Bartholdi zeta functions of graph bundles having regular fibers

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Abstract

As a continuation of computing the Bartholdi zeta function of a regular covering of a graph by Mizuno and Sato in J. Combin. Theory Ser. B 89 (2003) 27, we derive in this paper some computational formulae for the Bartholdi zeta functions of a graph bundle and of any (regular or irregular) covering of a graph. If the fiber is a Schreier graph or it is regular and the voltages to derive the bundle or the covering lie in an Abelian group, then the formulae can be simplified. As a byproduct, the Bartholdi zeta functions of Schreier graphs, Cayley graphs and the cartesian product of a graph and a regular graph are obtained.

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1. Introduction

In this paper we consider an undirected finite simple graph. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $v_G$ and $e_G$ denote the number of vertices and edges of $G$, respectively. An automorphism of $G$ is a permutation of the vertex set $V(G)$ that preserves adjacency. The set of automorphisms forms a permutation group, called the automorphism group $\text{Aut}(G)$ of $G$.
A \((v_0, v_n)\)-path \(P\) of length \(n\) in \(G\) is a sequence \(P = (v_0, v_1, \ldots, v_{n-1}, v_n)\) of \(n + 1\) vertices and \(n\) edges such that consecutive vertices share an edge (we do not require that all vertices are distinct). Sometimes, the path \(P\) is also considered as a subgraph of \(G\). We say that a path has a backtracking if \(v_{i-1} = v_{i+1}\) for some \(i, 1 \leq i \leq n - 1\). A \((v_0, v_n)\)-path is called a cycle if \(v_0 = v_n\). The inverse cycle of a cycle \(C = (v_0, v_1, \ldots, v_{n-1}, v_0)\) is the cycle \(C^{-1} = (v_0, v_{n-1}, \ldots, v_1, v_0)\).

A subpath \((v_1, \ldots, v_m, v_{m+1})\) of a cycle \(C = (v_1, \ldots, v_m, \ldots, v_1)\) is called a tail if \(\deg_C(v_1) = 1, \deg_C(v_i) = 2, 2 \leq i \leq m - 1,\) and \(\deg_C(v_m) \geq 3\), where \(\deg_C(v)\) is the degree of \(v\) in the subgraph \(C\). Each cycle \(C\) without backtracking determines a unique tailless, backtracking-less cycle \(C^*\) by removing all tails of \(C\). Note that any tail-less, backtracking-less cycle \(C\) is just a cycle such that both \(C\) and \(C^*\) have no backtracking (see [6, 12]). A cycle \(C\) is called reduced if it has no backtracking nor tail. Two reduced cycles \(C_1 = (v_1, \ldots, v_m)\) and \(C_2 = (w_1, \ldots, w_m)\) are called equivalent if there is \(k\) such that \(w_j = v_{j+k}\) for all \(j\), where the subscripts are modulo \(m\). Let \([C]\) be the equivalence class which contains a cycle \(C\). A reduced cycle \(C\) is prime if there is no cycle \(B\) such that \(C = B^r\) for \(r \geq 2\). Note that each equivalence class of prime, reduced cycles of a graph \(G\) corresponds to a unique element of the fundamental group \(\pi_1(G, v)\) of \(G\) at a vertex \(v \in V(G)\).

The Ihara zeta function [14] of a graph \(G\) is defined to be the function of \(u \in \mathbb{C}\) with \(|u|\) sufficiently small, given by

\[
Z(G, u) = Z_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime, reduced cycles of \(G\) and \(|C|\) denote the length of \(C\). Clearly, the zeta function of a disconnected graph is the product of the zeta functions of its connected components. Zeta functions of graphs were originated from zeta functions of regular graphs by Ihara [7], where their reciprocals are expressed as explicit polynomials. A zeta function of a regular graph \(G\) associated to a unitary representation of the fundamental group of \(G\) was developed by Sunada [15]. Hashimoto [6] treated multivariable zeta functions of bipartite graphs. Northshield [12] proved that the number of spanning trees in a graph \(G\) can be expressed in terms of the zeta function \(Z_G(u)\).

Let \(G\) be a connected graph. The adjacency matrix \(A(G) = (a_{ij})\) is the \(v_G \times v_G\) matrix with \(a_{ij} = 1\) if \(v_i\) and \(v_j\) are adjacent and \(a_{ij} = 0\) otherwise. Let \(D_G\) be the diagonal matrix whose \((i, i)\)-entry is \(d_G = \deg_G(v_i)\) for each \(1 \leq i \leq v_G\). The Ihara’s result on zeta functions of regular graphs is generalized as follows.

**Theorem 1** (Bass [2]). Let \(G\) be a connected graph. Then the reciprocal of the Ihara zeta function of \(G\) is given by

\[
Z_G(u)^{-1} = (1 - u^2)^{v_G - v_G} \det(I - uA(G) + u^2(D_G - I)). \quad \Box
\]

Note that Theorem 1 is still true for a disconnected graph \(G\).

Later, Stark and Terras [14] gave an elementary proof of Theorem 1 and discussed three different zeta functions of any graph. Mizuno and Sato [10] gave a decomposition formula for the zeta function of a group covering of a graph.
Let $G$ be a connected graph. For each $u, v \in V(G)$, let $[u, v]$ be the set of all $(u, v)$-paths in $G$. We say that a path $P = (v_0, \ldots, v_n)$ has a bump at the $v_i$ if $v_{i+1} = v_{i-1}$. The bump count $bc(P)$ of a path $P$ is the number of bumps in $P$. Furthermore, the cycle bump count $cbc(C)$ of a cycle $C = (v_0, \ldots, v_n)$ is
\[cbc(C) = |\{i : v_{i+1} = v_{i-1}, 1 \leq i \leq n\}|,\]
where $v_{n+1} = v_0$ and $|X|$ denotes the cardinality of a finite set $X$. In [1], Bartholdi defined a zeta function of $G$, called the Bartholdi zeta function, as follows: for two complex numbers $u$ and $t$ with $|u|$ and $|t|$ sufficiently small,
\[Z_G(u, t) = Z(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)}t^{|C|})^{-1},\]
where $[C]$ runs over all equivalence classes of prime cycles of $G$.

The following theorem shows how one can compute the Bartholdi zeta function of a graph $G$.

**Theorem 2** (Bartholdi [1]). Let $G$ be a connected graph. Then the reciprocal of the Bartholdi zeta function of $G$ is given by
\[Z_G(u, t)^{-1} = (1 - (1-u)^2t^2)^{G \times G} \times \det[I - tA(G) + (1-u)t^2(D_G - (1-u)I)].\]

In this paper, we aim to compute the Bartholdi zeta function of a graph bundle. In Section 2, a formula for the Bartholdi zeta function of a graph bundle is derived by Theorem 2 and a decomposition formula for it when the fiber is a Schreier graph. In Section 3, we compute the Bartholdi zeta function of a graph bundle when its voltages lie in an Abelian group. As a special case, we compute the Bartholdi zeta function of the cartesian product of a graph and a regular graph. In the last section, we show how the Bartholdi or Ihara zeta function of a regular graph can be computed by using its spectra and how it can be related to the characteristic polynomial of the graph with weights on vertices and edges.

## 2. Computing the Bartholdi zeta functions of graph bundles

Let $G$ be a connected graph and let $\tilde{G}$ be the digraph obtained from $G$ by replacing each edge of $G$ with a pair of oppositely directed edges. The set of directed edges of $G$ is denoted by $E(\tilde{G})$. By $e^{-1}$, we mean the reverse edge to an edge $e \in E(\tilde{G})$. We denote the directed edge $e$ of $\tilde{G}$ by $uv$ if the initial and terminal vertices of $e$ are $u$ and $v$, respectively. For a finite group $\Gamma$, a $\Gamma$-voltage assignment of $G$ is a function $\phi : E(\tilde{G}) \rightarrow \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\tilde{G})$. We denote the set of all $\Gamma$-voltage assignments of $G$ by $C^1(G; \Gamma)$.

Let $F$ be another graph and let $\phi \in C^1(G; \text{Aut}(F))$. Now, we construct a graph $G \times^\phi F$ with the vertex set $V(G \times^\phi F) = V(G) \times V(F)$, and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times^\phi F$ if either $u_1u_2 \in E(\tilde{G})$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. We call $G \times^\phi F$ the $F$-bundle over $G$ associated with $\phi$.
(or, simply a graph bundle) and the first coordinate projection induces the bundle projection 
\( p^\phi : G \times^\phi F \to G \). The graphs \( G \) and \( F \) are called the base and the fiber of the graph bundle \( G \times^\phi F \), respectively. Note that the map \( p^\phi \) maps vertices to vertices, but the image of an edge can be either an edge or a vertex. If \( F = \overline{K}_n \), the complement of the complete graph \( K_n \) of \( n \) vertices, then an \( F \)-bundle over \( G \) is just an \( n \)-fold graph covering over \( G \).

If \( \phi(e) \) is the identity of \( \text{Aut}(F) \) for all \( e \in E(\overline{G}) \), then \( G \times^\phi F \) is just the cartesian product of \( G \) and \( F \) (see [8]).

Let \( \phi \) be an \( \text{Aut}(F) \)-voltage assignment of \( G \). For each \( \gamma \in \text{Aut}(F) \), let \( \overline{G}_{(\phi, \gamma)} \) denote the spanning subgraph of the digraph \( \overline{G} \) whose directed edge set is \( \phi^{-1}(\gamma) \). Thus the digraph \( \overline{G} \) is the edge-disjoint union of spanning subgraphs \( \overline{G}_{(\phi, \gamma)} \), \( \gamma \in \text{Aut}(F) \). Let \( V(\overline{G}) = \{u_1, u_2, \ldots, u_{\nu_G}\} \) and \( V(F) = \{v_1, v_2, \ldots, v_{\nu_F}\} \). We define an order relation \( \leq \) on \( V(G \times^\phi F) \) as follows: for \( (u_i, v_k), (u_j, v_l) \in V(G \times^\phi F), (u_i, v_k) \leq (u_j, v_l) \) if and only if either \( k < \ell \) or \( k = \ell \) and \( i \leq j \). Let \( P(\gamma) \) denote the \( \nu_F \times \nu_F \) permutation matrix associated with \( \gamma \in \text{Aut}(F) \) corresponding to the action of \( \text{Aut}(F) \) on \( V(F) \), i.e., its \( (i, j) \)-entry \( P(\gamma)_{ij} = 1 \) if \( \gamma v_i = v_j \) and \( P(\gamma)_{ij} = 0 \) otherwise. Then for any \( \gamma, \delta \in \text{Aut}(F) \), \( P(\delta \gamma) = P(\delta)P(\gamma) \). The tensor product \( A \otimes B \) of the matrices \( A \) and \( B \) is considered as the matrix \( B \) having the element \( b_{ij} \) replaced by the matrix \( Ab_{ij} \). Kwak and Lee [9] expressed the adjacency matrix \( A(G \times^\phi F) \) of a graph bundle \( G \times^\phi F \) as follows.

**Theorem 3** (Kwak and Lee). Let \( G \) and \( F \) be graphs and let \( \phi \) be an \( \text{Aut}(F) \)-voltage assignment of \( G \). Then the adjacency matrix of the \( F \)-bundle \( G \times^\phi F \) is

\[
A(G \times^\phi F) = \left( \sum_{\gamma \in \text{Aut}(F)} A(\overline{G}_{(\phi, \gamma)}) \otimes P(\gamma) \right) + I_{\nu_G} \otimes A(F),
\]

where \( P(\gamma) \) is the \( \nu_F \times \nu_F \) permutation matrix associated with \( \gamma \in \text{Aut}(F) \) corresponding to the action of \( \text{Aut}(F) \) on \( V(F) \), and \( I_{\nu_G} \) is the identity matrix of order \( \nu_G \).

For any vertex \( (u_i, v_k) \in V(G \times^\phi F) \), its degree is \( d_G^G(u_i) + d_F^F(v_k) \), where \( d_G^G(u_i) = \text{deg}_G(u_i) \) and \( d_F^F(v_k) = \text{deg}_F(v_k) \). Therefore, \( D_{G \times^\phi F} = D_G \otimes I_{\nu_F} + I_{\nu_G} \otimes D_F \) and \( D_{G \times^\phi F} - (1 - u)I_{\nu_F} = (D_G \otimes I_{\nu_F} + I_{\nu_G} \otimes D_F) - ((1 - u)I_{\nu_F}) \otimes I_{\nu_F} = (D_G - (1 - u)I_{\nu_G}) \otimes I_{\nu_F} + I_{\nu_G} \otimes D_F \).

Notice that

\[
\varepsilon_{G \times^\phi F} - \nu_{G \times^\phi F} = \nu_F \varepsilon_G + \nu_F \varepsilon_F - \nu_G \varepsilon_F = (\varepsilon_G - \varepsilon_F)I_{\nu_F} + \nu_F \varepsilon_F.
\]

Now, the following theorem follows immediately from **Theorem 2**.

**Theorem 4.** Let \( G \) and \( F \) be two connected graphs and let \( \phi \) be an \( \text{Aut}(F) \)-voltage assignment of \( G \). Then the reciprocal of the Bartholdi zeta function of the graph bundle \( G \times^\phi F \) is

\[
Z_{G \times^\phi F}(u, t) = (1 - (1 - u)^2t^2)^{(\varepsilon_G - \varepsilon_F)I_{\nu_F} + \nu_F \varepsilon_F}
\]

\[
\times \det \left[ I_{\nu_G} - t \sum_{\gamma \in \text{Aut}(F)} (A(\overline{G}_{(\phi, \gamma)}) \otimes P(\gamma) + I_{\nu_G} \otimes A(F))
\]

\[
+ (1 - u)t^2((D_G - (1 - u)I_{\nu_G}) \otimes I_{\nu_F} + I_{\nu_G} \otimes D_F) \right].
\]
In the following, we consider three particular cases of Theorem 4: (i) $F$ is a Schreier graph, (ii) $F = \overline{K_n}$, (iii) $\phi = 1$ is trivial, or more generally all voltages lie in an Abelian group. The last cases will be treated in Section 3 later.

Let $B$ be a subgroup of a group $A$ and let $S = \{x_1, x_2, \ldots, x_n\}$ be a symmetric subset set of $A$. The Schreier graph is the graph whose vertex set is the right cosets of $B$ in $A$, and there is an edge between two vertices $Ba$ and $Bb$ if and only if $Bb = Bax_i$ for some $x_i \in S$. For the special case $B = \{1\}$, the Schreier graph is just the Cayley graph. Observe that the group $A$ acts transitively on the right cosets of $B$ by right multiplication. By the permutation group theory, one can say that a simple graph $F$ is a Schreier graph if there exists a symmetric subset $S$ of $\phi(S)$ such that any two vertices $v_i$ and $v_j$ are adjacent if and only if $v_j = v_i$ for some $s \in S$ (see Section 2.4.4 in [5]). We call such an $S$ the connecting set of the Schreier graph $F$. Notice that a Schreier graph $F$ is connected if and only if the subgroup $S$ generated by the connecting set $S$ acts transitively on $\{1, 2, \ldots, s\}$. Moreover, a Schreier graph with connecting set $S$ is a regular graph of degree $|S|$ and most regular graphs are Schreier graphs (see Section 2.3.4 in [5]).

Now, let $F$ be a Schreier graph with connecting set $S$. Then the adjacency matrix of $F$ is $A(F) = \sum_{s \in S} P(s)$. Hence, for any voltage assignment $\phi : E(\mathcal{G}) \to Aut(F)$, one can have

$$A(G \times^\phi F) = \left( \sum_{\gamma \in Aut(F)} A(\mathcal{G}(\phi, \gamma)) \otimes P(\gamma) \right) + I_{vG} \otimes \sum_{s \in S} P(s).$$

Let $\Gamma$ be the subgroup of $S_{vF}$ generated by $\{\phi(e), e \in E(\mathcal{G}), s \in S\}$. A representation $\rho$ of the group $\Gamma$ over the complex field $\mathbb{C}$ is a group homomorphism from $\Gamma$ to the general linear group $GL(r, \mathbb{C})$ of invertible $r \times r$ matrices over $\mathbb{C}$. The number $r$ is called the degree of the representation $\rho$. Clearly, the homomorphism $P : \Gamma \to GL(n, \mathbb{C})$ defined by $\gamma \mapsto P(\gamma)$ is a representation of $\Gamma$, called the permutation representation of $\Gamma$. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ and let $f_i$ be the degree of $\rho_i$ for $1 \leq i \leq \ell$, so that $\sum_{i=1}^{\ell} f_i^2 = |\Gamma|$. Then, the permutation representation $P$ can be decomposed as the direct sum of irreducible representations: say $P = \bigoplus_{i=1}^{\ell} m_i \rho_i$ with multiplicities $m_i$ (see [16]). Moreover, there exists an invertible matrix $M$ such that

$$M^{-1} P(\gamma) M = \bigoplus_{i=1}^{\ell} (\rho_i(\gamma) \otimes I_{m_i})$$

for any $\gamma \in \Gamma$, where $A_1 \oplus \cdots \oplus A_k$ denotes the block diagonal sum of matrices with block diagonals $A_1, \ldots, A_k$ consecutively. Furthermore, it is known [13] that $m_1 \geq 1$ since it is the number of orbits under the action of the group $\Gamma$. Note that $\sum_{i=1}^{\ell} m_i f_i = v_F$. Since $F$ is regular of degree $|S|$, one can see that

$$(1 - u)^2 ((D_G - (1 - u)I_{vG}) \otimes I_{vF} + I_{vG} \otimes D_F)$$

$$= (1 - u)^2 (D_G - (1 - u - |S|)I_{vG}) \otimes I_{vF}$$

and $\epsilon_F = \frac{v_F |S|}{}$. Hence,
(I_{V_G} \otimes M)^{-1} (1 - tA(G \times^\phi F) + (1 - u)t^2(D_{G \times^\phi F} - (1 - u)I))(I_{V_G} \otimes M)

= I_{V_{GF}} - \left[ \sum_{i=1}^t \ell \left( \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes \rho_i(\gamma) + I_{V_G} \otimes \sum_{s \in S} \rho_i(s) \right) \otimes I_{m_i} \right.

+ (1 - u)t^2(D_G - (1 - u - |S|)I_{V_G}) \otimes I_{V_{GF}}

= \ell \left[ I_{V_{GF}} - t \left( \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes \rho_i(\gamma) + I_{V_G} \otimes \sum_{s \in S} \rho_i(s) \right) \right.

+ (1 - u)t^2(D_G - (1 - u - |S|)I_{V_G}) \otimes I_{f_i} \left. \otimes I_{m_i} \right].

This proves the following theorem.

**Theorem 5.** Let G be a connected graph, F a Schreier graph with connecting set S and let \( \phi : E(\tilde{G}) \to \text{Aut}(F) \) be a voltage assignment. Let \( \Gamma \) be the subgroup of the symmetric group \( S_n \) generated by \( \{\phi(e), s : e \in E(\tilde{G}), s \in S\} \), and let \( \rho_1, \rho_2, \ldots, \rho_\ell \) be the irreducible representations of \( \Gamma \) having degree \( 1, f_2, \ldots, f_\ell \), respectively. Then the reciprocal of the Bartholdi zeta function of the F-bundle \( G \times^\phi F \) of G is

\[
Z_{G \times^\phi F}(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{\frac{\nu_G}{2} G} \prod_{i=1}^\ell [(1 - (1 - u)^2 t^2)^{(1 - u)^{-1} f_i} T(u, t, \rho_i, \phi)]^{m_i},
\]

where

\[
T(u, t, \rho_i, \phi) = \det \left[ I_{V_{GF}} - t \left( \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes \rho_i(\gamma) + I_{V_G} \otimes \sum_{s \in S} \rho_i(s) \right) \right.

+ (1 - u)t^2(D_G - (1 - u - |S|)I_{V_G}) \otimes I_{f_i} \left. \right],
\]

and \( m_i \) is the multiplicity of \( \rho_i \) in the permutation representation \( P \) of \( \Gamma \). \( \square \)

Notice that if G is a regular graph of degree \( d_G \), then one can simplify \( T(u, t, \rho_i, \phi) \) as

\[
T(u, t, \rho_i, \phi) = \det \left[ I_{V_{GF}} - t \left( \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes \rho_i(\gamma) + I_{V_G} \otimes \sum_{s \in S} \rho_i(s) \right) \right.

+ (1 - u)t^2(d_G + |S| + u - 1)I_{V_G} \otimes I_{f_i} \left. \right].
\]
Let $F = \overline{K_n}$ be the trivial graph on $n$ vertices. Then any $\text{Aut}(\overline{K_n})$-voltage assignment is just a permutation voltage assignment defined in [5], and $G \times^\phi \overline{K_n} = G^\phi$ is just an $n$-fold covering graph of $G$. In this case, it may not be a regular covering. Notice that $A(G) = \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma))$ and $\overline{K_n}$ is a Schreier graph with empty connecting set. Hence,

$$T(u, t, \rho, \phi) = \left[ I_{\nu} - t A(G) + (1-u)t^2(D_G - (1-u)I_{\nu}) \right]^{-1} \left[ Z_G(u, t) \right]^{-1}.$$

Now, the following comes from Theorem 5.

**Corollary 6.** The reciprocal of the Bartholdi zeta function of the connected covering $G^\phi$ of a graph $G$ derived from a permutation voltage assignment $\phi : E(G) \rightarrow S_n$ is

$$Z_{G^\phi}(u, t)^{-1} = Z_G(u, t)^{-1} \prod_{i=2}^\ell \left[ (1-(1-u)t)^2(D_G - (1-u)I_{\nu}) \right]^{m_i}.$$

where

$$T(u, t, \rho_i, \phi) = \left[ \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes \rho_i(\gamma) \right] + (1-u)t^2(D_G - (1-u)I_{\nu}) \otimes I_{f_i}.$$

It is clear from Corollary 6 that for any covering $\tilde{G}$ of $G$ the Bartholdi zeta function $Z_G(u, t)$ divides $Z_{G^\phi}(u, t)$.

If $\Gamma$ acts on itself by right multiplication, then $\Gamma$ can be identified as a regular subgroup of $S_T$ and the covering $G^\phi$ is a regular covering of $G$. In this case, the multiplicity $m_i$ is equal to the degree $f_i$ for each $i$. Therefore, we have Theorem 3 in [11] as a corollary.

**Corollary 7** (Mizuno and Sato). The reciprocal of the Bartholdi zeta function of the connected regular covering $G^\phi$ of $G$ derived from an ordinary voltage assignment $\phi : E(G) \rightarrow \Gamma$ is

$$Z_{G^\phi}(u, t)^{-1} = Z_G(u, t)^{-1} \prod_{i=2}^\ell \left[ (1-(1-u)t)^2(D_G - (1-u)I_{\nu}) \right]^{f_i}.$$

where $T(u, t, \rho_i, \phi)$ is given in Corollary 6.

Now, let $G$ be a Schreier graph. Then the $G$-bundle of the complete graph $K_1$ on one vertex is isomorphic to the fiber $G$. Hence, we have the following corollary.

**Corollary 8.** Let $G$ be a Schreier graph with connecting set $S$ and let $\Gamma$ be the subgroup of the symmetric group $S_G$ generated by $S$. Let $\rho_1 = 1$, $\rho_2$, ..., $\rho_\ell$ be the irreducible representations of $\Gamma$ having degree $1$, $f_2$, ..., $f_\ell$, respectively. Then the reciprocal of the Bartholdi zeta function of $G$ is
$Z_G(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{|S| - 1} \prod_{i=1}^{\ell} T(u, t, \rho_i, \phi)^{m_i}$,

where

$T(u, t, \rho_i, \phi) = \det \left[ I_{f_i} - t \sum_{s \in S} \rho_i(s) + (1 - u)t^2(|S| + u - 1)I_{f_i} \right]$.

and $m_i$ is the multiplicity of $\rho_i$ in the permutation representation $P$ of $\Gamma$. \hfill \square

In the following example, by using Corollary 8, we compute the Bartholdi zeta function of a Cayley graph. In this case, the multiplicity $m_i$ is equal to the degree $f_i$.

**Example 1.** Let $G$ be a Cayley graph $\text{Cay}(\mathcal{A}, S)$, where $S$ is a symmetric generating set of $\mathcal{A}$ which does not contain the identity. Then $G$ is regular of degree $|S|$ and $A(G) = \sum_{s \in S} P(s)$. Hence,

$Z_G(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{|S| - 1} \det[I - tA(G) + (1 - u)t^2(|S| + u - 1)I]$

$= (1 - (1 - u)^2 t^2)^{|S| - 1} \times \det \left[ I - t \sum_{s \in S} P(s) + (1 - u)t^2(|S| + u - 1)I \right]$

$= (1 - (1 - u)^2 t^2)^{|S| - 1} \times \prod_{i=1}^{\ell} \det \left[ I_{f_i} - t \sum_{s \in S} \rho_i(s) + (1 - u)t^2(|S| + u - 1)I_{f_i} \right]^{f_i}$

$= (1 - (1 - u)^2 t^2)^{|S| - 1} \left( 1 - t|S| + (1 - u)t^2(|S| + u - 1) \right) \times \prod_{i=2}^{\ell} \det \left[ I_{f_i} - t \sum_{s \in S} \rho_i(s) + (1 - u)t^2(|S| + u - 1)I_{f_i} \right]^{f_i}$,

where $\rho_i$ runs over all irreducible representations of $\mathcal{A}$ whose degree is $f_i$. Moreover, if $\mathcal{A}$ is Abelian, then every irreducible representation of $\mathcal{A}$ is linear and hence we have

$Z_G(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{|S| - 1} \left( 1 - t|S| + (1 - u)t^2(|S| + u - 1) \right) \times \prod_{i=2}^{\ell} \left[ 1 - t \sum_{s \in S} \chi_i(s) + (1 - u)t^2(|S| + u - 1) \right]$, 

where $\chi_i$ runs over all irreducible characters of the Abelian group $\mathcal{A}$. For example, let $Q_n$ be the hypercube. Then $Q_n = \text{Cay}(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \{e_1, \ldots, e_n\})$, where all coordinates of $e_i$ are 0 except the $i$-th one. Now, one can see that

$Z_{Q_n}(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{2n-1(n-2)} \times \prod_{k=0}^{n-1} \left[ 1 - (n - 2k)t + (1 - u)t^2(n + u - 1) \right]^{\binom{n}{2}}$. \hfill \square
3. Graph bundles having voltages in an abelian group

In this section, we consider the Bartholdi zeta functions of graph bundles $G \times^{\phi} F$ when the images of $\phi$ lie in an abelian subgroup $\Gamma$ of $\text{Aut}(F)$. Here $F$ need not be a Schreier graph, but regular of degree $d_F$. In this case, for any $\gamma_1, \gamma_2 \in \Gamma$ the permutation matrices $P(\gamma_1)$ and $P(\gamma_2)$ are commutative and $D_F = d_F I_n$.

It is well known (see [3]) that every permutation matrix $P(\gamma)$ commutes with the adjacency matrix $A(F)$ of $F$ for all $\gamma \in \text{Aut}(F)$. Since the matrices $P(\gamma), \gamma \in \Gamma$ and $A(F)$ are all diagonalizable and commute with each other, they are simultaneously diagonalizable. That is, there exists an invertible matrix $M_{\Gamma}$ such that $M_{\Gamma}^{-1} P(\gamma) M_{\Gamma}$ and $M_{\Gamma}^{-1} A(F) M_{\Gamma}$ are diagonal matrices for all $\gamma \in \Gamma$. Let $\lambda_{(\gamma,1)}, \ldots, \lambda_{(\gamma,\nu_F)}$ be the eigenvalues of the permutation matrix $P(\gamma)$ and let $\lambda_{(F,1)}, \ldots, \lambda_{(F,\nu_F)}$ be the eigenvalues of the adjacency matrix $A(F)$. Then

$$M_{\Gamma}^{-1} P(\gamma) M_{\Gamma} = \begin{bmatrix} \lambda_{(\gamma,1)} & 0 & \cdots & 0 \\ 0 & \lambda_{(\gamma,2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(\gamma,\nu_F)} \end{bmatrix}$$

and

$$M_{\Gamma}^{-1} A(F) M_{\Gamma} = \begin{bmatrix} \lambda_{(F,1)} & 0 & \cdots & 0 \\ 0 & \lambda_{(F,2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(F,\nu_F)} \end{bmatrix}.$$

From this, one can see that

$$(I_{\nu_G} \otimes M_{\Gamma})^{-1} \left( \sum_{\gamma \in \Gamma} A(\tilde{G}_{i(\phi,\gamma)}) \otimes P(\gamma) + I_{\nu_G} \otimes A(F) \right) (I_{\nu_G} \otimes M_{\Gamma})$$

$$= \bigoplus_{i=1}^{\nu_F} \left( \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\tilde{G}_{i(\phi,\gamma)}) + \lambda_{(F,i)} I_{\nu_G} \right)$$

and

$$(I_{\nu_G} \otimes M_{\Gamma})^{-1} ((D_G - (1-u)I_{\nu_G}) \otimes I_{\nu_F} + I_{\nu_G} \otimes D_F) (I_{\nu_G} \otimes M_{\Gamma})$$

$$= \bigoplus_{i=1}^{\nu_F} (D_G - (1-u - d_F)I_{\nu_G}).$$

Thus

$$\det[I - t A(G \times^{\phi} F) + (1-u)t^2(D_G \times^{\phi} F - (1-u)I)]$$

$$= \prod_{i=1}^{\nu_F} \det \left[ I_{\nu_G} - t \left( \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\tilde{G}_{i(\phi,\gamma)}) + \lambda_{(F,i)} I_{\nu_G} \right) + (1-u)t^2(D_G - (1-u - d_F)I_{\nu_G}) \right].$$
For each $i = 1, 2, \ldots, v_F$ and the voltage assignment $\phi$, we denote

$$T(u, t, i, \phi) = \det \left[ I_{v_G} - t \left( \sum_{y \in F} \lambda_{(y, i)} A(G_{\phi \cdot y}) + \lambda_{(F, i)} I_{v_G} \right) + (1 - u)t^2(D_G - (1 - u - d_F)I_{v_G}) \right].$$

Since $\varepsilon_F = \frac{\nu_{d_F}}{2}$, we have

$$Z_{G \times F}(u, t)^{-1} = \prod_{i=1}^{v_F} (1 - (1 - u)^2 t^2)^{\varepsilon_{G - v_G} + \frac{\nu_{d_F}}{2} + \varepsilon_{F - v_G}} T(u, t, i, \phi).$$

This proves the following theorem.

**Theorem 9.** Let $G$ be a connected graph and let $F$ be a connected regular graph of degree $d_F$. If the images of $\phi \in C^1(G; \text{Aut}(F))$ lie in an abelian subgroup of $\text{Aut}(F)$, then the reciprocal of the Bartholdi zeta function of the $F$-bundle $G \times^\phi F$ is

$$Z_{G \times F}(u, t)^{-1} = \prod_{i=1}^{v_F} (1 - (1 - u)^2 t^2)^{\varepsilon_{G - v_G} + \frac{\nu_{d_F}}{2} + \varepsilon_{F - v_G}} T(u, t, i, \phi),$$

where

$$T(u, t, i, \phi) = \det \left[ I_{v_G} - t \left( \sum_{y \in F} \lambda_{(y, i)} A(G_{\phi \cdot y}) + \lambda_{(F, i)} I_{v_G} \right) + (1 - u)t^2(D_G - (1 - u - d_F)I_{v_G}) \right]. \quad \square$$

Notice that the cartesian product $G \times F$ of two graphs $G$ and $F$ is the $F$-bundle over $G$ associated with the trivial voltage assignment $\phi$, i.e., $\phi(e) = 1$ for all $e \in E(G)$ and $A(G) = A(G)$. The following corollary comes from this observation.

**Corollary 10.** For any connected graph $G$ and a connected $d_F$-regular graph $F$, the reciprocal of the Bartholdi zeta function of the cartesian product $G \times F$ is

$$Z_{G \times F}(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{\varepsilon_{G - v_G} + \frac{\nu_{d_F}}{2} + \varepsilon_{F - v_G}} \prod_{i=1}^{v_F} \det[I_{v_G} - t(A(G) + \lambda_{(F, i)} I_{v_G}) + (1 - u)t^2(D_G - (1 - u - d_F)I_{v_G})].$$

In particular, if $G$ is a regular graph of degree $d_G$, then the reciprocal of the Bartholdi zeta function of the cartesian product $G \times F$ is

$$Z_{G \times F}(u, t)^{-1} = \prod_{i=1}^{v_F} \prod_{j=1}^{v_G} (1 - (1 - u)^2 t^2)^{\frac{d_F + d_G}{2} - 1} \prod_{i=1}^{v_F} \det[I_{v_G} - t(A(G) + \lambda_{(F, i)} I_{v_G}) + (1 - u)t^2(D_G + d_F + u - 1)t^2],$$

$$\times \prod_{i=1}^{v_F} \prod_{j=1}^{v_G} (1 - (1 - u)^2 t^2)^{\frac{d_F + d_G}{2} - 1}.$$
where \( \lambda_{(G,j)} (1 \leq j \leq v_G) \) and \( \lambda_{(F,i)} (1 \leq i \leq v_F) \) are the eigenvalues of \( G \) and \( F \), respectively. \( \square \)

**Example 2.** Let \( K_m \) be the complete graph on \( m \) vertices. Its eigenvalues are \( m - 1 \) with multiplicity 1 and \(-1\) with multiplicity \( m - 1 \). Therefore, by Corollary 10, one can see that

\[
Z_{K_m \times K_m}(u, t) = \frac{1}{(1 - u)(m + n + u - 3)t^2}\left[\prod_{i=1}^{v_G} (1 - \lambda_{(G,i)} t + (1 - u)(d_G - 1 + u)t^2)\right] - 1
\]

In particular,

\[
Z_{K_2 \times K_2}(u, t) = \left[1 - 2t + (1 - u^2)t^2\right]^{v_G} - 1
\]

which coincides with the computation in Example 1. \( \square \)

**4. Further remarks**

In this last section, by using formulæ introduced in this paper, we discuss further computational ideas for the zeta functions of graphs related to their spectrum.

First, by comparing Theorems 1 and 2, one can see that the computation of the Bartholdi zeta function \( Z_{G \times F}(u, t) \) of \( G \times F \) immediately gives the computation of Ihara zeta function \( Z_{G \times F}(u) \) of \( G \times F \), i.e., \( Z_G(0, u) = Z_G(u) \).

Next, if the fiber \( F \) is the one vertex, then \( G \times F = G \). The following corollary comes from Corollary 10.

**Corollary 11.** Let \( G \) be a regular graph of degree \( d_G \). Then

\[
Z_G(u, t) = \frac{1}{(1 - u^2)t^2} \prod_{i=1}^{v_G} (1 - \lambda_{(G,i)} t + (1 - u)(d_G - 1 + u)t^2)
\]

and

\[
Z_G(u) = \frac{1}{u^2} \prod_{i=1}^{v_G} (1 - \lambda_{(G,i)} u + (d_G - 1)u^2),
\]

where \( \lambda_{(G,i)} \), \( 1 \leq i \leq v_G \), are the eigenvalues of \( G \). \( \square \)

It follows from Corollary 11 that if \( G \) is regular and all eigenvalues of \( G \) are known, then one can get the Bartholdi zeta function \( Z_G(u, t) \) and the Ihara zeta function \( Z_G(u) \) of \( G \) without further computations.

As an example, consider the five Platonic solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Denote their graphs by \( T, Q, O, D, I \), respectively. They are all regular and their characteristic polynomials are known in [4] (see Table 1). Now, by...
Table 1

<table>
<thead>
<tr>
<th>Platonic graphs and their characteristic polynomials</th>
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<tbody>
<tr>
<td>Characteristic polynomial</td>
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<tr>
<td>---------------------------</td>
</tr>
<tr>
<td>$T$</td>
</tr>
<tr>
<td>$Q$</td>
</tr>
<tr>
<td>$O$</td>
</tr>
<tr>
<td>$D$</td>
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<td>$I$</td>
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Corollary 11, one can compute the zeta functions of the Platonic graphs. For instance, for the icosahedron graph $T,$

$$Z_T(u, t)^{-1} = [1 - (1 - u)^2t^2][1 - 5t + (1 - u)(u + 4)t^2]$$

$$\times [1 - \sqrt{5}t + (1 - u)(u + 4)t^2]^3$$

and

$$Z_T(u)^{-1} = (1 - u^2)^3(1 - 5u + 4u^2)^2(1 - \sqrt{5}u + 4u^2)^3(1 + u + 4u^2)^5$$

Finally, we observe that the Bartholdi zeta function of a graph can be expressed in terms of the characteristic polynomial of a weighted graph, which is discussed in [4]. Let $\omega : V(G) \cup E(G) \to \mathbb{C}$ be a function. Then the pair $(G, \omega) = G_\omega$ is called a vertex- and edge-weighted graph. The adjacency matrix $A(G_\omega)$ of the weighted graph $G_\omega$ is the square matrix of order $V_G$ defined by

$$a_{ij} = \begin{cases} 
\omega(v_i v_j) & \text{if } i \neq j, \\
\omega(v_i) & \text{if } i = j.
\end{cases}$$

Let $\Phi(G_\omega, \lambda) = \det(\lambda I - A(G_\omega))$ denote the characteristic polynomial of $G.$ For a given graph $G,$ we define a weight function $\omega_{u,t} : V(G) \cup E(G) \to \mathbb{C}$ by

$$\omega_{u,t}(v_i v_j) = \begin{cases} 
\lambda & \text{if } i \neq j, \\
(u - 1)(a_{ij}G + u - 1)t^2 & \text{if } i = j,
\end{cases}$$

where $a_{ij}$ is the $(i, j)$-entry of $A(G).$ Then, by Theorem 2, one can see that

$$Z_{G}(u, t)^{-1} = (1 - (1 - u)^2t^2)^{\eta_G - \nu_G} \Phi(G_{\omega_{u,t}}; 1)$$

and

$$Z_{G}(u)^{-1} = Z_{G}(0, u)^{-1} = (1 - u^2)^{\eta_G - \nu_G} \Phi(G_{\omega_{0,u}}; 1).$$

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References