Dynamical Behaviour on the Parameter Space:
New Populational Growth Models Proportional to Beta
Densities

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Abstract. We present new populational growth models, generalized logistic models which are proportional to beta densities with shape parameters $p$ and 2, where $p > 1$, with Malthusian parameter $r$. The complex dynamical behaviour of these models is investigated in the parameter space $(r, p)$, in terms of topological entropy, using explicit methods, when the Malthusian parameter $r$ increases. This parameter space is split into different regions, according to the chaotic behaviour of the models.

Keywords. Beta densities, population dynamics, dynamical behaviour.

1. Introduction

To forecast the future evolution of the dynamic of populations is an important issue in several areas, such as biological, ecological, social or economical sciences. Sophisticated random models are now available in population dynamics, but in many instances their mean functions are well known deterministic models, useful as a first approach in applied problems. The logistic (or Verhulst) model, [8], which incorporates in its parameters both the Malthusian growth rate $r$ and the retroaction due to the limitation of natural resources, is a natural candidate to model the dynamic of non-overlapping generations, namely when the unit of time is related to the life span of the individuals in the population. The logistic map has been used with success to model the population growth for some species, but is inadequate for other.

In several numerical studies, the families of unimodal maps have been used, allowing for exhaustive investigations in terms of symbolic dynamics, [1]. The unimodal maps theory can be used in many branches of science, namely in the ones mentioned above. In population dynamics, aiming to model the growth of a certain species, the use of the Verhulst Model, which is proportional to the $Beta(2,2)$ density, [7], has been a standard, although in some cases it doesn’t fit properly the observations.

In what follows, we introduce more general models, related to $X \sim Beta(p, q)$ ($p > 0$ and $q > 0$), with probability density function

$$f_X(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}I_{(0,1)}(x),$$

where $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \forall p,q > 0$ is the Euler’s beta function.

The generalized logistic models we shall use are proportional to the $Beta(p,2)$ densities, with $p \in [1, +\infty[$, [2]. This family provides more flexible models to theorize
population dynamics when the classical logistic or Verhulst Model fails to provide a good fit. After the theoretical deduction of these models and the characterization of the patterns of the respective family of unimodal maps,

\[ f_{r,p}(x) = rx^{p-1}(1-x), \quad x \in [0,1] \]

we analyze their behaviour as a function of the parameters \( r \) and \( p \), exhibiting several regions with interesting behaviour patterns in the parameter space.

The principal aim of the present paper is to exhibit the dynamical patterns that can arise for different combinations of \((r,p)\) in the parameter space \( \Theta \) of this maps family, the main source of inspiration being [4], where a similar research has been carried out for the logistic case \( r = 2 \). This leads us to split out the parameter space into six distinct regions (the second of which has two important sub-regions) where the population dynamics presents distinct patterns. Most of our work, which is primarily analytical, has been possible using symbolic calculus available in Mathematica 6.0. The description of the regions, which is constructive, sketches the proof of the main results, that cannot be presented in detail herein.

Finally, we observe that the complexity of the inherent models, for each fixed value of the parameter \( p \), increases as the Malthusian parameter \( r \) becomes higher, Theorem 1. This follows from measuring the chaotic behaviour of the maps in terms of the topological entropy.

2. New populational growth models proportional to the Beta\((p,2)\) densities, with \( p \in ]1, +\infty[ \)

The Verhulst model can be easily presented as an approximation obtained by a sensible truncation of a series expansion, [7]. Using a similar reasoning, now we present new models proportional to the Beta\((p,2)\) densities, with \( p \in \mathbb{N}\setminus\{1,2\}, \) [1] and [2]. These models can be advantageous in the modeling of populational growth of species whose evolution needs a greater growth rate than the one given by the logistic model.

Let the population size \( N(t) \) be a regular function, the series expansion is given by:

\[
\frac{d}{dt}N(t) = A_0 + A_1N(t) + \ldots + A_{p-1}N(t)^{p-1} + A_pN(t)^p + A_{p+1}N(t)^{p+1} + \ldots ,
\]

truncating the terms of order smaller than \( p-1 \) and the terms of order largest than \( p \), considering that they are irrelevant to the model, i.e., \( A_i = 0 \), for \( i \leq p-2 \) and \( i \geq p+1 \), we obtain the simplified model:

\[
\frac{d}{dt}N(t) = A_{p-1}N(t)^{p-1}\left(1 + \frac{A_p}{A_{p-1}}N(t)\right)
\]

with \( A_{p-1} > 0 \) and \( A_p < 0 \). Denoting by \( A_{p-1} = r^* \), the coefficient proportional to the instantaneous populational growth, and \( K = -\frac{A_p}{A_{p-1}} \), the carrying capacity, (1), which represents the populational growth rate, can be rewritten \( \frac{d}{dt}N(t) = r^*N(t)^{p-1}\left(1 - \frac{N(t)}{K}\right) \), with \( p \in \mathbb{N}\setminus\{1,2\} \). As in the classical logistic setup, we now consider the corresponding difference-equation

\[
N(t_{n+1}) = r^*N(t_n)^{p-1}\left(1 - \frac{N(t_n)}{K}\right).
\]

Considering that \( x_n = \frac{N(t_n)}{K} \) and \( r = r^*K^{p-2} \), this discretized model is given by:

\[
x_{n+1} = r^*x_n^{p-1}(1-x_n).
\]

So, we considere the family of unimodal maps \( f_{r,p} : [0,1] \to [0,1] \), with two parameters \( p \in \mathbb{N}\setminus\{1,2\} \) and \( r > 0 \) defined by:

\[
f_{r,p}(x) = rx^{p-1}(1-x),
\]

with right-hand side proportional to the beta density, with shape parameters \( p \) and 2.

Therefore, let us consider the family of unimodal maps \( f_{r,p} : [0,1] \to [0,1] \), with two parameters \( p \) and \( r \), whose maximum variation intervals are given respectively by \( p \in [1,p_M] \) and \( r \in [0,r(p_M)] \), defined by (2), and \( c \) is the critical point of \( f_{r,p} \), which satisfies the following conditions:

- \( f_{r,p} \in C^3([0,1]) \);
- \( f'_{r,p}(x) \neq 0, \forall x \neq c; \)
- \( f_{r,p}(c) = 0 \) and \( f''_{r,p}(c) < 0 \) meaning that \( f_{r,p} \) is strictly increasing in \([0, c[\) and strictly decreasing in \(]c, 1]\);
- \( f_{r,p}(0) = f_{r,p}(1) = 0; \)
- \( f_{0,p}(c) = 0 \) and \( f_{r(p_M),p}(c) = 1; \)
- The Schwarz derivative of \( f_{r,p}(x) \) is

\[
S(f_{r,p}(x)) = \frac{f'''_{r,p}(x)}{f''_{r,p}(x)} - \frac{3}{2} \left( \frac{f''_{r,p}(x)}{f'_{r,p}(x)} \right)^2 < 0
\]

\( \forall x \neq c, \) with \( p > 2 \) and \( x > x_d, \) with \( 1 < p < 2^1.\)

Note that the parameter \( p \) has to be greater than one, since \( f_{r,p} \) is unimodal. In this study, the maximum value considered for the parameter \( p, \) denoted by \( p_M, \) was the largest value for which we consider that the model can be realistic. The value \( r(p_M) \) is the value of the parameter \( r \) corresponding to the full shift for \( p = p_M. \) In these maps \( f_{r,p}, \)

\( r \) and \( p \) are both shape parameters, which are respectively related to the height and the skewness of the curve. For any fixed \( p > 1, \)

if \( r = 0 \) there is no curve; as the value of \( r \) increases, we get higher curves, until the value of \( r \) corresponds to the full shift, when the height of the curve attains its maximum value 1, see Fig. 1. Considering for each \( p \) the value of the parameter \( r \) for which we obtain the full shift, we conclude that the curve of the map \( f_{r,p} \) can have three different patterns of skewness, as shown in Fig. 2.

In this work, we consider that \( 1 < p \leq p_M = 20 \) and \( 0 < r \leq 53.001. \) Observing Figs. 1 and 2, we can verify that the unimodal maps \( f_{r,p}, \)

always have the fixed point \( x^* = 0, \)

for any \( r > 0 \) and \( p > 1. \) However, seeing the cases of the maps corresponding to \( p = 1.1 \) and \( p = 1.5 \) in Fig. 2, we can verify that these maps have another positive fixed point besides 0. We can see that there are two more fixed points besides 0, for \( p \in [2, p_M], \) for

\[^1\) In this case, \( S(f_{r,p}(x)) \) is not always negative in the all interval \( I = [0, 1]. \) In fact, the interval \([0, x_d], \)

where the Schwarz derivative is positive, has a very small range because \( x_d \) is near 0, so the positivity of the Schwarz derivative occurs for values in the beginning of the interval. Those do not disturb the dynamic behaviour of the map \( f_{r,p}. \)

3. Dynamical behaviour on the parameter space

In this section, we split the parameter
space \( \Theta = \{(r,p)\} \) of the unimodal maps belonging to the family of functions \( f_{r,p} \) proportional to \( \text{Beta}(p,2) \) densities into regions with distinct characteristic patterns of dynamical behaviour. We restrict ourselves to \( 1 < p \leq p_M = 20 \) and \( 0 < r \leq 53.001 \), since so far greater values of those parameters seem to have no practical interest. Similar results for \( f_{r,2} \), have been presented in [4], cf. also references therein.

We partition \( \Theta \) into six main regions \( \Theta = \bigcup_{k=1}^{6} \Theta_k \), based on the usual dynamic study of unimodal maps, and the ensuing bifurcation structure. There are grounds to further split \( \Theta_2 = \Theta_{2a} \cup \Theta_{2b} \), as we shall see. In each of these seven regions we observe distinct patterns of dynamic behaviour, that may be of interested to have a deeper insight of the population dynamics of some species. To achieve this, for each fixed value of \( p \) in the range considered, we determine the points \( (r_i(p),p) \), with \( i = 1,2a,2b,3,4,5 \), as described in the previous section. In Fig. 1 we can see the maps \( f_{r_i(4),4} \), with \( i = 1,2a,2b,3,4,5 \), for \( p = 4 \), and the values \( r_i(4) \) from the boundary curves of the regions described in what follows:

Connecting the points associated to each \( p \in [1,p_M] \) we define 6 curves, which delimit the 7 regions of interest, shown in Fig. 3. Note that each line on the graphic corresponds to the summarized information of a Feigenbaum diagram to the map \( f_{r,p} \) for a certain fixed \( p \).

**Sudden extinction region \( \Theta_1 \)**

\( \Theta_1 = \{(r,p): 0 < r < r_1(p), 2 < p \leq 20 \} \).

Its right boundary curve, which lies in \( \Theta_1 \), is the set of points with ordinates \( p \), for \( p \in [2,20] \), whose abscissas, denoted by \( r_1(p) \), are, for each \( p \), the first values for which the iterates of the map \( f_{r,p} \) are attracted to an unique positive fixed point. This function \( r_1(p) \), for \( p \in [2,20] \), defines a stable or attraction line. Note that, the curve \( r_1(p) \) belong to the next region \( \Theta_2 \).

Globally, the iterates of any map \( f_{r,p} \), whose parameters values belong to this region \( \Theta_1 \), are always attracted to the attractive fixed point \( x^* = 0 \). So, this is a region of extinction since a map \( f_{r,p} \), with \( (r,p) \in \Theta_1 \) can model only species that will become extinct, as soon as they appear they are doomed to disappear. The growth rate it is not big enough to stabilize the population size. The unimodal maps \( f_{r,p} \) of the region \( \Theta_1 \) do not have a chaotic behaviour, its topological entropy is null, [3]. The symbolic sequences associated to the orbits of the critical point \( c = \frac{p-1}{p} \) are of the type \( CL^\infty \).

**Stability or equilibrium region \( \Theta_2 = \Theta_{2a} \cup \Theta_{2b} \)**

The stability or equilibrium region is \( \Theta_2 = \{(r,p): r_1(p) \leq r < r_2(p), 1 < p \leq 20 \} \). In a generic way, we can say that the iterates of any map \( f_{r,p} \), whose parameters values belong to the region \( \Theta_2 \), converge to the larger positive attractive fixed point (it is unique if \( p \in [1,2] \)). So, this is a region of stability or equilibrium, since one map \( f_{r,p} \), with \( (r,p) \in \Theta_2 \), can model populational evolutions of species whose size is approximately constant in time.

In fact, for each value of the parameter \( p \), is observed a drastic change when \( r \in [r_1(p),r_2(p)] \), resulting from the possibility of reaching the equilibrium between the two competitive forces, reproduction by one side and resource limitation by another. The unimodal maps \( f_{r,p} \) of the region \( \Theta_2 \), also do...
not have a chaotic behaviour, its topological entropy being null, [3].

**Remark 1.** This region contains a super stable or super attractive curve, denoted by \( r^*(p) \), whose expression is given by

\[
r^*(p) = p \left( \frac{p}{p - 1} \right)^{p-2}.
\]

Therefore, this curve divides the region \( \Theta_2 \) in two sub-regions, denoted by \( \Theta_{2a} \) and \( \Theta_{2b} \), which are delimited by the curves \( r_1(p) \), \( r^*(p) \) and \( r_2(p) \), respectively.

In this region, the symbolic sequences associated to the critical point orbits are of the type \( C\mathcal{L}^\infty \) in the region \( \Theta_{2a} \) and in the region \( \Theta_{2b} \), are of the type \( C\mathcal{R}^\infty \).

The second boundary curve \( r_2(p) \) has points with ordinates \( p \) whose abscissas correspond, for each \( p \), to the first value of the parameter \( r \) for which we can observe an orbit of period 2. Thus \( r_2(p) \) is the curve where period doubling starts. Note that, this curve belong to the next region \( \Theta_3 \).

**Period doubling region \( \Theta_3 \)**

The third region, denominated period doubling region, is \( \Theta_3 = \{(r, p) : r_2(p) \leq r < r_3(p), 1 < p \leq 20\} \).

In other words, the region \( \Theta_3 \) shows population dynamics patterns describing the generations of species oscillating in cycles of period \( 2^n \), with \( n \in \mathbb{N} \). The unimodal maps \( f_{r,p} \) of the region \( \Theta_3 \) still do not exhibit chaotic behaviour, its topological entropy being null, [6].

Its right boundary curve \( r_3(p) \) is the set of points whose ordinates \( p \) correspond to abscissas \( r \) where the map \( f_{r,p} \) has no longer orbits of period \( 2^n \) — i.e., which do not correspond to Feigenbaum points — orbits of other periods starting at those values. Thus it is the chaos starting line, and it belongs to the next region \( \Theta_4 \).

**Chaotic region \( \Theta_4 \)**

The fourth region, denominated chaotic region and denoted by \( \Theta_4 \), is defined for \( 1 < p \leq 20 \) and \( r_3(p) \leq r < r_4(p) \).

So, the iterates of the maps \( f_{r,p} \) whose parameter values belong to the region \( \Theta_4 \) or origin orbits of the several types, which already present chaotic patterns of behaviour; so its topological entropy is positive. The value of the topological entropy increases with the value of the parameter \( r \), until it attains its maximum value \( \ln 2 \), [5].

The fourth boundary curve, \( r_4(p) \), has the points with ordinates \( p \) whose abscissas correspond, for each \( p \), to the first value of the parameter \( r \) exhibiting the natural Allee effect, [2]. The curve \( r_4(p) \) is thus named line of the Allee effect, and belongs to the region \( \Theta_5 \).

**Allee effect caused extinction region \( \Theta_5 \)**

The fifth boundary curve \( r_5(p) \) has the points with ordinates \( p \), with \( p \in ]1, 20[ \), whose abscissas are the corresponding values of the parameter \( r \) where full shift does occur. This full shift line belongs to the closed region \( \Theta_5 = \{(r, p) : r_4(p) \leq r \leq r_5(p), 1 < p \leq 20\} \).

In a generic way, we can say that the iterates of the maps \( f_{r,p} \), whose parameters values belong to this region \( \Theta_5 \), are always attracted to the attractive fixed point \( x^* = 0 \).

Therefore, the maps \( f_{r,p} \), with \( (r, p) \in \Theta_5 \), can model populational evolutions of species that once developed disorderly and now go to extinction, because there are already little individuals and are spatially very far away from each other, the reproduction becomes impossible, what leads to the extinction of these species. The unimodal maps \( f_{r,p} \) of the region \( \Theta_5 \) present chaotic behaviours, with maxim topological entropy \( \ln 2 \), [6].

**Differed extinction region \( \Theta_6 \)**

\( \Theta_6 = \{(r, p) : r_5(p) < r \leq 53, 1 < p \leq 20\} \) is a differed extinction region. The graphic of any map \( f_{r,p} \), with \( (r, p) \in \Theta_6 \), is no longer totally included in the invariant interval \( [0, 1] \times [0, 1] \). The dynamic completely looses its deterministic component, and the size of the population in successive generations behaves as a random numbers generator.
device, until ultimate extinction does occur. At this stage, Cantor sets become observable.

The results described above, in order to characterize the topological complexity of the dynamical systems in to each considered region, measured in terms of topological entropy, are stated in the following theorem.

**Theorem 1.** The topological entropy of the family of unimodal maps \( f_{r,p}(x) = rx^{p-1}(1-x) \), with \((r,p) \in \Theta\) is characterized by:

1. in the regions \( \Theta_1, \Theta_2 \) and \( \Theta_3 \), the topological entropy is null;
2. in the region \( \Theta_4 \), the sets where the topological entropy is constant are connected and indexed in a strictly monotonous and continuous way, by this topological invariant, except the null measure sets;
3. in the region \( \Theta_5 \) the topological entropy is constant and equal to its maximum value, \( \ln 2 \).

**Proof:** These claims follow easily from the fact that \( f_{r,p} : [0,1] \rightarrow [0,1] \) is a family of unimodal maps, having in mind the results in [3], [5] and [6], respectively.

Values of the parameters larger than the ones considered above (i.e., \( p > 20 \)) do not have a realistic meaning, since they are not adequate to model satisfactorily any known population. When the values of the parameter \( r \) become very high, we observe that:

- the range of the sudden extinction region increases considerably for large values of \( r \);
- the ranges of the equilibrium region, of the period doubling region and of the chaotic region decrease to 0 (i.e., these regions tend to disappear) when \( r \) increases;
- the range of the region where the Allee effect exists increases slowly with \( r \);
- the differed extinction region range decreases as a function of \( p \).

Thus, for very high values of the parameter \( p \), the populational growth pattern is extinction.

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**References**


