} \) and \( \Delta^{\text{left}} \) be \( \phi_l^{\text{left}} \) and \( \phi_l^{\text{right}} \), \( l = 0, \ldots, p \), respectively. Using the boundary intervals \( \Delta^{\text{left}} \) and \( \Delta^{\text{right}} \) (instead of the entire interval) we seek a periodic extension of the basis over the interval \([1 - \Delta, 1 + \Delta + 2d]\) (with periodicity of \(2\Delta + 2d\)) such that the extensions interpolate the basis functions over \( \Delta^{\text{right}} \) and \( 1 + d + \Delta \) at a sufficiently large number of equi-spaced grid points. In other words, we seek functions \( \hat{\psi}_l(x) \) of the form

\[
\hat{\psi}_l(x) = \sum_{j \in g(M)} a_j e^{\frac{\pi i j x}{\Delta^{\text{right}}}},
\]

(6)
where the coefficients $\hat{a}_j^\ell$ are found by requiring, in a least squares sense, that

$$\hat{\psi}_\ell(y_k) = \sum_{j \in g(M)} \hat{a}_j^\ell e^{\frac{\pi i}{\Delta} j (y_k)} = \phi^\text{right}_\ell(y_k)$$  \hspace{1cm} (7)$$

and

$$\hat{\psi}_\ell(y_k + \Delta + d) = \sum_{j \in g(M)} \hat{a}_j^\ell e^{\frac{\pi i}{\Delta} j (y_k + \Delta + d)} = \phi^\text{left}_\ell(y_k + \Delta + d)$$  \hspace{1cm} (8)$$

with

$$y_k = 1 - \Delta + \frac{(k + 1/2)\Delta}{Q} \hspace{1cm} k = 0, \ldots, Q - 1$$  \hspace{1cm} (9)$$

for an adequately large value of $Q$. Setting $\phi^\text{right}_\ell(y_k) = \phi^\text{left}_\ell(y_k + \Delta + d)$, $\ell = 0, \ldots, p$ and $k = 0, \ldots, Q - 1$, for periodicity of the function $f$ in $[0, 1 + d]$, along with the uniqueness of the continuation dictate that all odd modes be zero. The Fourier extension of the basis functions then reduces to

$$\hat{\psi}_\ell(x) = \sum_{j \in g(M)} \hat{a}_j^\ell e^{\frac{\pi i}{\Delta} j x}, \hspace{1cm} j \text{ even},$$  \hspace{1cm} (10)$$

where the coefficients $\hat{a}_j^\ell$ are obtained by solving the over-determined linear system

$$\hat{\psi}_\ell(y_k) = \sum_{j \in g(M)} \hat{a}_j^\ell e^{\frac{\pi i}{\Delta} j y_k} = \phi^\text{right}_\ell(y_k) \hspace{1cm} j \text{ even}.$$  \hspace{1cm} (11)$$

As noted in [6], this calculation should be performed in high-precision arithmetic, and the small number of associated coefficients should be stored for later use. To include the odd modes in the continuation of the basis function we consider functions $\tilde{\zeta}_\ell(x)$ such that they approximate, in a least squares sense, $\phi^\text{right}_\ell(x)$ and $-\phi^\text{left}_\ell(x + \Delta + d)$ in $\Delta^\text{right}$ and $1 + d + \Delta^\text{left}$, respectively:

$$\tilde{\zeta}_\ell(y_k) = \sum_{j \in g(M)} \hat{b}_j^\ell e^{\frac{\pi i}{\Delta} j y_k} = \phi^\text{right}_\ell(y_k)$$  \hspace{1cm} (12)$$

and

$$\tilde{\zeta}_\ell(y_k + \Delta + d) = -\sum_{j \in g(M)} \hat{b}_j^\ell e^{\frac{\pi i}{\Delta} j (y_k + \Delta + d)} = -\phi^\text{left}_\ell(y_k + \Delta + d).$$  \hspace{1cm} (13)$$

In this case all the even modes are zero and the Fourier extensions become

$$\tilde{\zeta}_\ell(x) = \sum_{j \in g(M)} \hat{b}_j^\ell e^{\frac{\pi i}{\Delta} j x}, \hspace{1cm} j \text{ odd},$$  \hspace{1cm} (14)$$

where the coefficients $\hat{b}_j^\ell$ are obtained by solving the linear system

$$\tilde{\zeta}_\ell(y_k) = \sum_{j \in g(M)} \hat{b}_j^\ell e^{\frac{\pi i}{\Delta} j y_k} = \phi^\text{right}_\ell(y_k) \hspace{1cm} j \text{ odd}.$$  \hspace{1cm} (15)$$
Having computed $\tilde{\psi}_\ell$ and $\tilde{\zeta}_\ell$ through Eqs. (10), (11), (14) and (15), we can now express the Fourier continuation approximation of the original function as follows: using the discrete inner products

\begin{align}
\hat{f}_\ell^{\text{right}} &= (f, \varphi_{\ell}^{\text{right}}), \\
\hat{f}_\ell^{\text{left}} &= (f, \varphi_{\ell}^{\text{left}}),
\end{align}

over $\gamma$ equi-spaced grid points in $\Delta^{\text{right}}$ and $\Delta^{\text{left}}$, respectively, the periodic extension $\tilde{f}$ of $f$ is obtained as

\begin{align}
\tilde{f}(x) &= f(x) \quad 0 \leq x \leq 1, \\
\tilde{f}(x) &= \sum_\ell \left( \frac{\hat{f}_\ell^{\text{right}} + \hat{f}_\ell^{\text{left}}}{2} \right) \tilde{\psi}_\ell(x) + \sum_\ell \left( \frac{\hat{f}_\ell^{\text{right}} - \hat{f}_\ell^{\text{left}}}{2} \right) \tilde{\zeta}_\ell(x) \quad 1 < x \leq 1 + d.
\end{align}

Note that unlike [6], where the FC method is defined using a set of equi-spaced points containing the two end points of the approximation interval, we consider the FC method defined over a set of purely internal grid points. Although these variants of the FC method display almost identical approximation properties, the latter case is preferred for the multi-domain FC method and hybrid FC-WENO formulation since the conditions at the domain interface are simplified as we shall discuss in the next section.

**Remark 1.** For our numerical examples we have taken the maximum Gram polynomial degree as $p = 5$, the number of boundary points $\gamma = p + 1 = 6$, and $d/\Delta = 26/6$. The extensions $\tilde{\psi}_\ell$ and $\tilde{\zeta}_\ell$ are computed using $Q = 150$ and $M = 64$. Note that these functions are computed only once and their values are stored at $x_k = 1 + (k + 1/2)/26, k = 0, \ldots, 25$ to be used for the computation of the periodic extension of a general function through Eq. (17b).

It is worth emphasizing that extensive experimentation with higher values of $p$ does not show significant advantages for the class of problems being considered in this work. This is mainly to the limitations in accuracy imposed by the WENO techniques for problems with discontinuous solutions. However, for problems with large regions of smooth behavior it is likely that one would see improvements by increasing this value, e.g., in [1] $p = 11$ was used with success.

**Remark 2.** It may seem counter intuitive that the high-order projection of a function into a polynomial spaces defined by a set of equi-spaced grid points does not exhibit the Runge phenomenon as the approximation approaches the boundary. The explanation for this is that since the polynomial order is kept constant – as the number of points per domains increases, the boundary interval shrinks and so does Runge zone in the complex plane. For details and rigorous proofs of related subjects, we refer to [3, 4].

**Remark 3.** It is interesting to note that, while the polynomial projection of the original function implies a finite order of approximation of the method, the
Fourier continuation method retains the all desirable properties normally associated with a Fourier spectral method, e.g., a differential operator with spectrum that scales linearly with the spatial resolution and, as we shall demonstrate in Sec. 4, no pollution error in the sense that for a certain accuracy level, the required number of points per wave length remains constant as the wave number of the solution increases.

3. Multi-domain Fourier-Continuation/WENO hybrid method

In the following we describe the four main elements of our multi-domain hybrid FC-WENO method for solution of conservation laws. Our method is based on use of the FC method for the smooth portions of the solutions, the WENO method for regions with steep gradients or discontinuous solutions, the smoothness indicator, which allows us to determine when to switch between the two techniques, and a multi-domain formulation ensuring stable and non-oscillatory transfer of data between adjacent domains. For the temporal discretization, we use a standard third-order total variation diminishing (TVD) Runge-Kutta method [28].

3.1. FC method for conservation laws with smooth solutions

We shall initially consider smooth solutions to a nonlinear conservation law of the form

$$\frac{\partial u(t,x)}{\partial t} + \frac{\partial f(u(t,x))}{\partial x} = 0,$$

(18)

defined on a one-dimensional spatial domain $\Omega = [a,b]$ in the time interval $[0,T]$, subject to appropriate initial and boundary conditions. Given a set of equi-spaced grid points, $x_k = a + (k+1/2)L/N$, $k = 0, \cdots, N-1$, with $L = b-a$, we seek $u_h(t,x_k)$, an approximation to $u(t,x_k)$, that satisfies the equation in a collocation sense

$$\frac{\partial u_h(t,x_k)}{\partial t} + \frac{\partial u_h(t,x_k)}{\partial x} \frac{\partial f(u_h(t,x_k))}{\partial u} = 0 \quad \forall k = 0, \cdots, N-1.$$  

(19)

Note that we have not expressed our equation in conservation form, since (19) leads to a more efficient algorithm in the smooth-region.

The spatial derivative $\frac{\partial u_h(t,x_k)}{\partial x}$ is computed using the Fourier continuation of the numerical solution. In view (2) we have the Fourier continuation expansion of $u_h(t,x_k)$,

$$u_h(t,x) = \sum_{j \in \mathcal{G}(M)} \hat{a}_j(t) e^{\frac{2\pi i}{L_d} jx},$$

(20)

with $L_d = b + d - a$. The spatial derivative is computed as

$$\frac{\partial u_h(t,x_k)}{\partial x} = \sum_{j \in \mathcal{G}(M)} \frac{2\pi ij}{L_d} \hat{a}_j(t) e^{\frac{2\pi i}{L_d} jx_k}.$$  

(21)
As we shall discuss in Sec. 4, the appearance of very small unstable eigenvalues through the Fourier continuation may render this approach weakly unstable. To remedy this, we apply a very weak exponential filter to damp out high-frequency modes and thus stabilize the numerical method with minimal adverse impact on accuracy [20]. Specifically, after computing the Fourier coefficients \( \hat{a}_j(t) \) on the extended domain, we compute the modified coefficients \( \tilde{a}_j(t) \), as

\[
\tilde{a}_j(t) = \exp(-\beta(j/N/2)^2)\hat{a}_j(t),
\]

leading to

\[
\frac{\partial u_h(t,x_k)}{\partial x} = \sum_{j \in g(M)} \frac{2\pi ij}{L_d} \tilde{a}_j(t)e^{(2\pi i/L_d)k}.
\]

In (22), we choose \( \beta \) such that the highest mode, \( \tilde{a}_{N/2}(t) \), vanishes that is, \( \beta = -\log(\varepsilon_M) \), where \( \varepsilon_M \) is the machine precision, typically \( 10^{-16} \). In this work, the filter order, \( q \), is typically chosen \( q \approx N/2 \) for \( N < 200 \). For larger \( N > 200 \), the order of the filter is required to be lowered, \( q < N/2 \), to maintain stability. The impact of such a weak filter is truly minimal, as is demonstrated clearly by the numerical examples discussed later.

Following [33, 32] we impose Dirichlet boundary conditions weakly through a penalty term, although strongly enforced boundary conditions are generally also possible in this context, see [1, 6, 7]. However, in the particular formulation introduced here the weak approach is a natural choice since the FC discretization is based on a set of internal grid points, excluding the two end points of the interval where boundary conditions are needed.

Assuming the given boundary value \( u(t,x=a) = g_D(t) \) for Eq. (18), the approximate solution \( U_h(t,x) \) is recast as

\[
U_h(t,x) = u_h(t,x) + (g_D(t) - u_h(t,x_0))L_0(x),
\]

where \( L_0(x) \) is the first Lagrange Fourier interpolating polynomials ([34]) based on the set of grid points on the extended domain, \( xe_k = a + kL_d/N \), with

\[
L_0(xe_0) = 1
\]

\[
L_0(xe_k) = 0 \quad k = 1, \cdots, N-1.
\]

The resulting modified solution derivative is

\[
\frac{\partial U_h(t,x)}{\partial x} = \frac{\partial u_h(t,x)}{\partial x} + (g_D(t) - u_h(t,x_0)) \frac{\partial L_0(x)}{\partial x}.
\]

When extending to a system of equations, it is essential to apply boundary conditions weakly using the characteristic compatibility method ([31]) ensuring that outgoing characteristics are not restricted by unphysical boundary conditions.

Remark 4. For nonlinear problems, the use of an exponential filter serves the additional purpose of removing energy accumulated at high frequencies, and thus stabilizing the calculations at high frequencies due to the nonlinearity of the governing equation [20].
Remark 5. Note that with \( p = 5 \) in the FC approximation, the derivative computation is technically fifth-order accurate, i.e., our FC method for the conservation law (18) with smooth solution is formally fifth-order accurate. However, as we discuss shortly, the behavior of the method away from the boundaries is closer to that of a Fourier spectral method and it is not impacted by the nominal fifth order of approximation.

Remark 6. Regarding the implementation of the derivative calculations, the Fourier coefficients in (20) are first computed using a fast Fourier transform (FFT) and point-values of the derivatives in (23) are obtained using an inverse FFT. Both steps, FFT and inverse FFT, can be computed with a complexity of \( \mathcal{O}(N \log(N)) \). Alternatively, the derivative calculation can be performed using a matrix-vector product with a cost of \( \mathcal{O}(N^2) \), where the matrix representing the effect of differentiation and the vector being the collocation values of the function [21]. For larger values \( N \), the former is clearly preferred and that is the approach we consider here.

Remark 7. Equation (19) is in non-conservative form. The equivalent conservative form (for smooth problems) is

\[
\frac{\partial u_h(t, x_k)}{\partial t} + \frac{\partial f(u_h(t, x_k))}{\partial x} = 0 \quad \forall k = 0, \ldots, N - 1.
\]

In all our numerical tests, both conservative and non-conservative FC discretization of the conservation laws have been tested and have yielded almost identical results. Our preference for the non-conservative form is guided by efficiency. While the non-conservative form requires only the continuation of \( u \) and its filtering, the conservative form requires continuation and filtering of both \( u \) and the flux, \( f(u) \) and, thus, renders the scheme more expensive.

3.2. WENO Methods for conservation laws with discontinuous solutions

WENO finite difference methods are finite difference schemes tailored to enable the accurate solution of conservation laws involving discontinuous solutions—or, more generally, solutions which contain steep gradients. The details of the WENO schemes are well-documented [18, 15], and we therefore offer only a brief presentation of the essential elements of these methods.

A traditional spatial finite difference discretization of (18) seeks point-wise enforcement of the equation at a set of equi-spaced grid points \( x_k = a + (k + 1/2)L/N, \ k = 0, \ldots, N - 1 \), as

\[
\frac{\partial u_h(t, x_k)}{\partial t} + \frac{\hat{f}(t, x_{k+1/2}) - \hat{f}(t, x_{k-1/2})}{\Delta x} = 0,
\]

where \( \Delta x = x_k - x_{k-1} \), \( u_h(t, x_k) \) is a numerical approximation of \( u(t, x_k) \) and where \( \hat{f}(t, x_{k+1/2}) \) represents a numerical flux. The numerical flux (with shorthand notation \( \hat{f}_{k+1/2} \)) is defined through a reconstruction based on the numerical solutions at \( x_k \), \( r \) solution to the left and \( s \) solutions to the right of \( x_k \):

\[
\hat{f}_{k+1/2} = \hat{f}(u_{k-r}, \ldots, u_{k+s}).
\]
Here \( \hat{f} \) is required to be Lipschitz continuous in all arguments and consistent with the physical flux \( f \), that is, \( \hat{f}(u, \cdots, u) = f(u) \). Based on the Lax-Wendroff theorem, the solution of this conservative scheme, if it converges, converges to the weak solution of (18).

The precise choice of the numerical flux \( \hat{f}_{i+1/2} \) is central and must follow a number of guidelines. To guarantee entropy dissipation, the flux is first split into positive and negative parts using Lax-Friedrichs splitting as

\[
f(u) = f^+(u) + f^-(u),
\]

with

\[
f^+(u) = (f(u) + \alpha u)/2, \quad f^-(u) = (f(u) - \alpha u)/2,
\]

and \( \alpha = \max_u \frac{df(u)}{du} \). The reconstruction is then applied for each positive and negative flux separately, before adding up to give the numerical flux.

A finite difference reconstruction of the flux, for instance for \( f^+(u_{j+1/2}) \), can be computed using solutions in \( q \) different stencils,

\[
S_r(k) = x_{k-r}, \cdots, x_{k-r+q-1} \quad r = 0, \cdots, q-1,
\]

as

\[
\hat{f}^+_r(u_{k+1/2}) = \sum_{n=0}^{q-1} C_{r,n} f^+(u_{k-r+n}).
\]

The constant coefficients \( C_{r,n} \) (given in [14] up to order seven) are chosen such that the approximate solution is formally accurate up to order \( q \) in regions in which the exact solution is sufficiently smooth. On the other hand, if in a stencil, the solution or one of its \( j \)-derivatives \( j \leq q \) contains a discontinuity, that stencil must be excluded from the flux approximation. This nonlinear procedure is referred to as an essentially non-oscillatory (ENO) scheme [17].

A slightly modified approach, offering computational advantages and added robustness (see [14]), is to consider a convex combination of all computed fluxes \( f^+_r(u_{j+1}) \) in \( q \) stencils as

\[
f^+(u_{k+1/2}) = \sum_{r=0}^{p-1} w_r f^+_r(u_{k+1/2}),
\]

where \( w_r \) is chosen such that the approximation is of order \( 2q - 1 \) accuracy in cases where the solution is sufficiently smooth in all stencils. An explicit expression for \( w_r \), based on a local smoothness indicator for orders \( q = 2 \), and 3, are given in [13] and for higher orders up to \( q = 6 \) in [16].

In the following we have implemented the fifth- and ninth-order WENO schemes. For the fifth-order scheme, we consider the standard WENO scheme introduced by Jiang and Shu ([13]) referred to as WENO5 and a modified version called WENO5-z ([19]) exhibiting slightly higher accuracy. The ninth-order variant was introduced in [16] and is referred to as WENO9 hereafter.

**Remark 8.** For system of conservation laws, such as the Euler equations, it is essential to apply the WENO reconstruction to the fluxes in the eigenvector space. This is performed through multiplication of the flux vectors with the left eigensystem. The reconstructed fluxes are then transferred back to the conservative fluxes through multiplication with the right eigensystem [13].
3.3. Detecting the discontinuities

Detecting discontinuities in the solution or its derivatives is an essential component and prerequisite for the hybrid strategy to be effective. For systems of nonlinear conservation laws like the Euler system of gas dynamics, the location of the discontinuities, e.g., shocks, are not known a priori and may emerge and propagate as the solution advances. We thus need to estimate the local smoothness from the solution field at regular temporal intervals. To this end, the multi-resolution (MR) analysis introduced by Harten [9] and later used in the context of a hybrid spectral-WENO method by Costa and Don [10] has proven to be effective. We present a brief description of this approach in what follows; a detailed account can be found in [9, 10].

Using the solution values \( f^0_k \) at equi-spaced grid points \( x^0_k \) (\( k = 1, \cdots, 2N + 1 \)) at which the approximate solutions are known, we construct the solution averages \( f^1_k \) at one level coarser grid points \( x^1_k = (x^0_{2k-1} + x^0_{2k+1})/2 \) with \( k = 1, \cdots, N \) as

\[
f^1_k = \frac{f^0_{2k} + f^0_{2k+1}}{2} \quad \forall k = 1, \cdots, N.
\]

(33)

Let \( p_s(x) \) denote the polynomial approximation of order \( s \) interpolating \( f^1_k \) at the coarser grid points \( x^1_k \). The approximation differences, \( d_k = f^0_k - p_s(x^1_k) \), have the property that if \( f(x) \) has \( r-1 \) continuous derivatives and a jump discontinuity at \( r \)th derivative, then

\[
d_k = \begin{cases} 
\Delta x^r \frac{d^r f_k}{dx^r} & s \geq r, \\
\Delta x^s \frac{d^s f_k}{dx^s} & s < r,
\end{cases}
\]

where \( \Delta x \) denotes the coarse grid spacing and where \( \lfloor \cdot \rfloor \) denotes jumps across a discontinuity. This implies that the higher degree of smoothness of the solution, and the higher the order of the polynomial approximation on the coarser grid (in the case of smooth function), the smaller the approximation differences, \( d_k \).

We thus adopt a tolerance \( \epsilon \), below which the function is assumed to be smooth and otherwise identify it as non-smooth. Once we have found the approximate locations of the discontinuities by this method, the domains containing smooth solutions are treated by the FC method and those with non-smooth solutions are treated by the WENO scheme. This simple strategy works well except for situations in which a discontinuity is very close to a boundary of a sub-domain, within a few grid points. In this case, the neighboring domain is also marked as a WENO domain to ensure conservative and non-oscillatory transfer of the discontinuity between neighboring domains. This strategy is similar to the strategy adopted in [10]. In our implementation, detection of a discontinuity within two grid points away from a sub-domain boundary is used to dictate the WENO treatment of the sub-domain.

Remark 9. Note that for low resolution (i.e., for representations of the solution using a small number of spatial grid points) the multi-resolution analysis may fail to distinguish between a discontinuity and a large smooth solution gradient. An illustration of this fact is presented below in the case of the Euler system. In such regions, the algorithm marks the solution as being non-smooth.
3.4. Multi-domain formulation

To accommodate the distinct treatment of smooth solution regions and discontinuous regions, we pursue a multi-domain solution algorithm. The computational domain is first decomposed into sub-domains with each sub-domain being discretized with either an FC or a WENO method depending on the local smoothness indication. A domain interface condition is adopted for stable and non-oscillatory transfer of data between adjacent sub-domains. The decomposition of the domain can be performed only once, leading to sub-domains with fixed sizes (sub-domains with fixed number of grid points), or repeated at each time step of the time marching yielding sub-domains with time-varying sizes (sub-domains with number of grid points varying in time). We adopt the fixed-sized domain strategy since it facilitates the extension of our multi-domain strategy to relatively complex geometries in multiple space dimensions in a straightforward block structured manner, see [1]. Figs. 2 (a) and (b) demonstrate a domain decomposition of an interval and the grid points for each sub-domain.

To ensure a stable approach for transfer of data between adjacent sub-domains, we utilize an overlapping grid stencil. Specifically, for an internal sub-domain marked as a smooth region, a derivative computation based on the FC method is carried out using an extended stencil consisting of the sub-domain’s own grid points and a few of the closest grid points of its neighbors (see Fig. 2 (c)). For a boundary domain, the extension is one-sided across the internal interface (see Fig. 2 (c)). Using this process, the derivatives of the solution are available for the extended stencil. However, only the computed derivatives of the actual sub-domain grid points are used for updating the solution field, and derivatives at the extended grid points are discarded. Extensive experimentation confirms that an extension of three grid points across each domain interface suffices to guarantee stability. For WENO sub-domains, a similar strategy is required with extensions of three and five grid points for fifth- and ninth-order schemes, respectively.

Note that this strategy of handling or circumventing interface conditions is uniform regardless of whether adjacent domains are two FC domains, or two WENO domains, or one FC and one WENO. This is a consequence of using equi-spaced internal grid points (not including end points, see Fig. 2 (b)) for both FC and WENO discretizations.

4. Numerical Examples

In this section we strive to carefully and rigorously evaluate all elements of the proposed algorithm. We first present results produced by the FC algorithm for the advection equation with smooth solutions, including comparisons with results produced by means of central difference (CD) methods to further emphasize the advantage of the FC based approach. We continue by considering a number of results for the inviscid Burgers’ equation, and finally, we present an extensive set of comparisons of our hybrid FC-WENO method with pure WENO
for shock-entropy-wave-interaction problems. As we shall see, we demonstrate in all cases substantial advantages over existing alternatives, both in terms of accuracy and computational performance.

4.1. Advection equation

We first consider the advection equation, Eq. (18) with $f(u(t,x)) = u(t,x)$, in a periodic spatial domain $[0, 2\pi]$ with a sinusoidal initial condition $u = \sin(\kappa x)$. This problem is solved using three different methods: a fifth-order Fourier continuation (FC5) method, a fifth-order central difference (CD5) and a sixth-order central difference (CD6) method [30].

The results at the final time $T = 100$ are compared in Fig. 3, showing the maximum error over all time as a function of the wave number $\kappa$. For the FC5 method we assume 10 points per wavelength (PPW) and CD5 and CD6 have 20 and 15 PPW, respectively. It should be noted that even though the problem is periodic, we still apply the full FC framework, i.e., we do not reduce the computation to a pure Fourier spectral method.

While error grows linearly with increasing wave number for the central difference methods, the situation is quite different for the FC method—for which the error remains constant at around 1%, see also [7]. This is in line with the classic analysis for Fourier spectral methods [21] and illustrates the significant advantage provided by the FC method over central difference methods for multi-scale problems with wide frequency spectra and long time integration.

We next study the rate of convergence of the FC5 method for solving the advection equation in the periodic spatial domain $[0, 2\pi]$ with the low-frequency exact solution $u(t,x) = \exp(\cos(x-t))$. In Fig. 4 the maximum error over time up to the total time $T = 100$ is plotted versus the grid spacing for the FC5 method and, for comparison, for the CD5 and CD6 methods as well. The FC5 method delivers fifth-order accuracy but with a constant that is about one to two orders of magnitude smaller than that found for the central difference scheme. Hence, for a specific accuracy requirement, the FC5 requires considerably less degrees of freedom, in particular when extending to multiple dimensions. Fur-
thermore, as we shall see shortly, the fifth order accuracy is a consequence of the periodic boundary condition.

The performance of the FC method for solving the advection equation in the presence of Dirichlet boundary condition, with and without the use of the exponential filter, is illustrated in Figs. 5(a) and (b). In Fig. 5(a), the maximum error over time is plotted for the filtered and unfiltered numerical solutions. While the unfiltered solution exhibits numerical instabilities characterized by rapid increase of the error, the filtered solution remains stable with small oscillations in error. Moreover, as seen in Fig. 5 (b), the filter does not have an adverse effect on the convergence rate, since the filtered solution exhibits a optimal sixth-order average convergence rate.

Finally, we demonstrate that the rate of convergence of the FC5 remains approximately unchanged when extended to the multi-domain formulation. To this end, we solve the advection equation in \([0, 2\pi]\) with the exact solution \(u(t, x) = \exp(6\cos(x - t))\) using \(ND = 4, 8, 16,\) and \(32\) domains and a fixed number of points per domains of \(NP = 20\). The results for the total time \(T = 100\) are depicted in Figure 6, showing an average rate of convergence of 5.6. This slight decrease in order of convergence is associated with a weak dependence of the error on the number of overlapping regions which increase linearly in number with increasing number of elements.
It is worth emphasizing that the sixth order accuracy of the FC approach is dictated by the order of the Gram polynomials and that one could in many cases choose a higher value, resulting in a scheme of formally higher order. For the sake of definiteness, however, we have chosen to maintain the original implementation [6, 7] of the continuation method and maintained the FC at fifth order which extensive computations have confirmed suffices to results in a balanced scheme for most of the problems considered here. However, there may well be other classes of problems where the situation is different and extending the FC approach to higher order may turn out to be beneficial. A further discussion and examples along this line can be found in [1].

4.2. Burgers’ equation

As a first test of the full hybrid scheme, we consider the inviscid Burgers’ equation obtained by setting \( f(u(t,x)) = u^2(t,x)/2 \) in (18). The computational domain is \([-1, 1]\) with periodic boundary conditions and the initial condition is taken to be \( u(0,x) = (1 + \sin(\pi x))/2 \). We first study the accuracy of the FC5 method for Burgers’ equation using both single-domain and multi-domain formulations. In Fig. 7, the maximum error is plotted versus the grid spacing for the final time \( T = 0.25 \), where the solution remains smooth. For the single-domain method, the number of points \( NP = 20, 40, 80, \) and \( 160 \), and for the
Figure 5: (a) Illustration of stability of the fifth-order Fourier continuation (FC5) as applied to the solution of the Dirichlet problem for the advection equation considered in the text, in the interval $[0, 2\pi]$, and with the exact solution given by $u = \exp(6\cos(x - t))$. For this test we have used $NP = 20$ and show results with and without the use of a weak exponential filter. (b) Convergence as a function of the grid size for the weakly filtered FC5 method for the problem consider in (a) when increasing $NP$. In (b), the best line fit to data and its slope are also shown to highlight optimal convergence.

We also consider Burgers’ equation at the later time $T = 0.75$, by which a discontinuity in the solution has appeared. These results are produced by two different methods: the proposed hybrid FC-WENO method with fifth-order accuracy for both the FC and the WENO parts (FC5-WENO5), and a pure fifth-order WENO (WENO5) scheme. In the FC-WENO hybrid calculations, the multi-resolution analysis is carried out at every stage of the Runge-Kutta time stepping procedure with the order of the interpolating polynomial on the coarse grid $s = 9$, and the tolerance $\epsilon \approx 1.0^{-4}$. In all the numerical experiments, to ensure linear stability, we pick time step size as $\Delta t \approx 0.5\delta x/C$, where $\delta x$ is the grid spacing and $C$ is the maximum wave speed.

The computed solutions are shown in Figure 8(a), while the convergence rates are depicted in Figure 8(b). As is evident both the hybrid FC5-WENO5 and the pure WENO5 methods yield almost identical solutions and convergence rates of 0.8 in the $L_1$ norm, reflecting that the overall error is dominated by the
Figure 6: Convergence of the fifth-order Fourier continuation (FC5) method as applied to the solution of the advection equation in the interval $[0, 2\pi]$ with an exact solution $u = \exp(6\cos(x - t))$. The final time is $T = 100$ and we use $ND = 4, 8, 16, \text{ and } 32$ domains, each with $NP = 20$ points. The best line fit to data and its slope are also shown to confirm the optimal convergence rate.

WENO segments of the scheme.

4.3. The Euler system

The one-dimensional Euler equations are given as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0,$$

where $\mathbf{u} = [\rho, \rho v, \rho E]$ and $\mathbf{f}(\mathbf{u}) = [\rho v, \rho v^2 + p, (\rho E + p)v]$. Here, $\rho$, $v$, $E$ and $p$, represent, respectively, the density, the velocity, the total energy and the pressure, and are subjected to appropriate initial and boundary conditions.

We consider two test problems, both featuring the interaction of a Mach 3 shock with an entropy wave. The first problem, comprising a shock/small-entropy-wave interaction case, features a small amplitude entropy wave, and allows for a quantitative comparison of our computed solutions with analytical results obtained using a linear analysis of the Euler equations. The second problem involves a shock/entropy-wave interaction configuration with an entropy wave with finite amplitude, leading to strongly nonlinear behavior of the solutions. No exact solution is known for this case.
Figure 7: Convergence for the fifth-order Fourier continuation (FC5) method for the solution of the Burgers' equation in $[-1, 1]$ with initial condition $u(0, x) = (1 + \sin(\pi x))/2$. Results are obtained at the final time $T = 0.25$ for both multi-domain and single domain formulations. For the single-domain method, the number of points is $NP = 20, 40, 80$, and $160$, while for the multi-domain $NP = 20$ is fixed and the number of domains $ND = 1, 2, 4, \text{and} 8$. For each data set, the best curve fit and its slope are also shown.

4.3.1. Shock/small-entropy-wave interaction test

To quantify the performance of the hybrid FC-WENO methods we consider a right-moving Mach 3 shock interacting with a very small entropy wave. The spatial domain is $[-10, 10]$ and the initial and boundary conditions are given by

$$
(r, v, p)(0, x) = \begin{cases} 
(3.857143, 2.629369, 10.33333) & x \leq -9.5, \\
(1.0, 0.0, 1.0) & -9.5 \leq x \leq -8.85, \\
(\exp(-0.01 \sin(13(x - 8.85))), 0, 1.0) & x > -8.85,
\end{cases}
$$

(36)

The effect produced by a strong shock as it passes through a very small entropy wave is characterized by a sudden rise in the wave frequency and a sudden decline in the wave amplitude. For sufficiently small entropy waves, these discontinuous changes in the frequency and amplitude, both of which are functions of the mean flow Mach number, can be obtained accurately through analysis of the linearized Euler equations [22]—thus allowing a quantitative assessment of the accuracy of our hybrid method.

We first consider results for solutions of the Euler system with the above initial conditions at the final time $T = 5.0$, and for three different settings. Considering first a high-resolution (fine) calculation using the pure ninth-order
Figure 8: (a) Discontinuous solution of Burgers’ equation at $T = 0.75$ computed using the hybrid FC5-WENO5 method; (b) Convergence of the pure WENO5 and hybrid FC5-WENO5 for the same test. In (b), the best curve fit to data and its slope are also shown.

WENO (WENO9) method with the total number of grid points $NP = 10240$. For this case we find that the computed entropy amplification, defined as the ratio of the maximum entropy wave amplitude before and after the shock, for the high resolution calculation is within 0.1% of the amplification factor of 0.841037 produced by the linear analysis [22]. The high-resolution result is thus considered exact up to this error tolerance. We proceed to compare these values to those obtained by means of two low-resolution (coarse) calculations using a WENO5-z method with $NP = 2560$ and the hybrid FC5-WENO5-z with $NP = 32$ and the number of domains $ND = 80$, hence an equivalent number of total grid points of 2560.

Figure 9 (a) depicts the computed entropies for the three methods and Fig. 9 (b) is a close-up view of the same data. As seen from Fig. 9 (b), right behind the shock ($x \approx 8.0$), both coarse grid calculations yield results very close to that of the high-resolution computation. Away from the shock ($x \approx 4.0$), however, the entropy using the WENO5-z method displays a much larger dissipation as compared to that of the hybrid FC5-WENO5-z. A WENO5-z [19] is used here over as more traditional WENO5 methods to improve accuracy although the differences are minimal.

To overcome the excessive damping, we replace the fifth-order WENO with a ninth-order WENO discretization for the coarse calculations and repeat the same test. The results are shown in Fig. 9 (c). Comparing the results in Fig. 9 (c) with those in Fig. 9 (b) clearly confirms that increasing the order of WENO scheme from five to nine yields more accurate entropy field in particular for
the pure WENO method and improved agreement with the result of the hybrid scheme. It also suggests that even at this high resolution the accuracy of the computed results dominated by the WENO scheme and not the FC parts of the hybrid scheme.

Figure 9: Entropy profiles for shock-small-entropy-wave-interaction test at the final time $T = 5.0$; (a) The pure WENO9 result with 10240 grid points, considered as the exact solution, and two coarse results obtained using a pure WENO5-z with 2560 grid points and the hybrid FC5-WENO5-z with $NP = 32$ and $ND = 80$; (b) close-up view of the same results. (c) close-up view of the entropy profiles obtained using the same setting as in (a) except replacing the fifth-order WENO a ninth-order scheme; (d) even a closer view than in (c).

We next offer a detailed quantitative comparison of the results produced by our hybrid FC-WENO solver and results obtained from the WENO method for the entropy amplification at two different Mach numbers. Tables 1 and 2 present results for Mach three and Table 3 for Mach six. In the upper half of Table 1, the error in entropy amplification at two locations near to and far from the shock
Table 1: Error in entropy amplification, Mach = 3.0, T = 5.0, ND = 80, NP = 32, ∆t = 4.4 × 10⁻⁴

<table>
<thead>
<tr>
<th>Method</th>
<th>Error near shock</th>
<th>Error far from shock</th>
<th>CPU time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WENO5</td>
<td>22.1%</td>
<td>56.2%</td>
<td>99.0</td>
</tr>
<tr>
<td>WENO5-z</td>
<td>10.5%</td>
<td>22.6%</td>
<td>99.0</td>
</tr>
<tr>
<td>FC5-WENO5-z</td>
<td>10.4%</td>
<td>10.2%</td>
<td>32.0</td>
</tr>
<tr>
<td>WENO9</td>
<td>4.4%</td>
<td>4.0%</td>
<td>188.0</td>
</tr>
<tr>
<td>FC5-WENO9</td>
<td>4.4%</td>
<td>4.1%</td>
<td>35.0</td>
</tr>
</tbody>
</table>

\((x ≈ 4.0 \text{ and } x ≈ 8.0, \text{ respectively})\) and the total CPU times (seconds) are given for the WENO5 ([13]), WENO5-z ([19]) and the hybrid FC5-WENO5-z. In the lower half the same quantities are listed for the WENO9 and the hybrid FC5-WENO9 method. For the hybrid method, we chose \(ND = 80\) and \(NP = 32\), and for the WENO methods we use the equivalent total number of points \(NP = 2560\). The error calculations are based on the analytical entropy amplification factors obtained using linear analysis, which are 0.841037 and 0.68485 for Mach numbers three and six, respectively (see [22], Eq. (7)).

Several points should be noted in connection with Table 1. First, the ninth-order WENO method yields a lower error than the fifth-order WENO version, as expected. Second, the hybrid FC-WENO schemes are faster than the corresponding pure WENO schemes, with a threefold advantage in the fifth-order case and a sixfold in the ninth-order case. Third, using WENO9 instead of WENO5 in a hybrid scheme significantly improves the accuracy with a minor increase in total cost. This confirms that, indeed, it is highly advantageous to exploit the fact that only a small percentage of domains need to be treated by the WENO method in the hybrid scheme since the overall error is dominated by the WENO schemes, even at this high resolution.

Table 2 reports data similar to that in Table 1, but with an additional level in the resolution: \(ND = 160\). As is clear from the table, the increased resolution yields increased accuracy, with errors in the WENO9 and FC-WENO9 results decreasing below one percent. Similar to the previous case, the hybrid approach is several-fold faster than the pure WENO method even for this relatively simply one-dimensional test case. Table 3, finally, presents data for a Mach six case, demonstrating similar trends as for the lower Mach number data. In this case, the error values are uniformly larger than those presented in Table 2 as is to be expected since higher Mach numbers give rise to larger solution gradients, and hence require higher resolution for the same level of accuracy.

For comparison, we also considered a different hybrid scheme in which a sixth-order central difference (CD6) method is combined with the WENO9 method ([11]). We consider the same test problem of a shock/small-entropy-wave interaction but with a lower Mach number of 1.25, and a larger computational domain \([-10, 30]\). This results in a much slower moving shock and the need to accurately model wave propagation over long physical distances since the solution features subsonic flow behind the shock with a very small amplitude acoustic wave traveling upstream, see Fig. 10 (a). The simulations are done
using FC5-WENO9, CD6-WENO9 for the final time of $T = 23$ using two resolutions for each method: fixed number of points per domain $NP = 96$ with number of domains being $ND = 20$ or 40.

Figure 10 (a) depicts the density profile behind the shock and Figs. 10 (b) and (c) show a zoomed-in view of the “entropy” and “acoustic” wave regions, respectively. We refer to the “acoustic” wave region as the region far upstream of the shock with only a left moving characteristic wave, and to the “entropy” wave region as the remainder of the domain in the neighborhood of the shock, with both left and right moving characteristic waves. In Figs. 10 (b) and (c), results using FC5-WENO9 with $ND = 20$ and CD6-WENO9 with two resolutions of $ND = 20$ and $ND = 40$ are shown. While the three density profiles in the “entropy” region display no discernible differences (Fig. 10 (b)), the density profile computed using the coarse CD6-WENO9 differs significantly from that of the FC5-WENO9, Fig. 10 (c). This is due to the much larger dispersion error of the finite difference method compared to the Fourier continuation method, highlighting that even though the formal order of the FC and the CD scheme are comparable, the former behaves much more like a spectral method in the interior of the domains. As is also clear from Fig. 10 (c), the CD6-WENO9 scheme requires twice as many grid points to recover a density profile with similar accuracy at the FC6-WENO9 solution. Keeping in mind that these results are all for one-dimensional tests, the potential for substantial savings in multiple dimensional tests is clear.

To quantify these observations, the errors in the entropy amplification behind the shock and the in computed density in acoustic wave, along with total CPU time for the two hybrid solvers are shown in Table 4. The error in the acoustic wave is computed using the fine solution of the FC-WENO solver considered the “exact” solution. Both hybrid methods yield comparable accuracy in the computed entropy amplification for the same resolution and total CPU
times for both methods at the same number of degrees of freedom are also comparable. This simply reflects that in the regions close to the shock, the accuracy of the solution is dominated by the WENO scheme which is the same in the two schemes. However, when considering the behavior of the solution far upstream, the different accuracy of the two schemes becomes very clear. For the coarse calculations the CD6-WENO9 solver yields a 30% error in density in acoustic region, while the FC5-WENO9 results in about 5% error. Increasing the resolution of the CD-based hybrid solver improves matters but still reflects a 10% error, being twice that obtained with the FC based scheme at the coarse grid.

Since the refined CD6-WENO9 computation requires approximately a three-fold higher total CPU time than that of the coarse FC5-WENO9 computations the FC based hybrid is clearly superior for comparable level of accuracy for problems with a mix of strong shocks and smooth regions with strong wave dynamics. Extending this to a the three-dimensional problem, the FC-based hybrid solver can be expected to yield an order of magnitude reduction in computational cost over similar CD-based solutions for a similar level of accuracy.

Furthermore, with the ability to handle complex, non-periodic domains at high order, discussed in detail in [6, 7], FC-based hybrid method is expected to be significantly more efficient and flexible than a CD-based in applications involving general geometries, strongly time dependent behavior, and a wide range of scales in two and three dimensions - all properties of problems of realistic complexity in science and engineering.

### 4.3.2. Shock/entropy-wave interaction test

The shock/entropy-wave interaction problem [17] is a widely used test to evaluate the performance of high-order shock capturing schemes in the presence of both shocks and significant oscillatory smooth structures. Owing to the coexistence of the shock discontinuities and smooth structures, this problem is also well suited for gauging the effectiveness of the multi-resolution strategy and the hybrid scheme in the strong nonlinear and highly dynamic case. The spatial domain is taken as $[-10, 10]$ and the initial and boundary conditions correspond to a right-moving Mach 3 shock located at $-9.5$ at time $T = 0.0$, approaching
Figure 10: Density profile behind the shock for the shock-small-entropy-wave-interaction test with $M = 1.25$, and at $T = 23.0$; (a) The FC5-WENO9 results with $ND = 20$, $NP = 96$; (b) a close-up view of the “entropy” wave region computed using the same FC5-WENO9 and the CD6-WENO9 with two grids of $ND = 20$ and 40 and the same $NP = 96$; (b) a close-up view of the “acoustic” wave region for three computations. The horizontal spacing of the dotted lines in the background of (b) and (c) matches two grid point spacings.

A sinusoidal density wave in $[-8.85, 10]$. Specifically,

$$\begin{align*}
(r, v, p)(0, x) = \begin{cases} 
(3.857143, 2.629369, 10.3333) & x \leq -9.5, \\
(1.0, 0.0, 1.0) & -9.5 \leq x \leq -8.85, \\
(1.0 + 0.2 \sin(5(x - 8.85)), 0, 1.0) & x > -8.85.
\end{cases}
\end{align*}$$

(37)
Figure 11: Density profile for the strong shock-entropy wave interaction. (a) \( T = 0.0013 \); (b) \( T = 1.25 \); (c) \( T = 2.5 \); (d) \( T = 3.75 \). The solid line represents the WENO5-z solution with 1280 grid points, the symbols show FC5-WENO5-z solution with 40 domains, each of 32 grid points; \( \Delta \) for the WENO domains and \( \square \) for the FC domains. The location of the vertical dotted lines signify domain boundaries in the hybrid method.

We solve the Euler system using both WENO5-z and FC5-WENO5-z solvers and consider the density profiles obtained for a total number of 1280 grid points, or 40 domains, each with 32 points each, and \( \Delta t = 8.8 \times 10^{-4} \). In Figs. 11(a)-(d), we show the solution at different times \( T = 0, 1.25, 2.5, \) and 3.75, respectively. As is clear from Fig. 11(a), on the onset \( (T = 0) \), the multi-resolution smoothness indicator correctly identifies both the shock discontinuities at \( x = -9.5 \) and the discontinuities in the density derivative at \( x = -8.85 \). Similarly, for the later times (Figs. 11(c)-(d)), the scheme successfully identifies all discontinuities and adapts the hybrid scheme appropriately.

As discussed previously, the smoothness indicator may mistakenly identify large but smooth gradients as discontinuities in cases where these gradients are insufficiently resolved. This is evident in the case of the shock/entropy-wave interaction for long integration times, for instance at \( T = 5 \), Fig. 12(a), where it is clear that the smooth structures in \([4.0, 6.0]\) are mistakenly identified as
non-smooth and treated as WENO domains. As shown in Fig. 12(b), increasing the resolution, from 40 domains to 80, yields a reliable shock detection strategy. Such increase in resolution may not be required solely for a reliable smoothness detection, but rather it may be necessary if a high level of accuracy is required. As shown in Fig. 12(c) and (d), while the lower-resolution calculations fails to capture the small discontinuities at $x \approx 4.5$ and 5.5 (highlighted in Figs 12(c) and (d) with blue squares), the general features of the solution are maintained, albeit at higher cost since more domains are designated to be WENO domains.

5. Concluding remarks

We have introduced a high-order method based on the hybridization of the Fourier continuation method and WENO finite difference discretization meth-
ods for the solution of non-linear conservation laws. The hybrid strategy is based on a multi-domain formulation with a smoothness detection procedure that effectively flags the domains containing discontinuities. The domains containing discontinuities are treated by means of the WENO scheme and those with smooth solutions are discretized using the Fourier continuation method. For applications with isolated discontinuities in space and time, this strategy offers significant advantages over the pure WENO methods, since the majority of the domains are discretized using the FC method which is considerably less expensive than the WENO reconstruction, yet offers a highly efficient and accurate procedure for complex but smooth portions of the solution.

Our hybrid FC-WENO method offers several advantages over alternative techniques based on finite difference methods or Chebyshev spectral methods. While the central difference methods suffer from significant pollution error, the FC methods are pollution-error-free as is characteristic of a Fourier method. This is particularly important for large scale wave dominated problems requiring long time integration as we have demonstrated. Unlike previous hybrid methods based on Chebyshev spectral methods, the FC method is based on an equi-spaced grid points. This not only avoids the significant cost of the extra interpolations to transfer data from the Chebyshev domains to the equi-spaced grids in the hybrid strategy, but also yields considerably less stringent stability constraint, and thus allows for use of larger time step size in an explicit time marching scheme.

We verified the fifth-order accuracy of the FC method for smooth problems and also demonstrated the effectiveness of the exponential filter in achieving a stable FC method for a long time integration. By solving the Euler system for a shock/small-entropy-wave interaction problem, we demonstrated that for a given accuracy level of 1%, our hybrid strategy is several-fold faster than the pure WENO scheme and that the hybrid approach is both accurate and robust for strongly nonlinear problems.

Performance tests confirms the potential for an order of magnitude acceleration when extending these techniques to the modeling of three-dimensional time-dependent problems. Based on these preliminary but nontrivial results we expect that our approach will prove to be an enabling methodology for the simulation of challenging multi-dimensional multi-scale problems such as shocked-induced transition, turbulence, and combustion—as we hope to demonstrate in future work.

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