A note on degree-constrained subgraphs
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Abstract
Elementary proofs are presented for two graph theoretic results, originally proved by H. Shirazi and J. Verstraëte using the combinatorial Nullstellensatz.
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In an undirected graph $G = (V, E)$ we denote by $d_G(v)$ the degree of $v \in V$. If $F(v) \subseteq \mathbb{N}$ is a set of forbidden degrees for every $v \in V$, then a subgraph $G' = (V, E')$ of $G$ is called $F$-avoiding if $d_{G'}(v) \not\in F(v)$ for all $v \in V$.

**Theorem 1** (Shirazi and Verstraëte [5]). If $G = (V, E)$ is an undirected graph and
\[ |F(v)| \leq d_G(v)/2 \quad \text{for every node } v, \]
then $G$ has an $F$-avoiding subgraph.

Theorem 1 appeared first under the name Louigi’s conjecture in [1]. A version with $d_G(v)/2$ replaced by $d_G(v)/12$ was given in [1], while $d_G(v)/8$ was proved in [2]. Louigi’s conjecture was first settled in the affirmative by Shirazi and Verstraëte [5]. Their proof is based on the combinatorial Nullstellensatz of Alon [3]. We give an elementary proof, which uses Theorem 2 below. In a directed graph $D = (V, \overrightarrow{E})$ we denote by $q_D(v)$ the in-degree of $v \in V$.

**Theorem 2.** If $G = (V, E)$ is an undirected graph and it has an orientation $D$ for which $q_D(v) \geq |F(v)|$ for every node $v$, then $G$ has an $F$-avoiding subgraph.

**Proof.** For an undirected edge $e$, let $\overrightarrow{e}$ denote the corresponding directed edge of $D$. We use induction on the number of edges. If $0$ is not a forbidden degree at any node, then the empty subgraph $(V, \emptyset)$ is $F$-avoiding. Suppose that $0 \in F(t)$
for a node $t$. Then $\varrho_D(t) \geq |F(t)| \geq 1$ and hence there is an edge $e = st$ of $G$ for which $\varpi$ is directed toward $t$. Let $G^- = G - e$ and $D^- = D - \varpi$. Define $F^-$ as follows. Let $F^-(t) = \{i - 1 : i \in F(t) \cap \{0\}\}$, $F^-(s) = \{i - 1 : i \in F(s) \cap \{0\}\}$, and for $z \in V - \{s, t\}$ let $F^-(z) = F(z)$. Since $|F^-(t)| = |F(t)| - 1$, $\varrho_{D^-}(v) \geq |F^-(v)|$ holds for every node $v$. By induction, there is an $F^-$-avoiding subgraph $G''$ of $G^-$. By the construction of $F^-$, the subgraph $G' := G'' + e$ of $G$ is $F$-avoiding. □

**Proof of Theorem 1.** It is well-known that every undirected graph $G$ has an orientation $D$ in which

$$\varrho_D(v) \geq |d_G(v)/2| \quad \text{for every node } v.$$  

Indeed, by adding a new node $z$ to $G$ and joining $z$ to every node of $G$ with odd degree, we obtain a graph $G^+$ in which every degree is even. Hence $G^+$ decomposes into edge-disjoint circuits and therefore it has an orientation in which the in-degree of every node equals its out-degree. The restriction of this orientation to $G$ satisfies (2). (An orientation with property (2) is also used in [5].) Therefore Theorem 2 implies Theorem 1. □

Hakimi [4] proved that, given a function $f : V \rightarrow \mathbb{Z}_+$, an undirected graph $G$ has an orientation for which $\varrho(v) \geq f(v)$ for every node $v$ if and only if $e_G(X) \geq \sum \{f(v) : v \in X\}$ holds for every subset $X \subseteq V$, where $e_G(X)$ denotes the number of edges with at least one end-node in $X$. By combining this with Theorem 2, one obtains the following.

**Corollary 3.** If $G = (V, E)$ is an undirected graph and $e_G(X) \geq \sum \{|F(v)| : v \in X\}$ holds for every subset $X \subseteq V$, then $G$ has an $F$-avoiding subgraph.

Along with Theorem 1, the following result was also proved in [5] via the Combinatorial Nullstellensatz. A graph is called empty if it has no edges.

**Theorem 4 (Shirazi and Verstraëte [5]).** If $G = (V, E)$ is an undirected graph, $0 \not\in F(v)$ for all $v \in V$ and $\sum_{v \in V} |F(v)| < |E|$, then $G$ has a nonempty $F$-avoiding subgraph $G'$.

**Proof.** Again, we use induction on the number of edges. If $d_G(v) \not\in F(v)$ for all $v \in V$, then the nonempty $G' = G$ will do. Otherwise there exists a node $t \in V$ where $d_G(t) \notin F(t)$. As $0 \not\in F(v)$, there is an edge $e$ of $G$ incident to $t$. Let $G^- = G - e$, let $F^-(t) = F(t) \setminus \{d_G(t)\}$ and for $z \in V - \{t\}$ let $F^-(z) = F(z)$. By induction, there is a nonempty $F^-$-avoiding subgraph $G'$ of $G^-$. As $d_G(t) < d_G(t)$, this $G'$ is also $F$-avoiding. □

We remark that Theorems 2 and 4 clearly hold for hypergraphs, as well, with the same proofs. Combining this with the hypergraph variant of Hakimi’s theorem, one concludes that also Corollary 3 applies to hypergraphs. However, in Theorem 1 one should replace the denominator 2 by the rank of the hypergraph (that is, the maximum size of a hyperedge). This is already observed by Shirazi and Verstraëte [5]. Note also that both proofs give rise to polynomial algorithms: such algorithms were not known before.

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**Note added in proof**

After submitting the paper, the authors learned that Adrian Bondy also formulated and proved Theorem 2. His proof goes along the same line as ours.

**References**