The vertex-connectivity of a distance-regular graph

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\begin{abstract}
The vertex-connectivity of a distance-regular graph equals its valency.
\end{abstract}

\section{Introduction}

In this paper we prove the following theorem.

\textbf{Theorem.} Let $\Gamma$ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals $k$, and the only disconnecting sets of vertices of size not more than $k$ are the point neighbourhoods.

The special case of this theorem where $\Gamma$ has diameter 2 was proved by Brouwer and Mesner \cite{4} more than twenty years ago. The case of diameter 3 was announced by the first author at the conference celebrating Eiichi Bannai’s 60th birthday.

The upper bound $k$ is tight. For example, an icosahedron (with $k = 5$) can be disconnected by removing a hexagon, leaving two triangles, and the line graph of the Petersen graph (with $k = 4$) can be disconnected by removing a 5-coclique, leaving two pentagons.

The edge-connectivity of distance-regular graphs was determined earlier.

\begin{proposition} (Brouwer and Haemers \cite{2}). Let $\Gamma$ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency $k$, and the only disconnecting sets of $k$ edges are the sets of edges incident with a single vertex.
\end{proposition}

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2. Tools

Given good information on the eigenvalues, expansion properties follow from the below version of Tanner’s bound.

**Proposition 2.1** (Haemers [5]). Let \( \Gamma \) be a regular graph of valency \( k \) with second largest eigenvalue \( \theta \) and smallest eigenvalue \( \theta' \). Let \( A \) and \( B \) be two separated sets in \( \Gamma \) of sizes \( a \) and \( b \), respectively. Then

\[
\frac{ab}{(v-a)(v-b)} \leq \left( \frac{\theta - \theta'}{2k - \theta - \theta'} \right)^2.
\]

If the separating set \( S \) has size \( s \), so that \( v - a = b + s \), then an equivalent formulation is

\[
\frac{ab}{vs} \leq \frac{(\theta - \theta')^2}{4(k-a)(k-b)}.
\]

For combinatorial work, the coding-theoretic argument below is useful. We will quote this as the ‘inproduct bound’.

**Lemma 2.2.** Among a set of \( a \) binary vectors of length \( n \) and average weight \( w \) there are two with inner product at least \( w (\frac{aw}{n} - 1)/(a - 1) = \frac{w^2}{n} - \frac{w(n-w)}{m(a-1)} \).

**Proof.** The sum of all pairwise inner products of the vectors is at least \( n \left( \frac{aw}{n} \right)^2 \).

**Lemma 2.3.** Let \( \Gamma \) be a distance-regular graph with a separation \( \Gamma \setminus S = A + B \), and let \( \alpha \in A \). If \( \alpha \) has \( s \) neighbours in \( S \), then \( |A| > v(1 - \frac{s}{k})(1 - \frac{k}{k_2}). \)

**Proof.** According to Lemma 4.3(i) (the ‘Shadow Lemma’) and subsequent remark in Brouwer and Haemers [2] one has \( |A| \geq v - \sum_{i=1}^{d} \frac{s_i}{k_i} (k_i + \cdots + k_d) \) where \( s_i = |S \cap T_i(\alpha)| \), and \( (k_i + \cdots + k_d)/k_i \geq (k_{i+1} + \cdots + k_d)/k_{i+1} \), so that \( |A| \geq v - \frac{s}{k} (v - 1) - \frac{k-s}{k_2} (v - 1 - k) > v(1 - \frac{s}{k})(1 - \frac{k}{k_2}). \)

3. Vertex-connectivity of a distance-regular graph

Let us say that a distance-regular graph \( \Gamma \) is OK when its vertex-connectivity equals its valency \( k \), and the only disconnecting sets of size \( k \) are the sets of neighbours of a vertex.

Let \( \Gamma \) be a distance-regular graph of diameter \( d \) at least 3, not a polygon, and suppose \( S \) is a set of vertices of size at most \( k \) such that \( \Gamma \setminus S \) is disconnected, say with separation \( A + B \). Suppose moreover that each of \( A \) and \( B \) contains at least two vertices. We shall obtain a contradiction. Notation is as in BCN [1].

Put \( a = |A|, b = |B|, s = |S| \).

**Lemma 3.1.** In any distance-regular graph of diameter more than 2 one has \( 3\lambda + 4 \leq 2k \).

**Proof.** Suppose \( \alpha \sim \beta \sim \gamma \sim \delta \) is a geodesic. If \( 3\lambda + 3 \geq 2k \), then \( \lambda \geq 2(b_1 - 1) \), so that not all common neighbours of \( \beta \) and \( \gamma \) are nonadjacent to \( \alpha \) or nonadjacent to \( \delta \). But then \( \alpha \) and \( \delta \) have a common neighbour, contradiction. \( \Box \)

**Lemma 3.2.** Neither \( A \) nor \( B \) is a clique.

**Proof.** Suppose \( A \) is a clique of size \( a \). Apply the inproduct bound to the \( a \) characteristic vectors of the sets \( \Gamma(\alpha) \cap S \) (for \( \alpha \in A \)) of length at most \( k \) and weight \( k - (a - 1) \), and find \( \lambda \geq a - 2 + \frac{(k-\alpha)(k-a)}{k} \). The right-hand side is minimal for \( a = (k+1)/2 \), and hence \( \lambda \geq \frac{3}{4}k - \frac{3}{2} - \frac{1}{4k} \). On the other hand, by Lemma 3.1, \( \lambda \leq \frac{3}{2}k - \frac{4}{3} \) and hence \( k = 3, \lambda = 0 \), contradiction. \( \Box \)

**Lemma 3.3.** \( a > 3 \).
If $k = 3$, then $A$ is a path of length 2, and $3 + k \geq a + |S| \geq 3 + 3k - 4 - 2\lambda - (\mu - 1) = 3k - 2\lambda - \mu$, so that $2b_1 \leq \mu + 1 \leq b_1$, impossible. □

Lemma 3.4. If $k = 3$ then $\Gamma$ is OK.

Proof. Suppose $k = 3$, and pick the separation $\Gamma \setminus S = A + B$ such that $S$ has minimal size (at most 3) and $A$ has minimal size larger than one (so that $|B| \geq |A| > 3$), given the size of $S$. If a point of $S$ has only one neighbour in $A$, then $A$ can be made smaller. If a point of $S$ has no neighbours in $A$, then $S$ can be made smaller. So, we may assume that each point of $S$ has precisely two neighbours in $A$ and one in $B$. But then there is a disconnecting set of at most three edges, not all on a single point, contradicting Proposition 1.1. □

Lemma 3.5. If $\lambda = 0$ and $\mu = 1$ then $a > 7$.

Proof. Each point of $A$ has $k$ neighbours in $A \cup S$, and each pair of vertices of $A$ at distance 2 have a common neighbour. We may assume that $A$ is connected, and then it has at least $a - 1$ edges. We find $ak - \binom{a}{2} + a - 1 \leq a + |S|$. Now use $k > 3$. □

Lemma 3.6. The icosahedron is OK.

Proof. This is a special case of the following lemma. □

Lemma 3.7. An antipodal 2-cover of a complete graph is OK.

Proof. Let $\Gamma$ be an antipodal 2-cover of a complete graph $K_{k+1}$. Since $|S| \leq k$ there is a pair of antipodal vertices neither of which is in $S$. If both are in $A$, then each vertex of $B$ is adjacent to some vertex of $A$, impossible. So, we have antipodal $a_0 \in A$ and $b_0 \in B$. Let $A' = A \setminus \{a_0\}$ and let $B'$ be the set of antipodes of $B \setminus \{b_0\}$. The graph $\Delta := \Gamma(a_0)$ is strongly regular and satisfies $k_\Delta = 2\mu_\Delta$ (BCN 1.5.3). The sets $A'$ and $B'$ are subsets of $\Delta$ and $|A'| + |B'| \geq k$ and every vertex of $A'$ is equal or adjacent to every vertex of $B'$. Now neither $A'$ nor $B'$ is a clique, so if $\alpha_1$, $\alpha_2$ are two nonadjacent vertices in $A'$ and $\beta_1$, $\beta_2$ two nonadjacent vertices of $B'$, then $k_\Delta = 2\mu_\Delta = \mu_\Delta(\alpha_1, \alpha_2) + \mu_\Delta(\beta_1, \beta_2) \geq |B'| + |A'| \geq k = v_\Delta$, impossible. □

Lemma 3.8. $k_2 > k$.

Proof. One always has $\mu \leq b_1$ and hence $k_2 \geq k$. If equality holds then by BCN 5.1.1(v) $\Gamma$ has diameter 3 and is an antipodal 2-cover ($k_3 = 1$), so is OK by Lemma 3.7. □

Lemma 3.9. $\max(\lambda + 2, \mu) \geq k(k + 1)/(a + k)$.

Proof. Apply the inproduct bound to the $a$ characteristic vectors of the sets $\{\alpha\} \cup \Gamma(\alpha)$ for $\alpha \in A$, of length at most $a + k$ and weight $k + 1$. □

Proposition 3.10. If $\lambda > 0$ and $\mu > 1$ and $\lambda + 2 \geq \mu$ then $\Gamma$ is OK.

Proof. By BCN 4.4.3 we have: either $\Gamma$ is the icosahedron, or $\lambda = 0$, or $\mu = 1$, or both $\theta_1 \leq b_1 - 1$ and $-\theta_d \leq \frac{1}{2}b_1 + 1$. In the latter case the separation bound gives

$$\frac{ab}{(v-a)(v-b)} \leq \left(\frac{\frac{3}{2}b_1}{2k+2 - \frac{1}{2}b_1}\right)^2.$$  

Put $a = ak$, $b_1 = \beta(k+1)$. Since $\lambda + 2 \geq \frac{k(k+1)}{a+\beta} = \frac{1}{1+\alpha}(k+1)$, we have $\beta \leq \frac{a}{1+\alpha}$. Let $\gamma$ be the RHS of the separation bound. Then $\gamma \leq \left(\frac{3\beta}{4(\beta+1)}\right)^2 \leq \left(\frac{3\alpha}{4(\alpha+1)}\right)^2$ and $ab \leq \gamma(v-a)(v-b) \leq \gamma(a+k)(b+k)$. Assuming $b \geq a$ we may multiply by $a/b$ and obtain $a^2 \leq \gamma(a+k)(a+k+\frac{b}{d}) \leq \gamma(a+k)^2$. But this is a contradiction. □
Now the proof is split into the three cases $\mu = 1$, and $\lambda = 0, \mu > 1$, and $2 < \lambda + 2 < \mu$. The first of these will be handled in Lemma 3.13 below. The second in Lemma 3.17.

Call a point in $A \cup B$ a deep point if it has no neighbours in $S$.

**Lemma 3.11.** If $\lambda = 0$ and $k_2 \geq 3k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma$, $\tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4}v(1 - \frac{\sigma}{\tau})$ and $b > \frac{3}{4}v(1 - \frac{\tau}{\sigma})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{3}{2}k$. Since $\lambda = 0$ we have $\sigma, \tau \leq \frac{1}{2}k$, so $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < 2k$. Since $\sigma, \tau \geq \mu$ we have $v \leq k + k^2/\mu$. If $\mu > 1$ then by Lemma 3.9, $\mu > k^2/(a + k) > \frac{1}{3}k$, so that $v < 4k$, contradiction. If $\mu = 1$, then by the same lemma $a + k > k(k + 1)/2$, but $a < 2k$ and hence $k \leq 4$. By Lemma 3.5 $a > 7$, contradiction. □

**Lemma 3.12.** If $k_2 \geq 4k$ and $v \geq 6k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma$, $\tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4}v(1 - \frac{\sigma}{\tau})$ and $b > \frac{3}{4}v(1 - \frac{\tau}{\sigma})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{3}{2}k$.

If $\sigma > \frac{3}{2}k$ then $\lambda$, $\mu > \frac{1}{3}k$ and $k_2 < 2k$, contradiction.

So $\sigma, \tau \leq \frac{3}{2}k$ and $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < \frac{3}{2}k$. On the other hand, $a > \frac{3}{4}v(1 - \frac{\sigma}{\tau}) \geq \frac{1}{4}v \geq \frac{3}{2}k$, contradiction. □

**Lemma 3.13.** If $\mu = 1$, then $\Gamma$ is OK.

**Proof.** Since $\mu = 1$ we have (by BCN 1.2.1) lines of size $\lambda + 2$, and $(\lambda + 1)|k$, hence $(\lambda + 1)|b_1$. Since $k_2 = b_1k$ we have $b_1 < 5$ by Lemma 3.12. This leaves the cases $(k, \lambda) \in \{(3, 0), (4, 0), (5, 0), (4, 1), (6, 1), (6, 2), (8, 3)\}$. The cases with $\lambda = 0$ are settled by Lemmas 3.4 and 3.11. This leaves $(k, \lambda) \in \{(4, 1), (6, 1), (6, 2), (8, 3)\}$.

(i) Suppose $(k, \lambda) = (4, 1)$. Now $\Gamma$ is the line graph of a cubic graph. There are four arrays with $d \geq 3$ (see e.g. Brouwer and Koolen [3]) namely $(4, 2, 1; 1, 1, 4)$ for the line graph of the Petersens graph on 15 vertices, $(4, 2, 2; 1, 1, 2)$ for the flag graph of the Fano plane on 21 vertices, $(4, 2, 2, 2; 1, 1, 1, 2)$ for the flag graph of $GQ(2, 2)$ on 45 vertices, and $(4, 2, 2, 2, 2; 1, 1, 1, 1, 2)$ for the flag graph of $GH(2, 2)$ on 189 vertices.

In these four cases the separation bound yields $a \leq 2$, $a \leq 3$, $a \leq 5$, $a \leq 9$, respectively. Since we have $k_2 = 2k = 8$, the shadow bound (Lemma 2.3) yields $a > v/8$. Since also $a > 3$, this settles the case $(k, \lambda) = (4, 1)$.

(ii) Suppose $(k, \lambda) = (6, 1)$ or $(k, \lambda) = (6, 2)$. If $k = 6$, $\lambda \in \{1, 2\}$, $\mu = 1$, $k_2 \in \{24, 18\}$, then $\sigma + \tau > 3$ (because of $v$), so $\sigma + \tau \geq 4$. Also $\sigma \leq 3$ (because of $\mu$) so $\tau > 0$ and $A, B$ do not have deep points. By the inner product bound (with $w = k = 6$ and $n = a + k$) we have $a \geq 9$. On the other hand, $a \leq k^2/(\sigma + \tau) \leq 9$. So $a = 9$, and $\sigma + \tau = 4$ and $\sigma a + \tau b \leq k^2$ and $a \leq 9$ imply $b = 9$. Now $v \leq a + b + k = 24$ and $v > 1 + k + k_2 \geq 25$, contradiction.

(iii) Suppose $(k, \lambda) = (8, 3)$. Then each point is in 2 cliques of size 5, and $\Gamma$ is the line graph of a graph of valency 5. If $d \geq 4$ then $v \leq k_2 + (k_3 + k_4) > 6k$, and $\Gamma$ is OK by Lemma 3.12. So, $d = 3$. Now by BCN 4.2.16, $\Gamma$ is the flag graph of $PG(2, 4)$ on 105 vertices, and we are done again since $v > 6k$. □

**Lemma 3.14.** If $d \geq 4$, or if $d = 3$ and $\Gamma$ is not bipartite, then $\mu \leq \frac{1}{2}k$.

**Proof.** If $d \geq 4$ this is trivial. Suppose $d = 3$ and $\Gamma$ is not bipartite and $\mu > k/2$. If $d(\alpha, \beta) = d(\beta, \gamma) = 2$ and $d(\alpha, \gamma) = 3$, then $\beta$ has $\mu$ common neighbours with each of $\alpha, \gamma$, and none occurs twice, so $\beta$ has more than $k$ neighbours. Contradiction. Hence $p^3_{22} = 0$, and the graph $F_2$ is (connected and) distance-regular with distances 0, 1, 2, 3 corresponding to 0, 2, 1, 3 in $\Gamma$. But then $k_2 \leq k$, contradiction. □
Lemma 3.15. If \( d \geq 4 \), or \( d = 3 \) and \( \Gamma \) is not bipartite, and \( \mu \geq \lambda + 2 \), then \( a > k \).

Proof. If \( a \leq k \) then, by Lemma 3.9, \( \max(\lambda + 2, \mu) > \frac{1}{2}k \). But this contradicts Lemma 3.14. □

Lemma 3.16. Suppose \( B \) has a deep point and \( A \) does not. If \( \lambda + 2 \leq \mu \) then there is a separating set smaller than \( S \).

Proof. Let \( B' \) be the set of points in \( B \) with a neighbour in \( S \). Put \( s := |S| \) and \( b' := |B'| \). Each point in \( A \cup B' \) has at least \( \mu \) neighbours in \( S \), so \( \mu(a + b') \leq k \). Since \( a + k > \frac{k^2}{\mu} \) (by Lemma 3.9) it follows that \( \mu(b' - s) < -(\lambda - \mu)(k - s) \leq 0 \) so that \( b' < s \). Since \( B \) has a deep point, \( B' \) is a separating set. □

Lemma 3.17. If \( \lambda = 0 \) and \( \mu > 1 \) then \( \Gamma \) is OK.

Proof. Suppose \( \mu = 0 \) and \( \mu > 1 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If \( A \) and \( B \) have deep points \( \alpha \) and \( \beta \), then \( a, b > v(1 - \frac{2}{k - 1}) \), so that \( 2\mu > k - 1 \). Now \( d \geq d(\alpha, \beta) \geq 4 \) and \( \mu = k/2 \) by Lemma 3.14. We have \( d = 4 \), otherwise \( k/2 = \mu \leq c_3 \leq b_2 \leq k/2 \) (using BCN 5.4.1) would give a contradiction. Now \( b_2 = k/2 \) and (by BCN 5.8.2) \( c_3 = k - 1 \) so that the graph is an antipodal 2-cover and \( \alpha \) and \( \beta \) are antipodes. Now \( |S| \geq k > k \), as desired.

If neither \( A \) nor \( B \) has a deep point (and \( S \) is minimal) then every point of \( A \) (or \( B \)) has distance 2 to some point of \( B \) (or \( A \)), and therefore has at least \( \mu \) neighbours in \( S \). Counting edges meeting \( S \) we find \( v - k \leq a + b \leq k^2/\mu \).

Now \( v \leq k + \frac{k^2}{\mu} \) and \( k_2 = \frac{k(k - 1)}{\mu} \) gives \( 1 + k_3 \leq v - k - k_2 \leq \frac{k}{\mu} \) so that \( k_3 < \frac{k - 1}{\mu} \) (because \( \mu > 1 \)). On the other hand, \( c_3 \leq k \) and \( b_2 \geq 1 \) imply \( k_3 = \frac{k(k - 1)b_2}{\mu c_3} \geq \frac{k - 1}{\mu} \), contradiction. □

Lemma 3.18. If \( d \geq 4 \) then \( \Gamma \) is OK.

Proof. By Proposition 3.10 and Lemmas 3.17 and 3.13 we may assume \( \lambda > 0 \) and \( \lambda + 2 \leq \mu \). By BCN Lemma 5.5.5 we have \( a_2 \geq \mu \), and since also \( b_2 \geq \mu \) (since \( d \geq 4 \)) it follows that \( \mu \leq k/3 \) and \( b_1 > 2k/3 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If both \( A \) and \( B \) have a deep point, then \( v > a + b > 2v(1 - \frac{2}{b_1}) > v \), contradiction.

If neither \( A \) nor \( B \) has a deep point, then \( 1 + k + \frac{k(k - 1)}{\mu} + k + 1 \leq v \leq k + \frac{k^2}{\mu} \), again a contradiction. □

Lemma 3.19. If \( \lambda > 0 \) then \( \theta_d \geq -\frac{1}{2}b_1 - 1 \geq -\frac{1}{2}k \).

Proof. By BCN 4.4.3(iii), if \( b_1/(\theta_d + 1) > -2 \), then either \( \lambda = 0 \) or \( \Gamma \) is the icosahedron, but the icosahedron is OK by Lemma 3.6. □

Proposition 3.20. Let \( (u_i) \) be the standard sequence for the second largest eigenvalue \( \theta_1 \). If \( u_{d-1} > 0 \) then \( \theta_1 < a_d \), and for each vertex \( \alpha \) the subgraph \( \Gamma_d(\alpha) \) is connected.

Proof. We have \( c_d u_{d-1} + a_d u_d = \theta_1 u_d \), and \( u_d < 0 \) (since \( (u_i) \) has precisely one sign change), so \( \theta_1 < a_d \). By interlacing \( \Gamma_d(\alpha) \) has eigenvalue \( a_d \) with multiplicity 1, and hence is connected. □

Proposition 3.21. If \( \lambda > 0 \) and \( \mu > 1 \) and \( \theta_1 < a_d \), then \( \Gamma \) is OK.

Proof. By Lemma 3.19 we have \( \theta_d \geq -\frac{1}{2}k \). By Proposition 3.10 we may assume \( \mu \geq \lambda + 2 \).

Put \( a = \alpha k \). Then by Lemma 3.9 \( c_d \geq \mu \geq \frac{k}{1 + \alpha} \), hence \( a_d = k - c_d < \frac{k}{1 + \alpha} \). Using \( \theta_1 < \frac{ak}{1 + \alpha} \) and \(-\theta_d \leq \frac{1}{2}k \) we find from the separation bound that

\[
ab \leq \frac{(3\alpha + 1)^2}{(3\alpha + 5)}.
\]
Let $\gamma$ be the RHS of the separation bound. Then $ab \leq \gamma(v - a)(v - b) \leq \gamma(a + k)(b + k)$. Assuming $b \geq a$ we may multiply by $a/b$ and obtain $a^2 \leq \gamma(a + k)(a + \frac{ab}{b}) \leq \gamma(a + k)^2$, so that $a \leq k$, contradicting Lemma 3.15. □

**Lemma 3.22.** Suppose $\lambda > 0$ and $\mu > 1$. If $\theta_1 \leq \frac{1}{2}k$, then $\Gamma$ is OK.

**Proof.** If $\theta_1 \leq \frac{1}{2}k$, then we can use the bound for separated sets again with $\theta \leq \frac{1}{2}k$ and $\theta' \geq -\frac{1}{2}k$. We find
\[
\frac{ab}{(v - a)(v - b)} \leq \frac{1}{4}
\]
so that $3ab \leq dk$, and if $a \leq b$ then $a \leq b \leq (a + k)k/(3a - k)$, so $(3a + k)(a - k) \leq 0$, that is, $a \leq k$. Now we are done by Lemma 3.15 and Proposition 3.10. □

**Lemma 3.23.** $\theta_1 \leq b_1 - 1$.

**Proof.** By BCN 4.4.3(ii) either $\theta_1 \leq b_1 - 1$ or $\mu = 1$ or $\Gamma$ is the icosahedron. But $\mu > 1$ by Lemma 3.13, and $\Gamma$ is not the icosahedron by Lemma 3.6. □

**Lemma 3.24.** Let $d = 3$. If $\frac{1}{2}k < \theta_1 \leq b_1 - 1$ then $\theta_1 < a_3$.

**Proof.** Firstly, $\theta_1 > \frac{1}{2}k$ is equivalent to $u_1 > \frac{1}{2}$. Secondly, $\theta_1 \leq b_1 - 1$ is equivalent to $u_0 - 2u_1 + u_2 \geq 0$. Since $u_0 = 1$ this implies that $u_2 \geq 2u_1 - u_0 > 0$. Now $\theta_1 < a_3$ follows by Proposition 3.20. □

**Theorem 3.25.** $\Gamma$ is OK.

**Proof.** The cases $\lambda = 0$ and $\mu = 1$ were done in Lemmas 3.17 and 3.13. By Lemma 3.18 we may assume $d = 3$. By Lemmas 3.22–3.24 we have $\theta_1 < a_3$ and now Proposition 3.21 completes the proof. □

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**References**