Abstract

We consider a generalized Pompeiu equation in the space of Schwartz distributions and as an application we find the locally integrable solutions of the equation.

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1. Introduction

In the previous papers [2–6,8,9,13] several functional equations have been studied in the space of Schwartz distributions. Making use of the Schwartz theory of distributions one can differentiate freely the underlying unknown functions with local integrability assumed, which is one of powerful advantages of the Schwartz theory and is applied to solve some classical functional equations [2–4,8,9,13]. Nevertheless, it is more convenient, using regularizing functions given distributional versions of functional equations or inequalities to classical ones [4,5] than the approach using differentiation [2–4,8,9,13]. In this work following the same approach as in [4] using tensor products of regularizing functions
we solve the generalized Pompeiu equation
\[ f(x + y + xy) = g(x) + h(y) + g(x)h(y), \quad x, y \in I \] (1.1)
in the space \( \mathcal{D}'(I) \) of Schwartz distributions, where \( I = (-1, \infty) \) and \( f, g, h : I \to \mathbb{C} \). Note that a distribution \( u \in \mathcal{D}'(I) \) is a linear form on the space \( C_c^\infty(I) \) of infinitely differentiable functions with compact supports such that for every compact set \( K \subset I \) there exist constants \( C \) and \( k \) satisfying
\[
|\langle u, \varphi \rangle| \leq C \sum_{0 \leq j \leq k} \sup \varphi^{(j)}
\]
for all \( \varphi \in C_c^\infty(I) \) with supports contained in \( K \). We refer the reader to [11,14] for more details of distributions.

We reformulate the functional equation (1.1) in the space of Schwartz distributions as in [2–5]:
\[
\psi \circ u = v \circ P_1 + w \circ P_2 + v \otimes w,
\]
where \( S(x, y) = x + y + xy, P_1(x, y) = x, P_2(x, y) = y, x, y \in I \), and \( u \circ S, v \circ P_1, w \circ P_2 \) are the pullbacks of \( u, v, w \) in \( \mathcal{D}'(I) \) by \( S, P_1 \) and \( P_2 \), respectively, and \( \otimes \) denotes the tensor product of distributions [11,16,14].

As a result, every nontrivial solution \( u, v, w \) in \( \mathcal{D}'(I) \) of Eq. (1.2) has the form
\[
\begin{align*}
\psi(x) &= \alpha \beta(x + 1)^a - 1, \\
v(x) &= \alpha(x + 1)^a - 1, \\
w(x) &= \beta(x + 1)^a - 1,
\end{align*}
\]
where \( \alpha, \beta, a \in \mathbb{C} \). As an application we prove that every nontrivial locally integrable solution \( f, g, h \) of Eq. (1.1) has the form
\[
\begin{align*}
f(x) &= \alpha \beta(x + 1)^a - 1, \\
g(x) &= \alpha(x + 1)^a - 1, \\
h(x) &= \beta(x + 1)^a - 1,
\end{align*}
\]
for all \( x \in I \), where \( \alpha, \beta, a \in \mathbb{C} \).

We also give a minor remark on the result of Baker [2], which slightly improves his results.

2. Main theorems

We introduce a regularizing sequence. Consider a nontrivial function \( \psi \in C^\infty(\mathbb{R}) \) such that
\[
\supp \psi \subset \{ x \in \mathbb{R} : |x| \leq 1 \}, \\
\psi(x) \geq 0 \text{ in } \mathbb{R} \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)dx = 1.
\] (2.1)

We employ the function \( \psi_t(x) := t^{-1}\psi(x/t), t > 0 \). Let \( u \in \mathcal{D}'(\mathbb{R}) \). Then for each \( t > 0 \), \( (u * \psi_t)(x) = (u_y, \psi_t(x-y)) \) is a smooth function in \( \mathbb{R} \) and \( (u * \psi_t)(x) \to u \) as \( t \to 0^+ \) in the sense of distributions, that is, for every \( \varphi \in C_c^\infty(\mathbb{R}) \),
\[
\langle u, \varphi \rangle = \lim_{t \to 0^+} \int (u * \psi_t)(x) \varphi(x)dx.
\]
Theorem 2.1. Every nontrivial solution \( u, v, w \) in \( \mathcal{D}'(I) \) of Eq. (1.2) has the form
\[
\begin{align*}
u &= \alpha(x + 1)^a - 1, \\
w &= \beta(x + 1)^a - 1,
\end{align*}
\]
where \( \alpha, \beta, a \in \mathbb{C} \).

Proof. We first convert the functional equation (1.2) to a more convenient one. For this purpose, let
\[
E : \mathbb{R} \to I \text{ with } E(x) = e^x - 1 \text{ and denote by }
\]
the pullbacks of \( u, v, w \) by \( E \). Then \( u^*, v^*, w^* \in \mathcal{D}'(\mathbb{R}) \). Note that
\[
\langle u^*, \varphi \rangle = \langle u, \varphi((\ln(x + 1))/2(x + 1)) \rangle, \quad \varphi \in C_c^\infty(\mathbb{R}).
\]

It can be checked that Eq. (1.2) is converted to the following equation:
\[
(1.2')
\]
where \( A(x, y) = x + y \). Convolving the tensor product \( \psi_t(x)\psi_s(y) \) in each side of (1.2') we have
\[
[(u^* \circ A) \ast (\psi_t(x)\psi_s(y))](\xi, \eta) = \langle u^* \circ A, \psi_t(\xi - x)\psi_s(\eta - y) \rangle
\]
\[
= \left\{ u^*, \int \psi_t(\xi - x + y)\psi_s(\eta - y)dy \right\}
\]
\[
= \left\{ u^*, \int \psi_t(\xi + \eta - x - y)\psi_s(y)dy \right\}
\]
\[
= \langle u^*, (\psi_t \ast \psi_s)(\xi + \eta - x) \rangle
\]
\[
= (u^* \ast (\psi_t \ast \psi_s))(\xi + \eta).
\]

Similarly we have
\[
[(v^* \circ P_1) \ast (\psi_t(x)\psi_s(y))](\xi, \eta) = (v^* \ast \psi_t)(\xi),
\]
\[
[(w^* \circ P_2) \ast (\psi_t(x)\psi_s(y))](\xi, \eta) = (w^* \ast \psi_s)(\eta),
\]
\[
[(v^* \otimes w^*) \ast (\psi_t(x)\psi_s(y))](\xi, \eta) = (v^* \ast \psi_t)(\xi)(w^* \ast \psi_s)(\eta).
\]

Thus Eq. (1.2') is converted to the following equation:
\[
(1.2'')
\]
for all \( x, y \in \mathbb{R}, t, s > 0 \). Now we exclude the trivial cases where \( w^* \ast \psi_s = -1 \) for all \( s > 0 \) or \( v^* \ast \psi_t = -1 \) for all \( t > 0 \), which implies
\[
u^* = w^* = -1, \quad v^* : \text{arbitrary},
\]
or
\[
u^* = v^* = -1, \quad w^* : \text{arbitrary}.
\]

In view of (2.2) it is easy to see that
\[
U(x) := \lim_{t \to 0^+} (u^* \ast \psi_t)(x),
\]
\[
V(x) := \lim_{t \to 0^+} (v^* \ast \psi_t)(x), \\
W(x) := \lim_{t \to 0^+} (w^* \ast \psi_t)(x),
\]
exist. Letting \( t, s \to 0^+ \) in (2.2) we have
\[
U(x + y) = V(x) + W(y) + V(x)W(y),
\]for all \( x, y \in \mathbb{R} \). In view of (2.2) it is easy to see that \( U, V, W \) are smooth functions and \( V(0) \neq -1 \) and \( W(0) \neq -1 \). Thus the solution \( U, V, W \) of Eq. (2.3) has the form
\[
\begin{align*}
U(x) &= \alpha \beta e^{ax} - 1, \\
V(x) &= \alpha e^{ax} - 1, \\
W(x) &= \beta e^{ax} - 1,
\end{align*}
\]
for some \( \alpha, \beta, a \in \mathbb{C} \). Let \( t \to 0^+ \) in (2.2) to get
\[
(u^* \ast \psi_s)(x + y) = ((w^* \ast \psi_s)(y) + 1)(V(x) + 1) - 1
\]
for all \( x, y \in \mathbb{R}, s > 0 \). Letting \( y = 0 \) and \( s \to 0^+ \) in (2.4) we have
\[
\begin{align*}
u^* &= \alpha \beta e^{ax} - 1. \\
w^* &= \beta e^{ax} - 1.
\end{align*}
\]
Changing the roles of \( v^* \) and \( w^* \) we have
\[
\begin{align*}
v^* &= \alpha e^{ax} - 1. \\
w^* &= \beta e^{ax} - 1.
\end{align*}
\]
Here \( \alpha = V(0) + 1, \beta = W(0) + 1 \). Thus it follows that
\[
\begin{align*}
u &= u^* \circ E^{-1} = \alpha \beta (x + 1)^a - 1, \\
v &= v^* \circ E^{-1} = \alpha (x + 1)^a - 1, \\
w &= w^* \circ E^{-1} = \beta (x + 1)^a - 1.
\end{align*}
\]
This completes the proof. \( \Box \)

**Corollary 2.2.** Every locally integrable solution \( f, g, h \) of Eq. (1.1) has the form
\[
\begin{align*}
f(x) &= \alpha \beta (x + 1)^a - 1, \\
g(x) &= \alpha (x + 1)^a - 1, \\
h(x) &= \beta (x + 1)^a - 1,
\end{align*}
\]
for all \( x \in I \), where \( \alpha, \beta, a \in \mathbb{C} \).

**Proof.** Note that every locally integrable function \( f : I \to \mathbb{C} \) can be regarded as a member of \( D'(I) \) via the equation
\[
\langle f, \varphi \rangle = \int f(x) \varphi(x) dx, \quad \varphi \in C_c^\infty(I).
\]
Thus by the above theorem the equalities (2.5)–(2.7) hold in the sense of distribution, which implies
\[
f(x) = \alpha \beta (x + 1)^a - 1,
\]
for all \( x \) in a set \( \Omega \subset I \) with the Lebesgue measure \( m(I - \Omega) = 0 \). Let \( x \in I \) be given. Define \( p : I \to I \) by \( p(t) = (1 + x)t + x \). Then \( m(I - p^{-1}(\Omega)) = m(p^{-1}(I - \Omega)) = 0 \), which implies \( \Omega \cap p^{-1}(\Omega) \neq \emptyset \), and there exists \( y \in \Omega \) such that \( x + y + xy \in \Omega \). Then it follows from Eq. (1.1) that

\[
g(x) = \frac{f(x + y + xy) - h(y)}{1 + h(y)} \]

which implies (2.6). Changing the roles of \( g, h \) we get (2.7). Put (2.6) and (2.7) in Eq. (1.1) to get (2.5). This completes the proof. \( \Box \)

**Remark 2.3.** General solutions \( f, g, h \) of real valued functions of Eq. (1.1) are well known [12,13]. Here we give general solutions of complex valued functions in a transparent way. Making the same change of variables as in the proof of Theorem 2.1, Eq. (1.1) is converted to the classical exponential–Pexider type functional equation

\[
F(x + y) = G(x)H(y),
\]

(2.9)

where \( F(x) = f(e^x - 1) + 1, G(x) = g(e^x - 1) + 1, H(x) = h(e^x - 1) + 1 \). The general solutions \( f, g, h \) follow if we find the solutions \( F, G, H \) of Eq. (2.9). Using Theorem 3 in [1, p. 54] we can check that every nontrivial solution of complex valued functions \( F, G, H \) of the exponential equation (2.9) has the form

\[
\begin{align*}
F(x) &= \alpha \beta e^{p(x) + iq(x)}, \\
G(x) &= \alpha e^{p(x) + iq(x)}, \\
H(x) &= \beta e^{p(x) + iq(x)},
\end{align*}
\]

where \( p : \mathbb{R} \to \mathbb{R} \) is an arbitrary solution of the Cauchy equation

\[
p(x + y) = p(x) + p(y),
\]

and \( q : \mathbb{R} \to \mathbb{R} \) is an arbitrary solution of the equation

\[
q(x + y) \equiv q(x) + q(y) \pmod{2\pi}.
\]

**Remark 2.4.** Our method employed in the Corollary 2.2 can be applied to some other functional equations such as the Pexider equations, another Pompeiu equation and the equation in [8]:

\[
\begin{align*}
f(xy) &= g(x) + h(y), \\
f(xy) &= g(x)h(y), \\
f(x + y - xy) &= g(x) + h(y) - g(x)h(y), \\
f(x + y) + g(x - y) &= h(xy).
\end{align*}
\]

Finally we give a minor remark on the results of Baker [2].
Remark 2.5. In [2], Baker considered the functional equations of Daróczy [7] and Heuvers [10], respectively,
\[
f \left( \frac{x + y}{2} \right) + f \left( \frac{2xy}{x + y} \right) = f(x) + f(y), \quad x, y \in I, \quad (2.10)
\]
\[
g(x + y) - g(x) - g(y) = g \left( \frac{1}{x} + \frac{1}{y} \right), \quad x, y > 0, \quad (2.11)
\]
in the space of distributions, where \( I \) is an open subinterval of \((0, \infty)\). As an application, he showed that locally integrable solutions \( f, g \) of Eqs. (2.10) and (2.11) are of the forms, respectively,
\[
f(x) = \alpha \log x + \gamma, \quad a.e. \ x \in I, \quad (2.10')
\]
\[
g(x) = \alpha \log x, \quad a.e. \ x > 0 \quad (2.11')
\]
for some \( \alpha, \gamma \in \mathbb{C} \). Here we remark that both the equalities (2.10') and (2.11') hold for all \( x \) in the domains. Indeed, as in the proof of Corollary 2.2, let \( \Omega \) be the set of all \( x \in I \) for which (2.10') holds. For given \( z \in I \), we define \( M, H : I \rightarrow I \) by \( M(t) = \frac{z+t}{2} \) and \( H(t) = \frac{2zt}{z+t} \). Then \( m(I - M^{-1}(\Omega)) = m(M^{-1}(I - \Omega)) = 0, m(I - H^{-1}(\Omega)) = m(H^{-1}(I - \Omega)) = 0 \), which imply \( \Omega \cap M^{-1}(\Omega) \cap H^{-1}(\Omega) \neq \emptyset \) and there exists \( y \in \Omega \) such that \( \frac{z+y}{2}, \frac{2zy}{z+y} \in \Omega \). Then it follows from Eqs. (2.10) and (2.10') that \( f(z) = \alpha \log z + \gamma \). This approach works also for Eq. (2.11).

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References