An approximation to the solution of hyperbolic equations by Adomian decomposition method and comparison with characteristics method

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Abstract

Adomian decomposition method has been applied to solve many functional equations so far. In this work, this method is applied to solve hyperbolic partial differential equations and results will be compared with characteristics method. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

Consider the second-order quasi-linear partial differential equation:

\[ a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0, \]  

(1)

where \( a, b, c \) and \( e \) may be functions of \( x, y, u, \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) but not of \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y} \) and \( \frac{\partial^2 u}{\partial y^2} \), i.e., the second-order derivatives occur only to the first degree. If

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$b^2 - 4ac > 0$, then Eq. (1) is called hyperbolic equation. Since the governing equations in many experiments in engineering as well as science leads to hyperbolic equations, as wave equation and telegraph equation, these equations have attracted much attention and solving the equations have been one of the interesting works for mathematicians. Numerical methods, which are commonly used as characteristics method needed large size of computation works and usually the round-off error causes the loss of accuracy. Adomian decomposition method has been applied to solve many functional equation and systems of functional equations. This method transforms Eq. (1) into a recursive relation. The Adomian decomposition method has proven to be very effective and results in considerable saving in computation time.

2. The Adomian decomposition method applied to hyperbolic equations

Consider hyperbolic Eq. (1) with the following indicated initial conditions:

$$u(x, 0) = f(x),$$  \hspace{1cm} (2)

$$\frac{\partial u(x, 0)}{\partial y} = g(x),$$  \hspace{1cm} (3)

Pay attention to initial conditions consider operator $L_{yy} = \frac{\partial^2}{\partial y^2}$. Therefore, we have:

$$\frac{\partial^2 u}{\partial y^2} = \frac{e}{c} - \frac{a}{c} \frac{\partial^2 u}{\partial x^2} - \frac{b}{c} \frac{\partial^2 u}{\partial x \partial y}. \hspace{1cm} (4)$$

The inverse operator of $L_{yy}$ is $L_{yy}^{-1} = \int_0^y \int_0^y (\cdot) \, dy \, dy$. Applying the inverse operator $L_{yy}^{-1}$ to both sides of (4), we get:

$$u(x, y) = u(x, 0) + \int_0^y \int_0^y \left( \frac{e}{c} + \frac{a}{c} \frac{\partial^2 u}{\partial x^2} + \frac{b}{c} \frac{\partial^2 u}{\partial x \partial y} \right) \, dy \, dy. \hspace{1cm} (5)$$

Substituting (2) and (3) into (5), we have:

$$u(x, y) = f(x) + g(x)y - \int_0^y \int_0^y \left( \frac{e}{c} + \frac{a}{c} \frac{\partial^2 u}{\partial x^2} + \frac{b}{c} \frac{\partial^2 u}{\partial x \partial y} \right) \, dy \, dy. \hspace{1cm} (6)$$

To solve Eq. (6) by Adomian decomposition method let consider, as usual in this method, the solution $u$ as of a series $u = \sum_{n=0}^{\infty} u_n$. So that the components $u_n$ will be determined recursive. And the integrand on the right side as the sum of a series as:

$$\frac{e}{c} + \frac{a}{c} \frac{\partial^2 u}{\partial x^2} + \frac{b}{c} \frac{\partial^2 u}{\partial x \partial y} = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n), \hspace{1cm} (7)$$
where \( A_n(u_0, u_1, u_2, \ldots, u_n) \) are called Adomian polynomials and should be computed. Adomian polynomials are calculated by using methods that introduced in [1,2,5,6]. Thus we have:

\[
\sum_{n=0}^{\infty} u_n = f(x) + g(x)y - \sum_{n=0}^{\infty} \int_0^y \int_0^y A_n(u_0, u_1, \ldots, u_n) \, dy \, dy.
\]

(8)

Therefore from (8) the following Adomian procedure can be defined:

\[
\begin{align*}
    u_0 &= f(x) + g(x)y \\
    u_{n+1} &= -\int_0^y \int_0^y A_n(u_0, u_1, \ldots, u_n) \, dy \, dy, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

(9)

We can determine the components \( u_n \) as far as we like to enhance the accuracy of the approximation. So, the \( n \)-terms \( \phi_n = \sum_{i=0}^{n-1} u_i \) can be used to approximate the solution.

3. Numerical results

**Example 1.** Consider the partial differential equation with the initial conditions [4]:

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + (1 - 2x) \frac{\partial^2 u}{\partial x \partial y} + (x^2 - x - 2) \frac{\partial^2 u}{\partial y^2} &= 0, \\
    u(x, 0) &= x, \\
    \frac{\partial u(x, 0)}{\partial y} &= 1.
\end{align*}
\]

By using (9) the Adomian scheme would be as follows:

\[
\begin{align*}
    u_0 &= x + y \\
    u_{n+1} &= \int_0^y \int_0^y \left( \left( \frac{2x - 1}{x^2 - x - 2} \right) \frac{\partial^2 u_0}{\partial x \partial y} - \frac{1}{x^2 - x - 2} \frac{\partial^2 u_0}{\partial x^2} \right) \, dy \, dy, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

and we have:

\[
\begin{align*}
    u_1 &= \int_0^y \int_0^y \left( \left( \frac{2x - 1}{x^2 - x - 2} \right) \frac{\partial^2 u_0}{\partial x \partial y} - \frac{1}{x^2 - x - 2} \frac{\partial^2 u_0}{\partial x^2} \right) \, dy \, dy = 0, \\
    u_2 &= 0, \\
    \vdots
\end{align*}
\]

Therefore \( u_n = 0, \, n \geq 1 \).

The solution is \( u(x, y) = x + y \), which is exact solution. In Table 1, for some values of \( x \) and \( y \) results this method and characteristics method are compared.
Example 2. Consider the following equation with initial conditions [4]:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} + 1 = 0,
\]

\[u(x, 0) = x,\]

\[\frac{\partial u(x, 0)}{\partial y} = x.\]

By using \(L_{yy}^{-1}\) yields

\[u = x + xy + \frac{y^2}{4} + \frac{1}{2} \int_0^y \int_0^y \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) dy dy.\]

From (9) we get:

\[
\begin{aligned}
    u_0 &= x + xy + \frac{y^2}{4}, \\
    u_{n+1} &= \frac{1}{2} \int_0^y \int_0^y \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial x \partial y} \right) dy dy, \quad n = 0, 1, 2, \ldots
\end{aligned}
\]

For the first few \(n\), we have:

\[
\begin{aligned}
    u_1 &= \frac{y^2}{4}, \\
    u_2 &= 0, \\
    u_3 &= 0, \\
    \vdots
\end{aligned}
\]

Therefore we would have the following exact solution:

\[u = u_0 + u_1 = x + xy + \frac{y^2}{2}.\]

Numerical results for some specified values of \(x\) and \(y\) are presented in Table 2.
Example 3. Let us solve the following partial differential equation [3]:

\[
\frac{\partial^2 u}{\partial x^2} - 4x^2 \frac{\partial^2 u}{\partial y^2} = 0,
\]

\[u(x, 0) = x^2, \quad \frac{\partial u(x, 0)}{\partial y} = 0.
\]

From (9) we obtain:

\[
\begin{align*}
\{ & u_0 = x^2, \\
& u_{n+1} = \frac{1}{4x} \int_0^y \int_0^y \frac{\partial u_0}{\partial x} \, dy \, dy, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

For the first few \(n\), we have:

\[
\begin{align*}
u_1 &= \frac{1}{4} y^2, \\
u_2 &= \frac{1}{32} y^4, \\
u_3 &= \frac{7}{640} y^6, \\
&\vdots
\end{align*}
\]

Five-term approximation to the solution will be as follows:

\[
u(x, y) \approx x^2 + \frac{1}{4} \frac{y^2}{x^2} + \frac{1}{32} \frac{y^4}{x^6} + \frac{7}{640} \frac{y^6}{x^{10}} + \frac{11}{2048} \frac{y^8}{x^{14}}.
\]

Some values of \(u(x, y)\) can be seen in Table 3.
4. Conclusions and discussion

The goal of this work has been to derive an approximation for solution of hyperbolic equations. We have achieved this goal by applying Adomian decomposition method. In some cases as Examples 1 and 2, after some steps the remaining terms would vanish and we derive the exact solution. In the cases as Example 3, the approximation can be obtained to any desired number of terms. The small size of computations in comparison with the computational size required in characteristics method and the rapid convergence show that the Adomian decomposition method is reliable and introduces a significant improvement in solving the hyperbolic equations. This method is very sensitive to initial conditions. If initial conditions are zero and there is no $e(x,y)$ term, the method would not function. Also, we use operator pay attention to initial condition. If initial conditions are $u(0,y) = f(y)$ and $\frac{\partial u(0,y)}{\partial x} = g(y)$, we consider $L_{xx} = \frac{\partial^2}{\partial x^2}$ and for $u(x,0) = f(x)$ and $u(0,y) = g(y)$ using from operator $L_{yy} = \frac{\partial^2}{\partial y^2}$.

References