Covers and Envelopes in Grothendieck Categories: 
Flat Covers of Complexes with Applications

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In the general setting of Grothendieck categories with enough projectives, we prove theorems that make possible to restrict the study of the problem of the existence of $ℱ$-covers and envelopes to the study of some properties of the class $ℱ$. We then prove the existence of flat covers and cotorsion envelopes of complexes, giving some examples. This generalizes the earlier work (J. Algebra 201 (1998), 86–102) and finishes the problem of the existence of flat covers of complexes.

Key Words: cover; envelope; cotorsion theory cogenerated by a set; flat complex of modules.

1. INTRODUCTION

The study of the existence of covers started with the introduction of the concept of a projective cover as a dual notion of the concept of an injective envelope. In 1960, H. Bass ([2]) published the characterization of the rings over which every module has a projective cover—the perfect rings.

But it was not until 1981 that the definition of covers and envelopes over arbitrary classes of modules was given ([6]). Then it could be noticed that projective covers and flat covers of modules over perfect rings coincide, and
the question of the existence of flat covers was raised. This question has been studied since 1981. Bican, El Bashir, and Enochs ([3]) have given the solution to the problem, proving that every module over any ring admits a flat cover.

In a natural way, flat covers (and covers by more general classes of objects) have been studied in more general settings than that of modules. A particular and important example is the study of flat covers of complexes of modules. These have been treated by different authors (see, e.g., [7] and [10]).

In this paper, we prove that any complex of modules over any ring admits a flat cover. We give a general construction for Grothendieck categories with enough projectives which makes it possible to restrict the general problem of the existence of $\mathcal{F}$-covers and envelopes to the study of some properties of the class $\mathcal{F}$.

We also give some applications using the previously developed arguments. One of these is a short proof of the Spaltenstein result [14] (for every complex $C$ there exists a quasi-isomorphism $C \to I$ with $I$ a DG-injective complex). We relate these proofs to the existence of DG-injective envelopes and exact covers of complexes.

Throughout this paper, unless otherwise stated, $\mathcal{A}$ denotes a Grothendieck category with a projective generator. Thus the category $\mathcal{A}$ has enough projectives. A class of objects $\mathcal{F}$ is always considered closed under isomorphisms. The letter $\aleph$ always denotes an infinite cardinal number, and the cardinality of a set $X$ is expressed by $|X|$.

A cotorsion theory in $\mathcal{A}$ is defined as a pair $(\mathcal{F}, \mathcal{C})$ of classes of $\text{Ob}(\mathcal{A})$ such that $\mathcal{F}^\perp = \mathcal{C}$ and $\perp^\perp = \mathcal{F}$. Recall that given a class of objects $\mathcal{F}$, its orthogonal class $\mathcal{F}^\perp$ is defined as the class of objects $C$ such that $\text{Ext}^1(F, C) = 0$ for all $F \in \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be cogenerated by a set $X \subset \mathcal{F}$ whenever $C \in \mathcal{C}$ if and only if $\text{Ext}^1(F, C) = 0$ for all $F \in X$.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects is said to have enough injectives (projectives) if, for any object $M$, there exists an exact sequence $0 \to M \to C \to F \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. Salce [13, Corollary 2.4] (or see [8, Proposition 7.1.7]) proved that if $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory of modules with enough injectives (projectives), then it has enough projectives (injectives).

Given a class of objects $\mathcal{F}$, an $\mathcal{F}$-precover of an object $M$ is a morphism $f: F \to M$ with $F \in \mathcal{F}$, such that $\text{Hom}(F', F) \to \text{Hom}(F', M)$ is exact for any $F' \in \mathcal{F}$. If, moreover, $f \circ g = f$ implies that $g$ is an isomorphism whenever $g \in \text{End}(F)$, then $f: F \to M$ is an $\mathcal{F}$-cover. $\mathcal{F}$-preenvelopes and $\mathcal{F}$-envelopes are defined dually. The definition of $\mathcal{F}$-cover and $\mathcal{F}$-envelopes in categories of modules was given in [6]. It is not hard to see that $\mathcal{F}$-envelopes and $\mathcal{F}$-covers, if they exist, are unique up to isomorphism.
A surjective \( \mathcal{F} \)-precover \( F \rightarrow M \) (an injective \( \mathcal{F} \)-preenvelope \( M \rightarrow F \)) is said to be special whenever \( \text{Ker} \ (F \rightarrow M) \in \mathcal{F}^\perp \) (\( \text{Coker} \ (M \rightarrow F) \in \mathcal{F}^\perp \)). (For further information about special covers and envelopes, see [15]).

Recall that a continuous chain of subobjects of a given object \( X \) is a set of subobjects of \( X, \{X_\alpha; \alpha < \lambda\} \) (for some ordinal number \( \lambda \)), such that \( X_\alpha \) is a subobject of \( X_\beta \) for all \( \alpha \leq \beta < \lambda \), and that \( X_\gamma = \sum_{\alpha < \gamma} X_\alpha \) whenever \( \gamma < \lambda \) is a limit ordinal.

Given a complex of modules \( C = (C^i, \delta^i) \), we use the usual terminology for cocycles, coboundaries, and the cohomology groups. Therefore, \( Z^i(C), B^i(C), \) and \( H^i(C) \) denote \( \text{Ker} \ \delta^i, \ \text{Im} \ \delta^i - 1, \) and \( \text{Ker} \ \delta^i / \text{Im} \ \delta^{i-1} \), respectively, for all \( i \in \mathbb{Z} \).

Henceforth, \( R \) is used to denote a ring, and all rings are associative with unity. All modules are unital left \( R \)-modules unless specified otherwise.

2. COVERS AND ENVELOPES IN GROTHENDIECK CATEGORIES

An object \( X \) of an abelian category is said to be \( \aleph \)-generated (where \( \aleph \) is an infinite cardinal number) if for any exact sequence \( \coprod_{i \in I} A_i \rightarrow X \rightarrow 0 \) there exists \( J \subseteq I, \ |J| \leq \aleph \) such that \( \coprod_{i \in J} A_i \rightarrow X \rightarrow 0 \) is also exact. \( \aleph \)-generated objects in abelian categories with exact direct limits have been characterized in [12, Proposition 2.1(I)]. It is not hard to see that in these categories, an object \( X \) is \( \aleph \)-generated if and only if, whenever \( X = \sum_{i \in I} X_i \), there exists \( J \subseteq I, \ |J| \leq \aleph \) such that \( X = \sum_{i \in J} X_i \). We then prove the following.

**Proposition 2.1.** Let \( \mathcal{A} \) be a locally small abelian category with exact direct limits and \( X \in \text{Ob}(\mathcal{A}) \). Then there exists a cardinal number \( \aleph \) such that \( X \) is \( \aleph \)-generated.

**Proof.** Since \( \mathcal{A} \) is locally small, we know that \( \ell(X) \) (the lattice of subobjects of \( X \)) is a set. Let \( \aleph = |\ell(X)| \). It is then clear that \( X \) is \( \aleph \)-generated. \( \blacksquare \)

**Proposition 2.2.** Let \( \mathcal{C} \) be a locally small abelian category with exact direct limits, \( \lambda \) the least ordinal number with \( |\lambda| > \aleph \), \( X \) an \( \aleph \)-generated object of \( \mathcal{C} \), and \( Y \in \text{Ob}(\mathcal{C}) \) such that \( Y = \sum_{\alpha < \lambda} Y_\alpha \) for \( \{Y_\alpha; \alpha < \lambda\} \), a continuous chain of subobjects of \( Y \). Then for any morphism \( f: X \rightarrow Y \) there exists \( \beta < \lambda \) such that \( f(X) \subseteq Y_\beta \).

**Proof.** If \( X \) is \( \aleph \)-generated, then \( f(X) \) is also \( \aleph \)-generated, and we have \( f(X) \subseteq \sum_{\alpha < \lambda} Y_\alpha \). Therefore, \( f(X) = (\sum_{\alpha < \lambda} Y_\alpha) \cap f(X) = (\sum_{\alpha < \lambda} Y_\alpha \cap f(X)) \), since direct limits are exact. We then know there exists a subset
$S \subseteq \lambda, |S| \leq 8$ such that $f(X) = \sum_{\alpha \in S} (Y_\alpha \cap f(X)) = (\sum_{\alpha \in S} Y_\alpha) \cap f(X)$, so $f(X) \subseteq \sum_{\alpha \in S} Y_\alpha$.

Since $S \subseteq \lambda$, we have that $|\alpha| \leq 8$ for all $\alpha \in S$, and if $\beta = \text{Sup } S$, then we see that $|\beta| \leq 8$ since $|S| \leq 8$, so we get $\beta < \lambda$. Therefore, $\sum_{\alpha \in S} Y_\alpha \subseteq Y_\beta$, and then $F(X) \subseteq Y_\beta$.

**Remark 2.3.** It is known that if a module $M$ can be written as the direct union of a continuous chain of submodules \( \{M_\alpha; \alpha < \lambda\} \) such that $\text{Ext}^1(M_0, C) = 0$ and $\text{Ext}^1(M_{\alpha+1}/M_\alpha, C) = 0$ whenever $\alpha < \lambda$, then $\text{Ext}^1(M, C) = 0$ (see, e.g., [4, Theorem 1.2]). That result shows that the proof is valid not only for the category of modules, but also for any category in which homology and direct limits make any sense, in particular for any Grothendieck category with a projective generator.

Thus we can prove the following result, whose proof closely follows the proof of [5, Theorem 2]. We have made only the modifications necessary to make the proof more categorical.

**Theorem 2.4.** Let $\mathcal{F}$ be a class of objects of $\mathcal{A}$ which is closed under direct sums. Suppose that for a fixed object $A \in \mathcal{F}$, $C \in \mathcal{F}^\perp$ if and only if $\text{Ext}^1(A, C) = 0$. Then for any object $M \in \text{Ob}(\mathcal{A})$ there exists an exact sequence in $\mathcal{A}$,

\[
0 \to M \to C \to C/M \to 0,
\]

with $C \in \mathcal{F}^\perp$ and $C/M \in \mathcal{F}^\perp$.

**Proof.** Let $0 \to K \to P \to A \to 0$ be an exact sequence in $\mathcal{A}$ with $P$ projective. If $M$ is any module and we let $\psi: K^{\text{Hom}(K, M)} \to M$ be the evaluation map, then we can construct the pushout

\[
\begin{array}{ccc}
K^{\text{Hom}(K, M)} & \xrightarrow{\psi} & M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi} & M_1
\end{array}
\]

and see that any morphism $f: K \to M$ can be extended to $f': P \to M_1$. Furthermore $M_1/M \cong (P/K)^{\text{Hom}(K, M)} \cong A^{\text{Hom}(K, M)} \in \mathcal{F}$, since $\mathcal{F}$ is closed under direct sums. We can now find another object $M_2$ in such a way that any morphism $K \to M_1$ can be extended to a morphism $P \to M_2$ and that $M_2/M_1 \in \mathcal{F}$. Repeating this procedure and letting $M_\beta = \sum_{\alpha < \beta} M_\alpha$ whenever $\beta$ is a limit ordinal, we see that for any ordinal $\mu$ we can construct a continuous chain of subobjects \( \{M_\alpha; \alpha < \mu\} \) such that $M_0 = M$, $M_{\alpha+1}/M_\alpha \in \mathcal{F}$ for all $\alpha + 1 < \mu$ and that any morphism $K \to M_\alpha$ extends to $P \to M_{\alpha+1}$ for all $\alpha + 1 < \mu$. 

Since every Grothendieck category is locally small, by Proposition 2.1 we know that $K$ is $\aleph$-generated for some cardinal number $\aleph$. We then choose the least ordinal number $\lambda$ such that $|\lambda| > \aleph$ and consider $C = \sum_{\alpha < \lambda} M_\alpha$. By Proposition 2.2, any $K \to C$ factors through $K \to M_\beta \to C$ for some $\beta < \lambda$. Then $K \to M_\beta$ extends to $P \to M_{\beta+1}$, since $\beta < \lambda$ (and so $\beta + 1 < \lambda$, since $\lambda$ is a limit ordinal), and we have an extension $P \to C$ of $K \to C$. This means that $\text{Ext}^1(A, C) = 0$, since $P$ is projective, and then that $C \in \mathcal{T}^\perp$.

It remains to be shown that the quotient object $C/M$ belongs to $\perp(\mathcal{T}^\perp)$. Now, $C/M = (\sum_{\alpha < \lambda} M_\alpha)/M = \sum_{\alpha < \lambda} (M_\alpha/M)$, where $\{M_\alpha/M; \alpha < \lambda\}$ is a continuous chain of subobjects of $C/M$, and we have

$$\text{Ext}^1(M_\beta/M, D) = \text{Ext}^1(0, D) = 0, \quad \text{Ext}^1\left(\frac{M_{\alpha+1}/M}{M_\alpha/M}, D\right) = 0$$

for all $D \in \mathcal{T}^\perp$ and all $\alpha < \lambda$ (since $\frac{M_{\alpha+1}/M}{M_\alpha/M} \cong M_{\alpha+1}/M_\alpha$). Thus, by Remark 2.3, we get $\text{Ext}^1(C/M, D) = 0 \forall D \in \mathcal{T}^\perp$ and $C/M \in \perp(\mathcal{T}^\perp)$. □

As in the case of cotorsion theories, we say that, given a class $\mathcal{T}$, the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is cogenerated by a set when there exists a set $X \subset \mathcal{T}$ such that $G \in \mathcal{T}^\perp$ if and only if $\text{Ext}^1(F, G) = 0$ for all $F \in X$. It is then clear that $(\mathcal{T}, \mathcal{T}^\perp)$ is cogenerated by a set if and only if there exists a single object, $F \in \mathcal{T}$, such that $G \in \mathcal{T}^\perp \iff \text{Ext}^1(F, G) = 0$. (We consider $\mathcal{T}$ to be closed under direct sums.)

We then have the following.

**Corollary 2.5.** Let $\mathcal{T}$ be a class of objects of $\mathcal{A}$ closed under direct sums. If the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is cogenerated by a set, then for any $M \in \text{Ob}(\mathcal{A})$ there exists an exact sequence

$$0 \to M \to C \to C/M \to 0,$$

with $C \in \mathcal{T}^\perp$ and $C/M \in \perp(\mathcal{T}^\perp)$. In particular, if $(\mathcal{T}, \mathcal{T}^\perp)$ is a cotorsion theory cogenerated by a set, then $(\mathcal{T}, \mathcal{T}^\perp)$ has enough injectives.

**Theorem 2.6.** Let $\mathcal{T}$ be a class of objects of $\mathcal{A}$ closed under direct sums, extensions, and continuous well-ordered unions. If the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is cogenerated by a set, then $(\mathcal{T}, \mathcal{T}^\perp)$ has enough injectives.

**Proof.** Let $M$ be any object of $\mathcal{A}$ and let

$$0 \to M \to C \to C/M \to 0$$

be exact with $C \in \mathcal{T}^\perp$ and $C/M \in \perp(\mathcal{T}^\perp)$ as in Theorem 2.4. The construction of $C/M$ shows that $C/M = \sum_{\alpha < \lambda} M_\alpha/M$, where $M_0 = M \in \mathcal{T}$, $M_{\alpha+1}/M_\alpha \in \mathcal{T}$ for all $\alpha < \lambda$ and $\{M_\alpha/M; \alpha < \lambda\}$ is a continuous chain of subobjects of $C/M$. Since $\mathcal{T}$ is closed under extensions and continuous well-ordered unions, we see that $M_\alpha/M \in \mathcal{T}$ for all $\alpha < \lambda$ and, in fact, that $C/M \in \mathcal{T}$. Therefore, $(\mathcal{T}, \mathcal{T}^\perp)$ has enough injectives. □
In [13, Lemma 2.2] Salce proved that in the category of abelian groups, any cotorsion theory \((\mathcal{F}, \mathcal{C})\) which has enough injectives (projectives) and such that \(\mathcal{F}\) is closed under extensions also has enough projectives (injectives). However, the argument used in the proof is valid not only for the category of abelian groups, but also for any Grothendieck category. Moreover, for the argument to hold, it is not necessary that \((\mathcal{F}, \mathcal{C})\) be a cotorsion theory. The only necessary condition is that \(\mathcal{F}\) contain all projective objects. Therefore, we have the next result.

**Corollary 2.7.** Let \(\mathcal{F}\) be a class of objects of \(\mathcal{A}\) as in Theorem 2.6, and suppose that \(\mathcal{F}\) contains all projective of \(\mathcal{A}\). If the pair \((\mathcal{F}, \mathcal{F}^\perp)\) is cogenerated by a set, then \((\mathcal{F}, \mathcal{F}^\perp)\) has enough injectives and projectives.

In [15, Section 2.2], Xu defined generators and minimal generators for \(\text{Ext}(\mathbb{L}, M)\), where \(M\) is an \(R\)-module and \(\mathbb{L}\) is a class of \(R\)-modules (Definition 2.2.1). We note that this definition makes perfect sense for any Grothendieck category (and even for more general abelian categories). Thus we have

**Definition 2.8.** Let \(\mathbb{L}\) be a class of objects of a Grothendieck category \(\mathcal{A}\), and let \(M\) be an object of \(\mathcal{A}\). An exact sequence \(0 \to M \to G \to L \to 0\) with \(L \in \mathbb{L}\) is called a generator for \(\text{Ext}(\mathbb{L}, M)\) if for any exact sequence \(0 \to M \to \overline{G} \to \overline{L} \to 0\) with \(\overline{L} \in \mathbb{L}\) there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & M & \to & G & \to & L & \to & 0 \\
\downarrow{id}_M & & \downarrow{g} & & \downarrow{f} & & \\
0 & \to & M & \to & G & \to & L & \to & 0.
\end{array}
\]

The foregoing generator is said to be minimal if for any commutative diagram

\[
\begin{array}{cccccc}
0 & \to & M & \to & G & \to & L & \to & 0 \\
\downarrow{id}_M & & \downarrow{g} & & \downarrow{f} & & \\
0 & \to & M & \to & G & \to & L & \to & 0,
\end{array}
\]

\(f\) is an isomorphism (and so \(g\) is as well).

Xu ([15]) related the existence of generators and minimal generators of \(\text{Ext}(\mathbb{L}, M)\) with the existence of \(\mathbb{L}^\perp\)-preenvelopes and \(\mathbb{L}^\perp\)-envelopes for \(M\). Indeed, he proved that, under certain restrictions on \(\mathbb{L}\), if \(\text{Ext}(\mathbb{L}, M)\) has a generator, then it has a minimal generator (see [15, Theorem 2.2.2]), or, equivalently, that if \(M\) has an \(\mathbb{L}^\perp\)-preenvelope, then it has an \(\mathbb{L}^\perp\)-envelope. The proof of Xu’s Theorem 2.2.2 was given in three steps (three lemmas), and it is easy to see that the proofs of those three lemmas still hold in Grothendieck categories. Xu constructs chains of submodules of a given
module $N$, with each of these chains having as many terms as one wants, contradicting the fact that the cardinality of $N$ is fixed. For the case of Grothendieck categories, we need only observe that, since they are locally small, the lattice $\ell(N)$ is a set (and so it has its fixed cardinality) and then apply Xu’s argument.

We state here Xu’s theorem.

**Theorem 2.9** ([15, Theorem 2.2.2]). *Suppose that $\mathcal{L}$ is closed under well-ordered direct limits. If $\text{Ext}(\mathcal{L}, M)$ has a generator, then it has a minimal generator.*

Similar results can be obtained for the case $\mathcal{L}$-precovers and $\mathcal{L}$-covers. We then have the following result, given by Xu for module categories and valid for Grothendieck categories.

**Theorem 2.10** ([15, Theorem 2.2.12]). *Assume that $\mathcal{L}$ is closed under well-ordered direct limits. If $M$ has an $\mathcal{L}$-precover, then it has an $\mathcal{L}$-cover $L \to M$. Furthermore, if $\mathcal{L}$ is closed under extensions, then $\text{Ker} (L \to M) \in \mathcal{L}^\perp$.*

We recall that given a class $\mathcal{F}$, a special $\mathcal{F}$-preenvelope of an object $A$ is an injective map $A \to F$ with $F \in \mathcal{F}$, such that $\text{Coker} (A \to F) \in \mathcal{F}^\perp$. (It is immediate that such a map is indeed a $\mathcal{F}$-preenvelope.)

**Corollary 2.11.** Let $\mathcal{F} \subset \text{Ob}(\mathfrak{A})$ be as in Theorem 2.6. If the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set, then every object $M \in \text{Ob}(\mathfrak{A})$ has a special $\mathcal{F}^\perp$-preenvelope. If, moreover, $\mathcal{F}$ is closed under well-ordered direct limits, then every $M \in \text{Ob}(\mathfrak{A})$ has an $\mathcal{F}^\perp$-envelope.

**Proof.** By Theorem 2.6, for any object $M \in \text{Ob}(\mathfrak{A})$ there exists an exact sequence

$$0 \to M \to C \to F \to 0$$

with $C \in \mathcal{F}^\perp$ and $F \in \mathcal{F}$. It is then immediate to see that $0 \to M \to C$ is a special $\mathcal{F}^\perp$-preenvelope. If $\mathcal{F}$ is closed under well-ordered direct limits, then Theorem 2.9 says that $M$ admits an $\mathcal{F}^\perp$-envelope.

**Corollary 2.12.** Let $\mathcal{F} \subset \text{Ob}(\mathfrak{A})$ be as in Corollary 2.7. If the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set, then every object $M \in \text{Ob}(\mathfrak{A})$ admits a special $\mathcal{F}$-precover. If, moreover, $\mathcal{F}$ is closed under well-ordered direct limits, then every $M \in \text{Ob}(\mathfrak{A})$ admits an $\mathcal{F}$-cover.

**Proof.** If $\mathcal{F}$ is as in Corollary 2.7 and $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set, then we know (by Corollary 2.7) that $(\mathcal{F}, \mathcal{F}^\perp)$ has enough projectives. Now just apply arguments dual to those of Corollary 2.11.
Corollary 2.13. Let $ℱ ⊂ Ob(𝒜)$ be closed under direct sums, direct summands, extensions, and continuous well-ordered unions and such that all projective objects of $𝒜$ are in $ℱ$. If the pair $(ℱ, ℱ^\perp)$ is cogenerated by a set, then $(ℱ, ℱ^\perp)$ is a cotorsion theory.

Proof. Let $M ∈ (ℱ^\perp)$ be any object. By Corollary 2.12, $M$ has a special $ℱ$-precover $F → M → 0$ (which is an epimorphism, since $ℱ$ contains all of the projectives), so that $K = \text{Ker}(F → M) ∈ ℱ^\perp$. Thus, the sequence

$$0 → K → F → M → 0$$

splits, and then $M ∈ ℱ$ since $F ∈ ℱ$, and $ℱ$ is closed under direct summands.

3. FLAT COVERS AND COTORSION ENVELOPES
OF COMPLEXES

This section is devoted to the study of the existence of flat covers and cotorsion envelopes for any complex of modules. Thus, throughout this section $ℱ$ denotes class of flat complexes over a fixed (but arbitrary) ring $R$. We recall that in this case, the class $ℱ^\perp$ is denoted by $C$ and is called the class of all cotorsion complexes. Given any complex $C$, by $C^i$ we mean the module in the $i$th component of $C$ for any $i ∈ ℤ$.

If $M$ is a module, then we denote by $\underline{M}$ the complex

$$\cdots → 0 → M → 0 → \cdots$$

with $M$ in the 0th component. By $\overline{M}$ we denote the complex

$$\cdots → 0 → M^l → M → 0 → \cdots$$

with $M$ in the 0th and (−1)st component. It is clear then that for any module $M$, $\underline{M}$ is a subcomplex of $\overline{M}$, and $\overline{M}$ is a projective complex if and only if $M$ is a projective module.

Flat complexes have been characterized in [10, Theorem 4.1.3] as those exact complexes $F = (F^i, δ^i)$ in which $Z^i(F)$ (i.e., $\text{Ker} δ^i$) is a flat submodule of $F^i$ for all $i ∈ ℤ$ or, equivalently, as those exact complexes $(F^i, δ^i)$ such that $Z^i(F) ≤ F^i$ is pure and $F^i$ is flat $∀ i ∈ ℤ$.

By [10, Theorem 4.1.3], we know that any direct limit of flat complexes of modules is flat, that all projective complexes are flat, and that $ℱ$ is closed under direct sums, direct summands, and extensions. Then, by Corollaries 2.12 and 2.11, if we prove that the pair $(ℱ, C)$ is cogenerated by a set, we will have that every complex has a flat cover and a cotorsion envelope. Furthermore, the pair $(ℱ, C)$ will be a cotorsion theory.
PROPOSITION 3.1. Let $|R| \leq \aleph$. Then for any flat complex $F$ and any element $x \in F^k (k \in \mathbb{Z}$ arbitrary), there exists a flat subcomplex $L$ of $F$ such that $x \in L^k, |L| \leq \aleph$, and $F/L$ is also a flat complex.

Proof. Let us suppose (without loss of generality) that $k = 0$ and $x \in F^0$. Consider then the exact complex

(S1) \[ \cdots \to A_1^i \delta_2 \to A_1^i \delta_1 \to Rx \to \delta^0(Rx) \to 0, \]

where $A_1^i$ is a submodule of $F^{-i}$ constructed as follows: $|Rx| \leq \aleph$, since $|R| \leq \aleph$, so we can find $A_1^i \leq F^{-1}$ such that $|A_1^i| \leq \aleph$ and $\delta^{-1}(A_1^i) = \text{Ker}(\delta^0|_{Rx})$. Then $A_1^i \leq F^{-2}, |A_1^i| \leq \aleph$, and $\delta^{-2}(A_1^i) = \text{Ker}(\delta^{-1}|_{A_1^i})$ and we repeat the argument.

Now $\text{Ker}(\delta^0|_{Rx}) \leq \text{Ker} \delta^i$, so we know by [3, Lemma 2] that $\text{Ker}(\delta^0|_{Rx})$ can be embedded into a pure submodule $S^i_2$ of $\text{Ker} \delta^0$ (taking $S_0$ of [3, Lemma 2] to be $\text{Ker}(\delta^0|_{Rx})$). Since $|\text{Ker}(\delta^0|_{Rx})| \leq \aleph$, we see, by the construction given in [3, Lemma 2], that $S^i_2$ can be chosen in such a way that $|S^0_2| \leq \aleph$. Then consider the exact complex

(S2) \[ \cdots \to A_2^i \delta_2 \to A_2^i \delta_1 \to Rx + S_2^i \to \delta^0(Rx) \to 0, \]

where $A_2^i$ are taken as before. It is clear that $\text{Ker}(\delta^0|_{Rx+S_2^i}) = S^i_2$, which is pure in $\text{Ker} \delta^0$, and that $|Rx + S_2^i| \leq \aleph + \aleph = \aleph$.

Observe now that $\delta^0(Rx) \leq \text{Ker} \delta^1$, so we can embed $\delta^0(Rx)$ into a pure submodule $S^1_3$ of $\text{Ker} \delta^1$ in such a way that $|S^1_3| \leq \aleph (|\delta^0(Rx)| \leq \aleph)$ and then take the exact complex

(S3) \[ \cdots \to A_3^i \delta_2 \to A_3^i \delta_1 \to A_3^i \delta_0 \to S^1_3 \to 0. \]

We see again that $\text{Ker}(\delta|_{S^1_3}) = S^1_3$, which is a pure submodule of $\text{Ker} \delta^1$.

We turn over and find $S^0_4 \leq \text{Ker} \delta^0$ pure with $|S^0_4| \leq \aleph$ and $S^0_4 \supseteq \text{Ker}(\delta^0|_{A_3^i})$, and then construct $A_4^{-i} \leq F^{-i}$ in $|A_4^{-i}| \leq \aleph \forall i$ such that

(S4) \[ \cdots \to A_4^i \delta_2 \to A_4^i \delta_1 \to A_4^i \delta_0 \to S^0_4 \to 0 \]

is exact. Once more, $\text{Ker}(\delta^0|_{A_3^i+S^0_4}) = S^0_4 \leq \text{Ker} \delta^0$ is a pure submodule. We then find $S^1_5 \leq \text{Ker} \delta^{-1}$ pure with $|S^1_5| \leq \aleph$, $\text{Ker}(\delta^{-1}|_{A_4^i}) \leq S^1_5$, and consider the exact complex

(S5) \[ \cdots \to A_5^i \delta_2 \to A_5^i \delta_1 \to A_5^i \delta_0 \to S^1_5 \to 0, \]

in which $\text{Ker}(\delta^{-1}|_{A_4^i+S^1_5}) = S^1_5 \leq \text{Ker} \delta^{-1}$ pure.

The next step is to find $S^2_6 \leq \text{Ker} \delta^{-2}$ pure such that $|S^2_6| \leq \aleph$ and $\text{Ker}(\delta^{-2}|_{A_5^i}) \subseteq S^2_6$, and then consider the exact complex

(S6) \[ \cdots \to A_6^i \delta_2 \to A_6^i \delta_1 \to A_6^i \delta_0 \to S^2_6 \to 0, \]

in which $\text{Ker}(\delta^{-2}|_{A_5^i+S^2_6}) = S^2_6 \leq \text{Ker} \delta^{-2}$ pure.
Therefore, we prove by induction that for any $n \geq 4$ we can construct an exact complex

\[
\cdots \rightarrow A_{n} \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow \cdots \rightarrow T_{n} \rightarrow T_{n-1} \rightarrow 0
\]

such that $\text{Ker } (\delta_{n+1}^{-1})$ is a pure submodule of $\text{Ker } \delta_{n+1}^{-m}$ for all $m$.

If we take the direct limit $L = \lim_{\rightarrow} (Sn)$ with $n \in \mathbb{N}$, then we see that the complex $L$ is exact and that $\text{Ker } (\delta_{n+1}^{-1})$ is a pure submodule of $\ker \delta_{n+1}^{-i}$ for all $i \leq 1$. Furthermore, $|L| \leq \aleph_0 \cdot \aleph = \aleph$ for any $i \leq 1$, so $|L| \leq \aleph$. We finally consider the complex $L$ to be

\[
L = \cdots \rightarrow L \rightarrow L \rightarrow \cdots \rightarrow L \rightarrow L \rightarrow 0 \rightarrow 0 \rightarrow \cdots,
\]

which is a subcomplex of $F$, $x \in L$, and $\text{Ker } (\delta_{n+1}^{-1})$ is a pure submodule of $\ker \delta_{n+1}$ for all $i \leq 1$. An easy computation shows that $\text{Ker } (\delta_{i}^{-1}) = \ker \delta_{i}/\text{Ker } (\delta_{i+1}^{-1})$, but by construction $\text{Ker } (\delta_{i}^{-1})$ is a pure submodule of $\ker \delta_{i}$ for all $i \in \mathbb{Z}$. Of course, $F/L$ is exact, since both $F$ and $L$ are exact, so $F/L$ is a flat complex.

**Remark 3.2.** If $F$ is a flat complex and $x \in F$, then by the last result we know that we can find a flat submodule $C$ of $F$ such that $x \in C$, $|C| \leq \aleph$, and the quotient complex $F/C$ is flat. Hence, by transfinite induction we can find a continuous chain of subcomplexes of $F$, say $\{C_{\alpha}; \alpha < \lambda\}$, for some ordinal number $\lambda$ such that $F = \bigcup_{\alpha < \lambda} C_{\alpha}$; $C_{0}, C_{\alpha+1}/C_{\alpha}$ are flat complexes; and $|C_{0}| \leq \aleph, |C_{\alpha+1}/C_{\alpha}| \leq \aleph \forall \alpha < \lambda$. But since flat complexes are closed under extensions and direct limits, we see that in fact each $C_{\alpha}$ is flat, and so we have that every flat complex is the direct union of a continuous chain of flat subcomplexes with cardinality less than or equal to $\aleph$.

Thus, if we let $X$ be a representative set of flat complexes $F$ with $|F| \leq \aleph$, then a complex $G \in \mathcal{C}$ if and only if $\text{Ext}^1(F, G) = 0 \forall F \in X$ by Remark 2.3; i.e., the pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set. Then, by Corollaries 2.12, 2.11, and 2.13 we get the following.

**Theorem 3.3.** Every complex of left (right) $R$-modules admits a flat cover and a cotorsion envelope, and the pair $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory.

**Example.** Given a module $M$, we construct the flat cover of the complex $M$. We first take the flat cover $\varphi: F \rightarrow M$ of $M$ ([3, Theorem 3 or Theorem 6]), and then construct the minimal cotorsion resolution of $F$,

\[
F = \cdots 0 \rightarrow 0 \rightarrow F \rightarrow C^{0}(F) \rightarrow C^{1}(F) \rightarrow \cdots.
\]
(Note that this resolution always exists since a module $N$ has a cotorsion envelope if and only if it has a flat cover by [15, Theorem 3.4.6].) We prove that the morphism of complexes

$$
\begin{array}{cccccccc}
F & \cdots & 0 & \rightarrow & 0 & \rightarrow & F & \stackrel{\delta^0}{\rightarrow} C^0(F) & \stackrel{\delta^1}{\rightarrow} C^1(F) & \stackrel{\delta^2}{\rightarrow} \cdots \\
\phi & & & & & & & & \\
M & \cdots & 0 & \rightarrow & 0 & \rightarrow & M & \rightarrow & 0 & \rightarrow & \cdots
\end{array}
$$

is a flat cover of $M$.

Of course, the diagram is commutative (i.e., $\phi$ is a morphism of complexes), so we first must show that $F$ is a flat complex. This is exact by construction, and it is clear that

$$\text{Ker } \delta^i = \text{Ker } (C^{i-1}(F) \rightarrow \text{Coker } \delta^{i-1}) \cong \text{Coker } \delta^{i-2} \quad \forall i \geq 2.$$  

But $\text{Coker } \delta^0 = C^0(F)/F$ is flat by [15, Theorem 3.4.2], so $\text{Coker } \delta^1$ is also flat by [15, Theorem 3.4.2], since $C^1(F)$ is the cotorsion envelope of $\text{Coker } \delta^0$. Thus, we get by induction that $\text{Ker } \delta^i$ is flat $\forall i \geq 2$. But $\text{Ker } \delta^0 = 0$ is flat and $\text{Ker } \delta^1 = \text{Im } \delta^0 = F$ is flat, so the complex $F$ is flat.

The kernel of the morphism $\phi$ is $\cdots 0 \rightarrow \text{Ker } \phi \rightarrow C^0(F) \rightarrow C^1(F) \rightarrow \cdots$, which is a complex bounded below and all of its terms are cotorsion, and so it is a cotorsion complex (see [10, Proposition 4.3.3]). Therefore, we see that we have a special flat precover.

To prove that $\phi$ is in fact a flat cover, suppose that we have a morphism of complexes $f : F \rightarrow F$ such that

$$
\begin{array}{ccc}
F & \xrightarrow{f} & F \\
\phi & & \phi \\
M & \downarrow & \\
& &
\end{array}
$$

is commutative. Then $\varphi f^0 = \varphi$, but $\varphi$ is a cover, so $f^0$ is an isomorphism. Now $\delta^0 : F \rightarrow C^0(F)$ is a cotorsion envelope and $f^0$ is an isomorphism, so $\delta^0 f^0 = f^1 \delta^0$ is also a cotorsion envelope of $F$. But envelopes are unique up to isomorphism, and hence $f^1$ must be an isomorphism. Then $f^1$ induces another automorphism on $C^0(F)/\delta^0(F)$, and $C^1(F)$ is the cotorsion envelope of $C^0(F)/\delta^0(F)$. Thus, applying the same argument as before, we see that $f^2$ is also an automorphism. If we continue this procedure, we then prove by induction that $f$ is an automorphism of complexes and that $\phi$ is a flat cover.

The flat cover of $\overline{M}$ is much simpler. It is $\overline{F} \rightarrow \overline{M}$ with the obvious map.
4. APPLICATIONS

The proof of Proposition 3.1 provides a way to find a large class of examples of covers and envelopes. If we consider $\mathscr{D}$ the class of all exact complexes which have all their terms flat (so this class properly contains the class of flat complexes), then, using a different argument of the proof of Proposition 3.1 but based on the same idea, we prove that the pair $(\mathscr{D}, \mathscr{D}^\perp)$ is cogenerated by a set.

**Proposition 4.1.** Let $|R| \leq \aleph$. Then for any complex $F \in \mathscr{D}$ and any element $x \in F^k (k \in \mathbb{Z}$ arbitrary), there exists a subcomplex $L \leq F$ such that $x \in L^k$, $|L| \leq \aleph$, and $L' \geq F'$ is a pure submodule for all $j \in \mathbb{Z}$.

**Proof.** Let us suppose (without loss of generality) that $k = 0$ and $x \in F^0$. We know by [3, Lemma 2] that $Rx$ can be embedded into a pure submodule $S_0$ of $F^0$ such that $|S_0| \leq \aleph$. Then we can consider the exact complex

\[(S1) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow A_i^{-1} \delta^{-1} \rightarrow S_i^0 \delta^0 (S_i^0) \rightarrow 0,
\]

where $A_i^{-j}$ is a submodule of $F^{-i}$ of cardinality less than or equal to $\aleph$ such that $\delta^{-i}(A_i^{-i}) = \text{Ker} (\delta^{-i+1}|_{A_i^{-i+1}})$ for all $i < 0$. (We let $A_0^0 = S_0^0$.) Now, we can embed $\delta^0(S_0^0)$ into a pure submodule $S_1^1$ of $F^1$. (We take $S_0$ of [3, Lemma 2] to be $\delta^0(S_0^0)$.) Since $|\delta^0(S_0^0)| \leq \aleph$, we see by the construction given in [3, Lemma 2] that $S_1^1$ can be chosen in such a way that $|S_1^1| \leq \aleph$. Then consider the exact complex

\[(S2) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow A_i^{-1} \delta^{-1} \rightarrow A_i^0 \delta^0 \rightarrow S_i^1 \delta^1 (S_i^1) \rightarrow 0,
\]

where each $A_i^{-j}$ is taken as before. If we embed $A_0^0$ into a pure submodule $S_1^1 \leq F^0$ with $|S_1^1| \leq \aleph$ and construct $A_1^{-i}$ as before, we then get another complex,

\[(S3) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow A_i^{-1} \delta^{-1} \rightarrow S_i^0 \delta^0 + \delta^0(S_i^0) \delta^1 (S_i^1) \rightarrow 0,
\]

which is also exact. (It is exact by construction at $S_i^0$, $\delta^1(S_i^1)$, and $A_i^{-i}$ for $i < 0$, and it is also exact at $\delta^0(S_i^0) + S_i^1$ because $\text{Ker} (\delta^1|_{S_i^1}) = \delta^0(A_i^0) \subseteq \delta^0(S_i^0)$.) Find now a pure submodule $S_4^{-1} \leq F^{-1}$ containing $A_3^{-1}$ such that $|S_4^{-1}| \leq \aleph$ and consider the complex

\[(S4) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow S_i^0 \delta^0 + \delta^{-1}(S_i^0) \delta^1 \rightarrow S_i^1 + \delta^0(S_i^0) \delta^1 (S_i^1) \rightarrow 0,
\]

which is exact by the same argument as in (S3). Now we turn over and get exact complexes

\[(S5) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow A_i^{-1} \delta^{-1} \rightarrow S_i^0 \delta^0 + \delta^0(S_i^0) \delta^1 (S_i^1) \rightarrow 0,
\]

\[(S6) \quad \cdots \rightarrow A_i^{-2} \delta^{-2} \rightarrow A_i^{-1} \delta^{-1} \rightarrow A_i^0 \delta^0 \rightarrow S_i^1 \delta^1 (S_i^1) \rightarrow 0,
\]
Proposition 4.1, and it is clear that \( \delta \) pure containing \( \mathcal{B} \) of complexes (see [1]).

The homological dimensions of complexes have been studied using this class before for \( i \).

Let \( \mathcal{C} \) be the direct limit of \( \mathcal{S}_i \) for any \( i \). Thus, by transfinite induction, we then find exact complexes \( \mathcal{S}_1 \) for all \( m \in \mathbb{N} \) in such a way that there are infinitely many \( m \) with \( \mathcal{S}_m \) a pure submodule of \( F \) for each \( i \). Furthermore, \( x \in (\mathcal{S}_m)^0 \forall m \in \mathbb{N} \) and \( |(\mathcal{S}_m)^0| \leq 8 \forall m \in \mathbb{N}, \forall i \in \mathbb{Z} \).

Let \( L \) be the direct limit of \( (\mathcal{S}_m), m \in \mathbb{N} \). Then \( L \) is an exact complex such that \( x \in L_0 \), \( L_j \leq F_j \) is a pure submodule \( \forall j \in \mathbb{Z} \) and \( |L_j| \leq 8 \cdot \mathbb{N}^{\cdot \mathbb{N}} \leq 8 \) for any \( j \in \mathbb{Z} \), so \( |L| \leq 8 \).

If we now consider a complex \( F \in \mathcal{L} \) and \( x \in F \), we then find \( L \) as in Proposition 4.1, and it is clear that \( F/L \in \mathcal{L} \). Thus, by transfinite induction, \( F = \sum_{\alpha < \lambda} C_\alpha \) for some ordinal \( \lambda \), where \( \{ C_\alpha : \alpha < \lambda \} \) is a continuous chain of complexes such that \( C_0 \) and \( C_{\alpha+1}/C_\alpha \) are as in Proposition 4.1. Then we see that the pair \( (\mathcal{L}, \mathcal{L}^+) \) is cogenerated by a set (any set of representatives of complexes \( L \), as in Proposition 4.1).

Now, it is immediate that the class \( \mathcal{L} \) is also closed under direct sums and summands and that it contains all projective complexes. Thus, we can apply Corollaries 2.12, 2.11, and 2.13 to get the next result.

**Theorem 4.2.** Every complex of modules admits an \( \mathcal{L} \)-cover and an \( \mathcal{L}^+ \)-envelope. Moreover, the pair \( (\mathcal{L}, \mathcal{L}^+) \) is a cotorsion theory.

The same type of argument holds for the class \( \mathcal{R} \) of complexes \( L \) (not necessarily exact) with \( L_j \) flat \( \forall j \in \mathbb{Z} \). This is an important example, since homological dimensions of complexes have been studied using this class \( \mathcal{R} \) of complexes (see [1]).

**Theorem 4.3.** Every complex of modules has a \( \mathcal{R} \)-cover and a \( \mathcal{R}^+ \)-envelope, and the pair \( (\mathcal{R}, \mathcal{R}^+) \) is a cotorsion theory.

Spaltenstein ([14]) has proved that any complex \( C \) of modules has a \( K \)-injective resolution. An analogous result says that for every such \( C \) there exists a quasi-isomorphism \( C \to I \), where \( I \) is a DG-injective complex. (We note, however, that the terminology “DG-injective” and “DG-projective” complexes was not used by Spaltenstein, but rather was introduced by Avramov and Foxby in [1]). A version of this result says that for any complex \( C \) there exists an exact sequence of complexes,

\[ 0 \to I \to E \to C \to 0, \]
with $E$ an exact complex and $I$ a DG-injective complex (see, e.g., [10, Theorem 2.2.4]). If we denote by $\mathcal{E}$ the class of all exact complexes, then it has been proved that the class $\mathcal{E}^\perp$ is the class of all DG-injective complexes ([9, Theorem] or [10, Proposition 2.3.4]). Thus, the existence of $\mathcal{E}$-covers and $\mathcal{E}^\perp$-envelopes (DG-injective envelopes) has been studied and proved ([9, Theorem] or [10, Theorem 2.3.11]). Here we apply the techniques developed in this paper to give a short proof of the existence of such covers and envelopes. We also give similar applications for the pair ($\mathcal{E}^\perp$, $\mathcal{E}$) to find exact envelopes and DG-projective covers. (Recall from [9, Theorem] or [10, Proposition 2.3.5] that $\mathcal{E}^\perp$ is precisely the class of all DG-projective complexes.)

Given a complex $C$ and an integer number $n$, the complex $C[n]$ is that whose $k$th term is $C[n]^k = C^{n+k}$ and whose $k$th boundary operator is $\delta[n]^k = (-1)^k \delta^{n+k}$. Therefore, we see that for a module $M$, $M[n]$ is the complex

\[
\cdots 0 \to 0 \to M \to 0 \to 0 \to \cdots
\]

with $M$ at the $(-n)$th position.

Let us start with the study of the pair ($\mathcal{E}$, $\mathcal{E}^\perp$).

The next result can be proved following the same argument of Proposition 4.1, with the only modification being that the submodules $S_i$ that we get in the process are not necessarily pure, since now the $E_i$ are not necessarily flat. We then prove the following.

**Theorem 4.4.** Suppose that $|R| \leq \aleph_0$. For any complex $E \in \mathcal{E}$ and any element $x \in E^k (k \in \mathbb{Z})$ there exists an exact subcomplex $L$ such that $x \in L^k$ and $|L| \leq \aleph_0$.

**Corollary 4.5.** The pair of classes ($\mathcal{E}$, $\mathcal{E}^\perp$) has enough injectives and enough projectives.

**Proof.** It is an immediate consequence of Theorem 4.4 that any exact complex $E$ can be written as the direct union of a continuous chain of exact subcomplexes $\{E_\alpha; \alpha < \lambda\}$ such that $|E_\alpha| \leq \aleph_0$ for all $\alpha < \lambda$. Here we need the fact that if $E$ is exact and if $L \subseteq E$ is an exact subcomplex, then $E/L$ is exact. Therefore, ($\mathcal{E}$, $\mathcal{E}^\perp$) is cogenerated by a set of representatives of exact complexes of cardinality less than or equal to $\aleph_0$. Hence, applying Corollary 2.7, we see that ($\mathcal{E}$, $\mathcal{E}^\perp$) has enough injectives and projectives. \[
\]

**Corollary 4.6.** Any complex of modules has an exact cover and a DG-injective envelope. Moreover, ($\mathcal{E}$, $\mathcal{E}^\perp$) is a cotorsion theory.

**Proof.** ($\mathcal{E}$, $\mathcal{E}^\perp$) has enough injectives and projectives. Then, by Corollary 2.12, Corollary 2.11, and Corollary 2.13, every complex has an $\mathcal{E}$-cover (i.e., an exact cover) and an $\mathcal{E}^\perp$-envelope (i.e., a DG-injective envelope), and ($\mathcal{E}$, $\mathcal{E}^\perp$) is a cotorsion theory. \[
\]
We now recall that a DG-injective envelope is a quasi-isomorphism ([9, Theorem 3.12]).

Let us study the pair \( (\perp, \mathcal{E}) \).

**Theorem 4.7.** The pair \( (\perp, \mathcal{E}) \) is cogenerated by a set.

**Proof.** Let

\[
C = \cdots \to C^{-1} \to C^0 \to C^1 \to \cdots
\]

be any complex of \( R \)-modules, and consider the complex \( R \). It is immediate that \( \text{Hom}(R, C) \cong Z^0(C) \), so if \( f \in \text{Hom}(R, C) \) corresponds to \( x \in Z^0(C) \), then it is easy to see that \( f \) extends to a morphism \( \overline{R} \to C \) if and only if \( x \in B^0(C) \), and then \( \text{Ext}^1(\overline{R}/R, C) = 0 \iff H^0(C) = 0 \). But \( \overline{R}/R \) is the complex \( R[1] \), so we have \( H^0(C) = 0 \iff \text{Ext}^1(R[1], C) = 0 \). Then we easily observe that \( H^0(C) = 0 \iff \text{Ext}^1(R[n+1], C) = 0 \), and then that \( C \in \mathcal{E} \) if and only if \( \text{Ext}^1(R[n], C) = 0 \forall n \in \mathbb{Z} \).

Therefore, the pair \( (\perp, \mathcal{E}) \) is cogenerated by the set \( X = \{R[n] \mid n \in \mathbb{Z}\} \).

The class \( \perp \mathcal{E} \) is closed under direct sums, extensions, and well-ordered unions, and of course it contains all projective complexes. Therefore, we can now apply Corollary 2.7 to get that the pair \( (\perp, \mathcal{E}) \) contains enough injectives and projectives. But although \( \perp \mathcal{E} \) is closed under well-ordered unions, it is not closed in general under well-ordered direct limits. This means that we will not be able to find \( \perp \mathcal{E} \)-covers (i.e., DG-projective covers) and exact envelopes in general. However, by Corollaries 2.11 and 2.12, we do have the following general result.

**Corollary 4.8.** Every complex of modules has an exact preenvelope and a DG-projective precover.

**Corollary 4.9.** The pair \( (\perp, \mathcal{E}) \) is a cotorsion theory.

**Proof.** Apply Corollary 2.13, taking note that \( \perp \mathcal{E} \) is also closed under direct summands.

It is known that the class \( \perp \mathcal{E} \) is closed under well-ordered direct limits if and only if the ring \( R \) is perfect. Thus, it is immediate that every complex of left \( R \)-modules has a DG-projective cover and an exact envelope if and only if \( R \) is left perfect.

**Remark 4.10.** If \( X \) is a ringed space with sheaf of rings \( \theta \) (see [11]), then a \( \theta \)-module \( F \) is flat if and only if \( F_x \) is a flat \( \theta_x \)-module for each \( x \in X \). Then if \( \mathcal{F} \) is the class of flat \( \theta \)-modules in the category \( \mathcal{A} \) of \( \theta \)-modules, Theorem 2.6 holds for \( \mathcal{A} \), even though in general \( \mathcal{A} \) does not have enough projectives. Hence Salce’s argument ([13]) cannot be used to argue that the pair \( (\mathcal{F}, \mathcal{F}^\perp) \) has enough projectives. If the pair \( (\mathcal{F}, \mathcal{F}^\perp) \) has
enough projectives, then the argument used for the proof of Theorem 2.9 would say that every $\theta$-module has a flat cover.

These remarks say that the first proof of the flat cover conjecture given in [3] does not carry over to prove the existence of flat covers of $\theta$-modules. It seems likely, however, that El Bashir’s argument given in the second proof of the flat cover conjecture in [3] can be modified to prove the existence of flat covers in this setting.

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