Strong edge-magic graphs of maximum size

J.A. MacDougall\textsuperscript{a}, W.D. Wallis\textsuperscript{b}

\textsuperscript{a}School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia
\textsuperscript{b}Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA

Received 9 July 2003; received in revised form 31 March 2005; accepted 18 December 2006
Available online 3 June 2007

Abstract

An edge-magic total labeling on \( G \) is a one-to-one map \( \lambda \) from \( V(G) \cup E(G) \) onto the integers \( 1, 2, \ldots, |V(G) \cup E(G)| \) with the property that, given any edge \( (x, y) \), \( \lambda(x) + \lambda(x, y) + \lambda(y) = k \) for some constant \( k \). The labeling is strong if all the smallest labels are assigned to the vertices. Enomoto et al. proved that a graph admitting a strong labeling can have at most \( 2|V(G)| - 3 \) edges. In this paper we study graphs of this maximum size.

Keywords: Graph; Magic labeling

1. Preliminaries

All graphs are finite, simple and undirected. The graph \( G \) has vertex-set \( V(G) \) and edge-set \( E(G) \). Unless otherwise noted, \( |V(G)| = v \) and \( |E(G)| = e \).

A labeling of a graph is any map that carries some set of graph elements to numbers (usually to the positive or non-negative integers). Magic labelings are one-to-one maps onto the appropriate set of consecutive integers starting from 1, with some kind of “constant-sum” property. An edge-magic total labeling on \( G \) is a one-to-one map \( \lambda \) from \( V(G) \cup E(G) \) onto the integers \( 1, 2, \ldots, |V(G) \cup E(G)| \) with the property that, given any edge \( (x, y) \),

\[
\lambda(x) + \lambda(x, y) + \lambda(y) = k
\]

for some constant \( k \). Edge-magic total labelings were first discussed by Kotzig and Rosa [5]. Magic labelings are surveyed in [9]; a recent discussion of edge-magic total labelings can be found in [10].

2. Strong labelings

An edge-magic total labeling will be called strong if it has the property that the vertex-labels are the integers \( 1, 2, \ldots, v \), the smallest possible labels. A graph with a strong edge-magic total labeling will be called strongly edge-magic (abbreviated SEM).
Consider any edge-magic total labeling $\lambda$ of $G$. Summing (1) over all $e$ edges, we have

$$\sum_{(x,y) \in E(G)} (\lambda(x) + \lambda(x, y) + \lambda(y)) = ke.$$ 

In this sum, each edge label occurs once, and if $x_i$ has degree $d_i$ then $\lambda(x_i)$ occurs $d_i$ times. So if the labeling is to be strong, it is necessary to find a permutation $\{a_i\}$ of the first $v$ integers such that

$$e \text{ divides } \sum_{i=1}^{v+e} i + \sum_{i=1}^{v} (d_i - 1)a_i$$

(2)

(and set $\lambda(x) = a_i$). It follows that the even cycles are not strongly edge-magic. However, every odd cycle has a strong labeling [2]. Here is a strong labeling of the odd cycle with vertices $x_1, x_2, \ldots, x_{2n+1}$ (named in sequence round the cycle):

$$\lambda(x_i) \equiv 1 + ni \pmod{2n+1}$$

(3)

(the members of $\mathbb{Z}_{2n+1}$ being taken as $1, 2, \ldots, 2n + 1$). Even though even cycles are not strongly edge-magic, we recently showed [6] that any cycle (odd or even) with a single chord added is strongly edge-magic.

It was observed in [3] that a graph $G$ is strongly edge-magic if and only if there is a map $\lambda$ from $V(G)$ onto $\{1, 2, \ldots, v\}$ such that

$$S = \{\lambda(x) + \lambda(y) \mid xy \in E(G)\}$$

(4)

is a set of consecutive integers. The strong edge-magic total labeling is constructed from $\lambda$ as follows: if $s$ is the largest member of $S$, the constant $k$ is $v + 1 + s$, and $\lambda(xy) = k - \lambda(x) - \lambda(y)$ for each edge $xy$. We call this the edge-magic total labeling induced by the vertex labeling $\lambda$; the use of the same symbol $\lambda$ for both labelings should cause no confusion. It is convenient to call $\lambda(x) + \lambda(y)$ the weight of the edge $xy$.

There is a natural duality for strong edge-magic total labelings. If $\lambda$ is a vertex labeling for which (4) is consecutive, then the map $\lambda^*$, defined by

$$\lambda^*(x) = v + 1 - \lambda(x),$$

also has this property. We call $\lambda^*$ the strong dual of $\lambda$. Strong duality will be useful in complete listings of small strong labelings. Sometimes the strong dual is identical to the original labeling (examples are the labelings in Fig. 1); in this case the labeling is strongly self-dual.

3. Graphs with the maximum number of edges

From now on, when a strong labeling is given, we shall use the name $i$ for the vertex that receives label $i$ unless otherwise noted.

There is a limit on the size of a strongly edge-magic graph. Enomoto et al. proved the following useful result.
Lemma 1 (Enomoto et al. [2]). Any strongly edge-magic graph with \( v > 1 \) satisfies \( e \leq 2v - 3 \).

We are interested in the maximal strongly edge-magic graphs: those with \( e = 2v - 3 \), the largest possible number of edges. (To avoid trivial cases, we shall not consider \( K_1 \) to be maximal.) According to (4), the weights form a consecutive set, and for there to be \( 2v - 3 \) consecutive weights from \( v \) vertices, they must form the set \( \{3, 4, \ldots, 2v - 1\} \). As was observed in [3], the weights 3, 4, 2\( v - 2 \) and 2\( v - 1 \) are expressed uniquely as 3 = 1 + 2, 4 = 1 + 3, 2\( v - 1 \) = \( v + (v - 1) \) and 2\( v - 2 \) = \( v + (v - 2) \). That is, in any strongly edge-magic graph, the following pairs of vertices must be adjacent:

\((1, 2), (1, 3), (2, v - 2), (v - 1, v)\).

Every weight from 5 to 2\( v - 3 \) can be obtained by more than one choice of vertex pair and a labeled graph can be produced by choosing one of the possible pairs for each weight. For example, for \( v = 4 \), the one additional weight is 5 and choosing the edge \((1, 4)\) produces a different labeling than does choosing edge \((2, 3)\). In this case we get two different labelings of the same graph; they are shown in Fig. 1. The possible vertex-pairs for \( v = 5 \) and 6 are given in the tables

<table>
<thead>
<tr>
<th>Weight</th>
<th>Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>4</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>5</td>
<td>(1, 4), (2, 3)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 5), (2, 4)</td>
</tr>
<tr>
<td>7</td>
<td>(2, 5), (3, 4)</td>
</tr>
<tr>
<td>8</td>
<td>(3, 5)</td>
</tr>
<tr>
<td>9</td>
<td>(4, 5)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Weight</th>
<th>Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>4</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>5</td>
<td>(1, 4), (2, 3)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 5), (2, 4)</td>
</tr>
<tr>
<td>7</td>
<td>(1, 6), (2, 5), (3, 4)</td>
</tr>
<tr>
<td>8</td>
<td>(2, 6), (3, 5)</td>
</tr>
<tr>
<td>9</td>
<td>(3, 6), (4, 5)</td>
</tr>
<tr>
<td>10</td>
<td>(4, 6)</td>
</tr>
<tr>
<td>11</td>
<td>(5, 6)</td>
</tr>
</tbody>
</table>

and a choice of one pair for each weight determines a labeled graph. This simple observation is a convenient viewpoint, and some properties of strong edge-magic labelings are readily apparent because of it.

The number of pairs with weight \( t \) is \( \left\lfloor \frac{(t - 1)}{2} \right\rfloor! \left\lfloor \frac{n - t}{2} \right\rfloor! \left( \left\lfloor \frac{n - 1}{2} \right\rfloor! \right)^2 \) for \( t \leq v + 1 \) and \( \left\lfloor \frac{(2v - 1 - t)}{2} \right\rfloor! \) for \( t > v + 1 \), and multiplying the number of choices for each row leads to the following result, part of which is a special case of Theorem 24 of [3]. Define the function

\[ g(n) = \left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n - 2}{2} \right\rfloor! \left( \left\lfloor \frac{n - 1}{2} \right\rfloor! \right)^2. \]

Proposition 1. There are \( g(v) \) strong edge-magic labelings of graphs of order \( v \) and size \( 2v - 3 \), including \( \left\lfloor \frac{v}{2} \right\rfloor! \right\rfloor! \) that are strongly self-dual.

**Proof.** To count the self-dual labelings, we note that if a pair \((a, b)\) is chosen, then its complementary pair \((v + 1 - a, v + 1 - b)\) must also be chosen, so that it is only necessary to count the ways of making selections for weights up to weight \( v + 1 \).

For order 5, there are eight ways to select the pairs to determine a labeling. These labelings, four of which are self-dual, are shown in Fig. 2. All of the size 7 graphs are represented here. Fig. 2 already illustrates the fact that some graphs permit more (often many more) than one labeling.

At order 6 we find 48 distinct labelings, 12 of which are self-dual and 36 which come in 18 dual pairs. Of the 20 connected graphs with \( v = 6 \) and \( e = 9 \), eight have no strong labeling and the other 12 among them account for the 48 labelings. While Proposition 2 determines the number of labelings of order \( v \) graphs, it is a much harder problem to determine exactly how many graphs of order \( v \) admit a labeling.

If an edge with vertices \((a, b)\) appears in a labeling of a graph \( G \), then a new labeling is produced by deleting \((a, b)\) and replacing it by another edge \((c, d)\) with the same weight. The result may be a labeling of a different graph or a new labeling for \( G \). Consequently,
Proposition 2. Any strong edge-magic labeling of an order \( v \) graph can be obtained from any other by a sequence of single edge replacements.

Proposition 3. Every maximal strongly edge-magic graph with at least 4 vertices contains at least two triangles.

Proof. The proposition is true for \( v = 4 \) and 5 (see Figs. 1 and 2), so we may assume \( v \geq 6 \). Let \( G \) be strongly edge-magic with \( 2v - 3 \) vertices and labeling \( \lambda \). The weights of the edges of \( G \) (under \( \lambda \)) are 3, 4, \ldots, \((2v - 1)\).

If both 1 and \( v \) have degree \( v - 1 \), then \( G \) contains all \( v - 2 \) triangles \((1, a, v)\), for \( 1 < a < v \). So assume there is some vertex not adjacent to \( v \). Let \( u \) be the smallest such vertex. \( G \) contains an edge of weight \( u + v \), so there must be two vertices, \( c \) and \( d \) say, satisfying \( c + d = u + v \); clearly \( 1 < c, d < v \), so \( c, d > u \). By the minimality of \( u \), both \( c \) and \( d \) are adjacent to \( v \), so \((c, d, v)\) is a triangle.

Now suppose \( w \) is the smallest integer such that \( G \) contains no edge of weight \( w \) with one endpoint 1. Trivially \( w \leq v + 2 \), so \( w \leq 2v - 1 \) (as \( v > 3 \)), so there is an edge of weight \( w \), which must be an edge \((a, b)\) with \( a + b = w \), \( a, b > 1 \). Then \((1, a, b)\) is a triangle in \( G \). The two triangles are distinct, as neither \( c \) nor \( d \) can equal 1. \( \square \)

In particular, no bipartite graphs can admit a strong edge-magic total labeling.

It is easy to see that the result of Proposition 3 is best possible in that there exists a strongly edge-magic graph of order \( v \) containing exactly two triangles, for each \( v > 3 \). (An example will be provided by the family of graphs that we shall construct in the next section, and is illustrated in the second graph of Fig. 3.)

Proposition 4. Every maximal strongly edge-magic graph of order \( v \) can be extended to one of order \( v + 1 \).

Proof. Let \( G \) have order \( v \). A strong edge-magic labeling for \( G \) must contain the edge \((v - 1, v)\) with weight \( 2v - 1 \). Adjoin a new vertex with label \( v + 1 \), making it adjacent to these two vertices only. Then we have added edges with
weights $2v$ and $2v+1$ and the sequence of weights for the new graph is consecutive. Hence it determines an edge-magic labeling. □

**Theorem 1.** Let $G_1$ and $G_2$ be any maximal SEM graphs of orders $v$ and $w$, respectively. Then there are SEM graphs of orders $v + w - 2$, $v + w - 1$, and $v + w$ which contain $G_1$ and $G_2$ as induced subgraphs.

**Proof.** (1) Add $v - 2$ to all the vertex labels of $G_2$. Identify the vertices labeled $v$ and $v - 1$ from $G_1$ with those labeled $v$ and $v - 1$ from $G_2$. Then the new graph has order $v + w - 2$. By the discussion in Section 3, the vertices labeled $v$ and $v - 1$ must be adjacent in $G_1$ and the vertices that were originally labeled 1 and 2 in $G_2$ (and are now labeled $v$ and $v - 1$) must also be adjacent. From $G_1$ we get all the weights from 3 up to $2v - 1$ and from $G_2$ we get all the weights from $2v - 1$ up to $2(v + w - 2) - 1$. Since the repeated weight $2v - 1$ comes from the identified edge $(v, v - 1)$, each weight appears once and thus the labeling is strongly edge-magic.

(2) Add $v - 1$ to all the vertex labels of $G_2$ and identify the vertex labeled $v$ in $G_1$ with the vertex labeled $v$ from $G_2$. Then the new graph has order $v + w - 1$ and all weights from 3 up to $2(v + w - 1) - 1$ are accounted for except weight $2v$. Add any edge (necessarily between $G_1$ and $G_2$) of weight $2v$ such as $(v - 1, v + 1)$ and the resulting graph is strongly edge-magic of maximum size.

(3) Add $v$ to all the vertex labels of $G_2$. Then $G_1$ accounts for all the weights $[3, 4, \ldots, 2v - 1]$ and $G_2$ accounts for the weights $[2v + 3, \ldots, 2v + 2w - 1]$. The missing weights are $2v, 2v + 1, 2v + 2$ and these can be obtained by inserting the edges $(v - 1, v + 1)$ and $(v, v + 2)$ between $G_1$ and $G_2$. □

We could have obtained the results (2) and (3) of this theorem simply by applying Proposition 4 to the construction of part (1), but the embeddings described in (2) and (3) are different and more interesting.

4. Strong labelings of some families of graphs

In this section we construct labelings for several infinite families of maximal SEM graphs. The join of $K_1$ with any star $K_{1,v-2}$ is a graph with $v$ vertices and $2v - 3$ edges. Chen [11] showed that this graph is SEM in order to prove that the bound $2v - 3$ could always be attained; the labeling is shown in the first graph of Fig. 3. It is easy to prove that this is the unique SEM labeling for this graph. In fact, more is true.

**Proposition 5.** Let $G$ be a maximal SEM graph of order $v > 4$ having two vertices of degree $v - 1$. Then $E(G) = \{(1, i)|1 < i \leqslant v\} \cup \{(v, i)|1 \leqslant i < v\}$.

**Proof.** Since edges $(1, 2), (1, 3)$ must be in $G$, the vertex labeled 1 must be one of those of degree $v - 1$. Similarly $(v - 2, v)$ and $(v - 1, v)$ must be in $G$, so the other vertex of degree $v - 1$ must be labeled $v$. Thus the edges described above must be included, and since these represent all weights from 3 to $2v - 3$ once each, the result follows. □

The vertices labeled 1 and $v$ must have degree $v - 1$ in $G$, and the edge $(1, v) \in G$ has weight $v + 1$. By the discussion in the previous section, $(1, v)$ can be replaced by any other edge of weight $v + 1$ and a maximal SEM will result. This can clearly be done in $[v/2]$ different ways and because of the symmetry of $G$, all of these replacements will result in labelings of the same graph. One of these is shown in the second graph of Fig. 3. It is easy to show that there can be no other SEM labelings for this graph.

4.1. Triangulations of the $v$-cycle

The graph in Fig. 1 and the graph in the top row of Fig. 2 are the smallest order graphs in an important family of graphs with $2v - 3$ edges: the triangulations of the $v$-cycle. For large $v$, there are many non-isomorphic triangulations. We show that for every $v$ there is at least one which is strongly edge-magic. Define the graph $F_v$ as follows: let $V(F_v) = \mathbb{Z}_v$ and include as edges all pairs of the forms $(i, i + 1 \text{ (mod } v)), (i, v - i)$ and $(i, v - i - 1)$. Then an SEM labeling
of $F_v$ (which, in fact, is self-dual) is as follows:

$$\lambda(i) = 2i + 1, \quad 0 \leq i \leq \left\lfloor \frac{v-1}{2} \right\rfloor,$$

$$\lambda(i) = 2(v-i), \quad \left\lfloor \frac{v-1}{2} \right\rfloor < i \leq v - 1.$$ 

Already the cases of $v = 4$ and $5$ (Figs. 2 and 3) illustrate that there will be numerous strong labelings for this same graph. Labelings of other triangulations can be obtained from any of these by single edge replacement as described in Section 3. However, not all graphs in this family are SEM since it was shown in Theorem 8 of [3] that the fan $f_n = P_n \cup \{x\}$ is not SEM for $n > 6$. The proof of this result is now transparent using our point of view. For whatever label $l$ is chosen for vertex $x$, since $x$ is adjacent to all other vertices, either the three smallest weights or the three largest weights do not involve $l$ and therefore induce either $K_{1,3}$ or $K_3$ as a subgraph of $f_n - \{x\} = P_n$.

Determining precisely which triangulations of a cycle can be labeled appears to be a difficult problem.

4.2. Generalized prisms

There is only one graph with $v = 3$ having $e = 2v - 3 = 3$ edges, namely $K_3$, and it is trivially seen to be strongly edge-magic. A natural generalization of the correct size is the graph $K_3 \times P_n$ (which gives us the familiar triangular prism when $n = 2$). It is straightforward to give a strong labeling for $G = K_3 \times P_n$. $G$ consists of $n$ triangles which we identify as $T_1$ to $T_n$ with edges joining vertices in $T_i$ to the corresponding vertices of $T_{i-1}$ and $T_{i+1}$. Assign the labels $\{3i - 2, 3i - 1, 3i\}$ to triangle $T_i$ and make these vertices adjacent to the vertices $\{3i + 3, 3i + 1, 3i + 2\}$, respectively, in $T_{i+1}$. In triangle $T_i$ we get the edge sums $6i - 3, 6i - 2$ and $6i - 1$, which gives all weights $\equiv 3, 4, 5 \pmod{6}$. The edges between $T_i$ and $T_{i+1}$ have weights $6i, 6i + 1$ and $6i + 2$, giving all weights $\equiv 0, 1, 2 \pmod{6}$. Thus we get the consecutive set of weights $\{3, \ldots, 6n - 1\}$ as required. We note that this labeling can be regarded as a repeated application of Theorem 1 (part 3) with $G_2 = K_3$. The labeling of $K_3 \times P_3$ is illustrated in Fig. 4.

4.3. Graphs with large cliques

For a given $r$, what is the smallest possible order $s(r)$ for an SEM graph containing an $r$-clique? We shall obtain an upper bound on $s(r)$.

For the vertices of the clique to be labeled so that no weight is repeated, the labels must be chosen from a set of positive integers in which the sums of pairs of distinct elements are all different. Such a set is called a weak Sidon set. (Some authors simply call it a Sidon set; others reserve “Sidon” for the case when the restriction is removed that the elements be distinct.) When the members of a weak Sidon set are placed in ascending order, the resulting sequence is called a Sidon sequence or well-spread sequence.

The question requires us to choose such a sequence $S = (s_1, s_2, \ldots, s_r)$ in which the largest element $s_r$ is as small as possible. Let us write $\sigma(r)$ for this smallest possible value of $s_r$, taken over all Sidon sequences $S$. The evaluation of $\sigma(r)$ remains unsolved in general; an account can be found in [7], and a recent survey of this and more general problems is [8]. Trivially, a Sidon sequence of length $r$ with largest element $\sigma(r)$ must have smallest member 1.

It is easy to see that the Fibonacci sequence $(f_n)$, defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, is a Sidon sequence, provided $f_0$ is deleted. There are Sidon sequences with $s_r$ smaller than $f_r$—for example, $\sigma(7) = 19$, while $f_7 = 21$—but the Fibonacci numbers provide a reasonably good upper bound for the function $\sigma$. 

Fig. 4. Strong labeling of $K_3 \times P_3$. 

**Theorem 2.** There is a maximal connected SEM graph of order $f_r$ that contains a clique of order $r$.

**Proof.** Beginning with an $r$-clique whose vertices are labeled with $f_1, f_2, \ldots, f_r$, we adjoin vertices labeled with the remaining positive integers smaller than $f_r$. These are connected to the vertices of the clique as follows (we refer to vertices by their labels):

- If $f_{r-1} < x < f_r$, then $x$ is joined to $f_r$.
- If $f_{i-1} < x < f_i$, then $x$ is joined to $f_j$ whenever $j > i$.

The greatest edge weight in this graph is $2f_r - 1$. We need to verify that every integer from 3 (the least weight, which occurs on the edge joining 1 to 2) to $2f_r - 1$ arises as a weight. In order to see this we note that every edge joins vertices whose greater endpoint (label) is a Fibonacci number; if that number is $f_i$, where $i < r$, then the other endpoint is no greater than $f_{i-1}$.

Suppose $y$ is an integer between $f_{i-1}$ and $f_i$. If $y = f_{i-1} + z$ then $0 < z < f_{i-2}$, so there is an edge from $z$ to $f_{i-1}$, with weight $y$. This is the only edge of weight $y$ whose greater endpoint is $f_{i-1}$; any edge with its greater endpoint less than $f_{i-1}$ will have weight at most $f_{i-3} + f_{i-2} = f_{i-1} < y$, while those with greater endpoint $f_i$ or greater have weight greater than $y$.

Finally, edges with weight a Fibonacci number occur uniquely in the clique; and weights greater than $f_{r-1} + f_r$ occur uniquely with greater endpoint $f_r$. □

As an example, there is an SEM graph on $21 = f_7$ vertices containing a clique of size 7. It is shown in Fig. 5, as an illustration of the above construction.

However, we can do better.

**Theorem 3.** For every positive integer $n \geq 3$, there is an SEM graph of order $\sigma(n)$ and size $2\sigma(n) - 3$ containing a $K_n$.

**Proof.** We construct a graph with vertices 1, 2, \ldots, $\sigma(n)$. Suppose $S$ is a Sidon sequence of length $n$ and largest element $\sigma(n)$. Consider any positive integer $w$ less than $2\sigma(n)$. If $w$ is the sum of two elements $s_i$ and $s_j$ of $S$, join vertices $s_i$ and $s_j$. Otherwise, if $w \leq \sigma(n)$, join 1 to $(w - 1)$; if $w > \sigma(n)$, join $w - \sigma(n)$ to $\sigma(n)$. The $2n - 3$ edges constructed have $2\sigma(n) - 3$ consecutive weights, and the elements of $S$ form a $K_n$. □

As an example, Fig. 6(a) shows the graph for $n = 7$.

The graphs constructed in Theorem 3 are typically not connected. It is easy to construct connected examples: for $n \leq 6$, $\sigma(n) = f_n$, so the existence follows from Theorem 2. The following table exhibits suitable labelings for $n = 7, 8, 9$ and 10. The left-hand column contains the members of a minimal Sidon sequence, and all these vertices are adjacent,
forming the clique. On the right are shown the labels of vertices adjacent to the corresponding left-hand vertex. The case \( n = 7 \) is illustrated in Fig. 6(b).

\[
\begin{array}{cccc}
\sigma(7) = 19 & \sigma(8) = 25 & \sigma(9) = 35 & \sigma(10) = 46 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 8 \\
5 & 4 & 5 & 11 \\
9 & 4 & 9 & 14 \\
19 & 6 - 8, 10 - 13, 15 - 18 & 25 & 27 \\
16 - 19, 21 - 24 & 30 & 42 \\
& 44 & 46 & 5, 13, 15 - 17, 19, 21, 24, 26, 28 - 39, 41, 43, 45
\end{array}
\]

Acknowledgement

The authors wish to thank the referees for their helpful comments, and in particular one referee who suggested the direction of the proof of Theorem 3.

References