Derivation of the Biot-Savart Law from Coulomb's Law and Implications for Gravity

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Electricity: Coulomb's Law

In 1785, Charles-Augustin de Coulomb used a torsion balance (Fig. 1) to discover that the electrostatic force between charges q, q_0 separated by a distance $\vec{r} = r\hat{r}$ has the same form as Newton's law of gravitation. Coulomb's law can be written:

$$\vec{F}_e = \frac{q_0 q}{4\pi\varepsilon_0 r^3}\vec{r} \qquad (1)$$

where ε_0 is a constant (the electric permittivity of free space). This equation is the starting point for all introductory E&M courses.

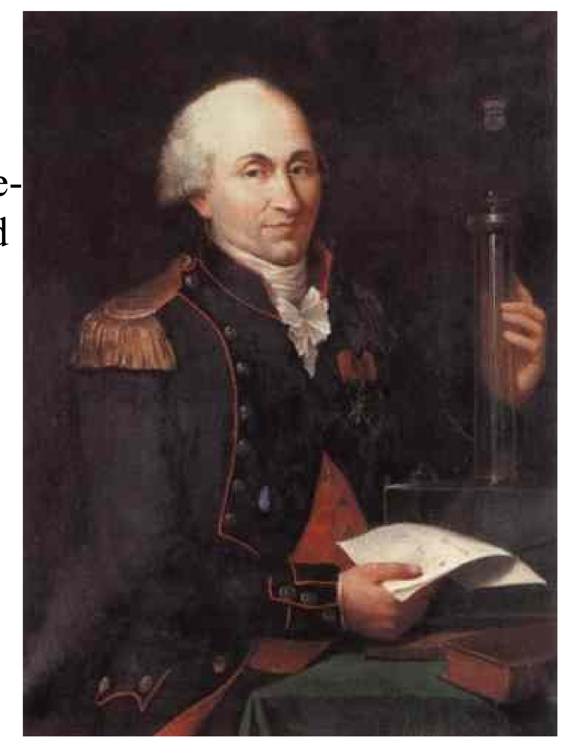

Fig. 1: Coulomb with torsion balance

Magnetism: the Biot-Savart Law

In 1820, Jean-Baptiste Biot and Félix Savart (Fig. 2) experimentally derived a quite different law governing the magnetic force between *moving* charges. Replacing the notion of charges q_0 moving at velocity $\vec{v}$ with *current* I along $d\vec{\ell}$ (i.e. $Id\vec{\ell} = q_0\vec{v}$), they found that the force exerted by this current on a charge q moving at $\vec{u}$ and located at $\vec{r} = r\hat{r}$ is given by:

Fig. 2: Biot (left) and Savart (right)

$$\vec{F}_m = q\vec{u} \times \vec{B}_m \quad \text{where} \quad \vec{B}_m = \frac{\mu_0 I d\vec{\ell} \times \vec{r}}{4\pi r^3} \qquad (2)$$

is the *magnetic field* associated with I and μ_0 is a constant (the magnetic permeability of free space).

To this point, it did not occur to anybody to suspect that $\vec{F}_e$ and $\vec{F}_m$ might be related.

Maxwell to Einstein: Electromagnetism

In 1861-2, James Clerk Maxwell unified all electrical and magnetic phenomena in a set of four equations. Two of Maxwell's equations, Gauss' law and Ampère's law with Maxwell's addition of the displacement current, contain Coulomb's law and the Biot-Savart law, respectively.

Maxwell's equations predict that disturbances in the combined electromagnetic field (i.e., light waves) move at a speed $c = 1/\sqrt{\epsilon_0\mu_0}$ that is *constant*, regardless of the speed of the source or the observer! This violates common sense but is actually observed to be true. It was this fact that led Einstein to special relativity in 1905. In this theory, c is guaranteed to stay the same for all observers. The price is that space and time are no longer independent: they transform into each other.

In Maxwell's theory, which is fully consistent with special relativity, electric and magnetic fields transform into each other in exactly the same way. Magnetism is essentially "moving electricity," and vice versa.

This relationship between electricity and magnetism is mentioned at the end of some upper-level undergraduate textbooks [1] and explored in more detail in some graduate texts [2]. However, most students continue to learn about electricity and magnetism as these concepts developed historically. Toward the end of their courses they may see that $\vec{E}$ and $\vec{B}$ are related through the laws of Ampere and Faraday, but not that they are manifestations of the *same thing*. They thereby miss out on perhaps the most powerful example of the unifying power of physics.

Relativistic Transformation of Forces

In fact, the Biot-Savart law can be derived from Coulomb's law in a way that is perfectly accessible to undergraduates. The starting point is the *Lorentz transformation equations*:

$$x'(t) = \gamma(x - vt), \qquad y' = y, \qquad z' = z,$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad (3)$$

where primes denote quantities observed in a frame moving at speed v in the x-direction. Differentiating, we obtain the *velocity transformations*, where $\vec{u} = d\vec{x}/dt$:

$$u_x{}' = \frac{u_x - v}{1 - u_x v/c^2}, \qquad u_y{}' = u_y\frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \qquad (4)$$

(The z -transformation is similar, with $y, y' \to z, z'$.) We will also need the inverse transformations:

$$u_x = \frac{u_x' + v}{1 + u_x{}'v/c^2}, \qquad u_y = u_y{}'\frac{\sqrt{1 - v^2/c^2}}{1 + u_x{}'v/c^2} \qquad (5)$$

(with the z -transformation as before.) Combining these expressions, we obtain the *speed transformation*:

$$u' = \frac{\sqrt{(u_x - v)^2 + (1 - v^2/c^2)(u_y{}^2 + u_z{}^2)}}{1 - u_x v/c^2} \qquad (6)$$

Mass transformations are best derived by considering an inelastic collision between two point masses moving toward each other with speed u (we are indebted to Ref. [3] for this argument). From (4) with $v = -u$ one has $u' = \frac{2u}{1 + (u/c)^2}$.

Then, assuming conservation of mass and momentum ($m_u{}'u' = M_u u$ and $m_u{}' + m_o = M_u$), it follows that:

$$m_u{}' = m_o\left(\frac{u'}{u} - 1\right)^{-1} = m_o\frac{1 + (u/c)^2}{1 - (u/c)^2} = \frac{m_o}{\sqrt{1 - u'^2/c^2}}$$

In order to express everything in the primed frame solely in terms of unprimed quantities, we substitute (6) in for u' in the above equation and arrive at the *mass transformation*:

$$m_u' = m_u\frac{1 - u_x v/c^2}{\sqrt{1 - v^2/c^2}} \qquad (7)$$

Since $\vec{F} = d\vec{p}/dt$ we use our velocity (4) and mass (7) transformations to write the *momentum transformation* and differentiate to obtain the *force transformation:*

$$\vec{p}' = m_u{}'\vec{u}' = \gamma(p_x - m_u)\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k},$$
$$\vec{F}' = \left(1 - \frac{u_x v}{c^2}\right)^{-1}\left[\left(F_x - \frac{dm_u}{dt}v\right)\hat{\imath} + \frac{F_y}{\gamma}\hat{\jmath} + \frac{F_z}{\gamma}\hat{k}\right] \qquad (8)$$

To find dm_u/dt we again follow Ref. [3] and use the work-energy theorem $K = \int \vec{F} \cdot d\vec{\ell} = \int \vec{F} \cdot \vec{u}\,dt$ where relativistic kinetic energy $K = E - E_0 = m_u c^2 - m_0 c^2$. Differentiating, and using the fundamental theorem of calculus, we have $\frac{dm_u}{dt} = \frac{1}{c^2}\vec{F} \cdot \vec{u}$. Putting this into (8) and using the inverse velocity transformations (5) , we find after some algebra that we can express force in the primed frame as $\vec{F}' = \vec{F}_1{}' + \vec{F}_2{}'$ where:

$$\vec{F}_1{}' = F_x\hat{\imath} + \gamma F_y\hat{\jmath} + \gamma F_z\hat{k} \qquad (9)$$
$$\vec{F}_2{}' = \frac{-\gamma v}{c^2}\left[F_y u_y{}' + F_z u_z{}'\right]\hat{\imath} + \frac{u_x' v}{c^2}\left[\gamma F_y\hat{k} + \gamma F_z\hat{k}\right]$$
$$= \vec{u} \times \left[\frac{\vec{v}}{c^2} \times \vec{F}_1{}'\right] \qquad (10)$$

The recognition of the vector triple product in the last step is the central "miracle" here.

From Coulomb to Biot-Savart

To apply the above formalism to Coulomb's law, we put (1) into (9) and (10), transforming $\vec{r}$ with the aid of (3) and assuming that $v \ll c$ so that $\gamma \approx 1$. This results in:

$$\vec{F}_1{}' = \frac{q_0 q}{4\pi\varepsilon_0}\frac{\gamma x'\hat{\imath} + \gamma y'\hat{\jmath} + \gamma z'\hat{k}}{\left(\gamma^2 x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}} = \frac{q_0 q}{4\pi\varepsilon_0 r'^3}\vec{r}' \qquad (11)$$
$$\vec{F}_2{}' = \vec{u} \times \left[\frac{\vec{v}}{c^2} \times \frac{q_0 q}{4\pi\varepsilon_0 r'^3}\vec{r}'\right] = q\vec{u} \times \left[q_0\vec{v} \times \frac{\vec{r}'}{4\pi\varepsilon_0 c^2 r'^3}\right]$$

$\vec{F}_1{}'$ is just the Coulomb force (1) in the primed frame, and $\vec{F}_2{}'$ is just the Biot-Savart law (2) in the form $\vec{F}_2{}' = q\vec{u} \times \vec{B}_m{}'$ since $q_0\vec{v} = Id\vec{\ell}$ and $1/\varepsilon_0 c^2 = \mu_0$. Electricity has been transformed into electricity + magnetism!

From Newton to Gravitomagnetism

What about gravity? Newton's law of gravitation reads:

$$\vec{F} = -\frac{Gm_0 m}{r^3}\vec{r} \qquad (12)$$

Putting this into (9) and (10) with the aid of (3), exactly as with Coulomb's law (2), we find that we can express force in the primed frame as $\vec{F}' = \vec{F}_1{}' + \vec{F}_2{}'$ where:

$$\vec{F}_1{}' = -\frac{Gm_0 m}{r'^3}\vec{r}' \qquad (13)$$
$$\vec{F}_2{}' = m\vec{u} \times \left[m_0\vec{v} \times \frac{-G\vec{r}'}{c^2 r'^3}\right] = m\vec{u} \times \left(\frac{Gm_0}{c^2 r'^3}\right)\vec{r}' \times \vec{v}$$

As before, $\vec{F}_1{}'$ is just Newton's law (12) in the transformed frame. The *gravitational analog of the Biot-Savart law* can

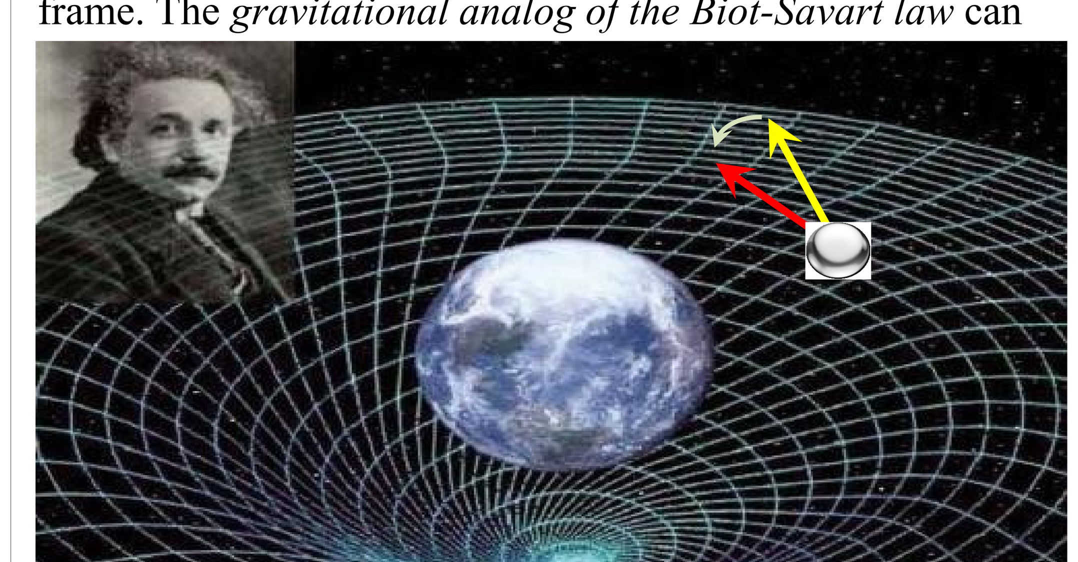

Fig. 3: In Einstein's theory of general relativity, gravity includes non-Newtonian terms associated with the mass and spin of the central body. Known as the geodetic and frame-dragging effects, these cause a gyroscope to precess as it orbits the Earth [5].

can be written $\vec{F}_2{}' = q\vec{u} \times \vec{B}_g{}'$ where the *"gravitomagnetic field"* of a mass M is defined by:

$$\vec{B}_g \equiv \left(\frac{GM}{c^2 r^3}\right)\vec{r} \times \vec{v} \qquad (14)$$

This quantity has units of s^{-1} or *spin* (in rad/s), as can easily checked for a test body in circular orbit around M, which has $v = \sqrt{\frac{GM}{r}}$ and $B_g = \left(\frac{GM}{c^2 r}\right)^{3/2}\frac{c}{r}$. Gravitomagnetism is thus a manifestation of spinning mass, just as magnetism is a manifestation of spinning charge.

Geodetic Precession

Eq. (14) turns out to be precisely $^2/_3$ of an effect known as *geodetic precession* $\vec{\Omega}_{geo}$ within general relativity, the extension of special relativity to curved spacetime [4]. This effect causes the spin axis of a test body to twist near a large mass (Fig. 3). It has been confirmed by the Gravity Probe B experiment (Fig. 4), which flew 4 gyroscopes in low-earth orbit at r = 7018 km, confirming that $\Omega_{geo} = \frac{3}{2}B_g$ = 6606 milliarcseconds/year to within an uncertainty of 0.28% [6].

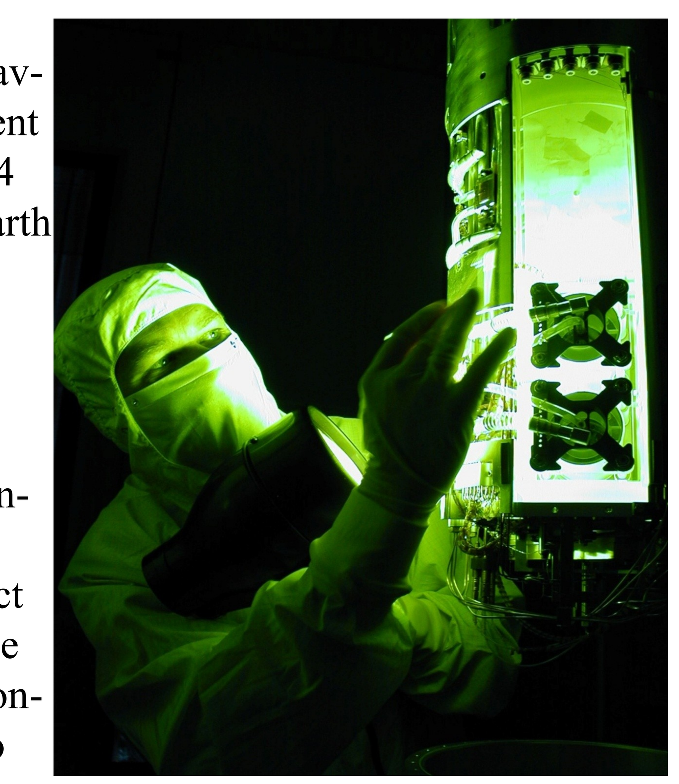

Fig. 4: a technician inspecting the gyroscope housing on the Gravity Probe B satellite

The geodetic effect is usually said to arise from two different contributions, one due to space curvature and the other a spin-orbit coupling between the spin of the gyroscope and the "mass current" of the central mass (as seen from the gyro's rest frame), an effect analogous to Thomas precession in electrodynamics. Various authors have differed on the relative importance of these effects, with the spin-orbit contribution said to make up anywhere from 0 to 4/3 of the total, possibly depending on the coordinates chosen ([7] and references therein). Insofar as our treatment is purely special relativistic, our results suggest that the spin-orbit effect should be held responsible for 2/3 of the total. This however remains a topic for further investigation.

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