Representation functions and the Neggers–Stanley condition for weight-shaped-posets

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\textbf{A R T I C L E  I N F O}

\textbf{Article history:}
Received 28 May 2007
Received in revised form 28 August 2008
Accepted 10 September 2008

\textbf{Keywords:}
(Hasse-) ws-poset
Hasse function
Representation function

\textbf{A B S T R A C T}

We introduce the class of weight-shaped-posets and we define the representation functions of such ws-posets as generalizations of the representation polynomials of finite posets. We define the Brylawski decomposition of ws-posets and use it to construct the algorithm for the computation of the representation functions. The class of ws-posets contains a class of fuzzy posets for example. The Neggers–Stanley conjecture for ws-posets is also discussed.

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Let $P$ be a poset (partially ordered set), i.e., a set equipped with a relation $<$ where $x < y$ implies $y \not< x$ and $x < y, y < z$ implies $x < z$. The relation $\leq$ as usual means $x = y$ or $x < y$. For details on the theory of posets we refer the reader to [6,7]. In these texts further references are supplied as well. Let $w : P \to \mathbb{R}$ be a weight function on the elements of $P$, where $\mathbb{R}$ is the set of all real numbers. Let $\alpha : E \to \mathbb{R}$ be a weight (length) function on the “edges” $(x, y)$ where $x < y$, i.e., on the (strict) order relation denoted by $<$. Furthermore, let $\gamma : G \to \mathbb{R}$ be a weight (gap) function on the “gaps” $\{x, y\}$, where $x$ and $y$ are not comparable (free, parallel) denoted by $x \circ y$ ($x \parallel y$). For example, the antichain $3$ has Hasse diagram:

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {$\bullet$};
    \node (2) at (1,0) {$\bullet$};
    \node (3) at (2,0) {$\bullet$};
    \draw (1) -- (2) node at (1.5,0) {$\gamma(1, 2)$};
    \draw (2) -- (3) node at (2.5,0) {$\gamma(2, 3)$};
    \draw (3) -- (1) node at (-0.5,0) {$\gamma(1, 3)$};
    \node at (0,-.5) {$w(1)$};
    \node at (1,-.5) {$w(2)$};
    \node at (2,-.5) {$w(3)$};
    \node at (0,-1) {$1$};
    \node at (1,-1) {$2$};
    \node at (2,-1) {$3$};
\end{tikzpicture}
\end{center}

while the chain $C_3$ has Hasse diagram:

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {$\bullet$};
    \node (2) at (1,0) {$\bullet$};
    \node (3) at (2,0) {$\bullet$};
    \draw (1) -- (2) node at (1.5,0) {$\alpha(1, 3)$};
    \draw (2) -- (3) node at (2.5,0) {$\alpha(2, 3)$};
    \draw (3) -- (1) node at (-0.5,0) {$\alpha(1, 2)$};
    \node at (0,-.5) {$w(1)$};
    \node at (1,-.5) {$w(2)$};
    \node at (2,-.5) {$w(3)$};
    \node at (0,-1) {$1$};
    \node at (1,-1) {$2$};
    \node at (2,-1) {$3$};
\end{tikzpicture}
\end{center}

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doi:10.1016/j.camwa.2008.09.037
when more generally we have a situation such as

\[
\begin{array}{c}
\alpha(2, 3) \\
\gamma(2, 3) \\
\alpha(3, 4) \\
\alpha(2, 4) \\
\alpha(3) \\
\alpha(1, 4) \\
\alpha(1, 3) \\
\alpha(1, 2) \\
\end{array}
\]

involving all three functions. If the underlying poset is locally finite, then we shall consider the function \( \alpha : E \to \mathbb{R} \) to be a Hasse function if \( \alpha(x, y) = 0 \) whenever there is a \( z \) such that \( x < z < y \) (i.e., \( x < y \) but \( y \) does not cover \( x \)).

A poset \( (P; w, \alpha, \gamma) \) is a weight-shaped-poset (ws-poset). If \( \alpha \) is a Hasse function, then \( (P; w, \alpha, \gamma) \) is a Hasse-ws-poset.

If \( G = \text{Dom} \gamma = \emptyset \), then the ws-poset \( P = (P; w, \alpha, \gamma) = (P; w, \alpha) \) is a ws-chain. Let \( \mu \) denote a mean process with \( \mu(1, \ldots, 1) = 1, \mu(0, \ldots, 0) = 0 \) and if \( (x_1, \ldots, x_n) \geq (y_1, \ldots, y_n) \), then \( \mu(x_1, \ldots, x_n) \geq \mu(y_1, \ldots, y_n) \) where \( n = 1, 2, \ldots \). Then, we define as follows:

\[
\alpha^*_\mu(x, y) = \begin{cases} 
\mu((\alpha(u, v)|u \leq x, y \leq v)) & \text{if } y \text{ covers } x; \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that \( \alpha^*_\mu : E \to \mathbb{R} \) is a Hasse function and \( (P; w, \alpha^*_\mu) = P^*_\mu \) is a Hasse-ws-poset, which is a ws-chain, i.e., it is a Hasse-ws-chain. For example, if \( \mu(a_1, \ldots, a_n) = \frac{a_1 + \cdots + a_n}{n} \), the ws-chain

\[
\begin{array}{c}
\alpha(3, 4) \\
\alpha(2, 3) \\
\alpha(1, 4) \\
\alpha(1, 3) \\
\alpha(1, 2) \\
\end{array}
\]

generates the Hasse-ws-chain

\[
\begin{array}{c}
\alpha(2, 4) \\
\alpha(2, 3) \\
\alpha(1, 4) \\
\alpha(1, 3) \\
\alpha(1, 2) \\
\end{array}
\]

\[
\begin{array}{c}
\frac{\alpha(3, 4) + \alpha(2, 4) + \alpha(1, 4)}{3} \\
\frac{\alpha(2, 3) + \alpha(1, 3) + \alpha(2, 4) + \alpha(1, 4)}{4} \\
\frac{\alpha(1, 2) + \alpha(1, 3) + \alpha(1, 4)}{3}
\end{array}
\]

Thus if \( \alpha^*_\mu = \chi_E, \alpha(x, y) = 1 \) if \( x < y \), then \( \alpha^*_\mu = \chi_{E^*} \), where \( E^* = \{(x, y)|y \text{ covers } x\} \), regardless of the choice of mean process \( \mu \).
If \( P = (P; w, \alpha) \) is a Hasse-\( ws \)-chain, then
\[
F_P(z) = z^{\sum w(u) \left[ 1 + z \right]^{\sum \alpha(u,v)}}
\]
provided \( \sum w(u) \) and \( \sum \alpha(u,v) \) are both finite, is the representation function of \( P \). Thus, e.g., if \( w(u) \equiv 1, \alpha(u,v) \equiv 1 \) when \( v \) covers \( u, \) then \( \sum \alpha(u,v) \) is the height of \( P, \) and \( \sum w(u) = |P| \) is the order of \( P \). Accordingly,
\[
F_P(z) = z^{|P| \left[ 1 + z \right]^{h(P)}} = z \left[ 1 + z \right]^{p-1},
\]
where \( h(P) = |P| - 1 \), is the previously defined representation polynomial.

If \( P = (P; w, \alpha) \) is a \( ws \)-chain, then \( (P; w, \alpha^*) = P^*_\mu \) is a Hasse-\( ws \)-chain and we set
\[
F_P(z) = F_{P^*_\mu}(z) = z^{\sum w(u) \left[ 1 + z \right]^{\sum \alpha^*_\mu(u,v)}}
\]
i.e., \( \sum \alpha^*_\mu(u,v) \) is taken over \( E^* \), since \( \alpha^*_\mu(u,v) = 0 \) when \( (u,v) \in E - E^* \).

The number \( |G|, \) the cardinal number of \( G \) measures the number of free pairs, a number “correlated” to the width, even though it is not equivalent to it. If \( \text{width}(P) = m \), then we know that \( |G| \geq \binom{m}{2} \). Also, if \( |P| = n \), then \( \binom{n}{2} \geq |G| \).

Now, consider a \( ws \)-poset \( (P; w, \alpha, \gamma) \) with \( |G| < \infty \). Select an element \( \{x, y\} \) of \( G \), with \( \gamma \rightarrow \gamma(x, y) \). Let \( P_1 = (P; w, \alpha, \gamma_1) \) be obtained by introducing \( (x, y) \in E_1 \), i.e., by letting \( x < y, \) Notice that since \( x < z < y \) is in any case impossible with \( \{x, y\} \in G \), it follows in fact that \( (x, y) \in E_1^+ \). Introducing this relation and taking the transitive closure produces \( E_1 \) by removing elements from \( G \). Thus \( G_1 = G - A \) and \( E_1 = E \cup A \) for \( A \subseteq G \). Hence, define \( \gamma_1 = \gamma | G_1 \) and \( \alpha_1 | A = \gamma | A \).

Similarly, \( P_2 = (P; w, \alpha_2, \gamma_2) \) is obtained by taking \( (y, x) \in E_2^+ \subseteq E_2 \), i.e., by setting \( y < x \).

Finally, \( P_3 = (Q; \hat{w}, \alpha, \hat{\gamma}) \) is obtained by setting \( x = y, \hat{w}(x = y) = \mu(w(x, w(y), \gamma(x, y)), \tilde{\alpha}(u, x = y) = \mu(\alpha(u, x), \alpha(u, y)), \hat{\alpha}(x = y, v) = \mu(\alpha(x, v), \alpha(y, v)), \hat{\alpha}(u, v) = \alpha(u, v) \) otherwise, \( \hat{\gamma}(\{u, x = y\}) = \mu(\gamma(u, x), \gamma(u, y)), \hat{\gamma}(\{u, y\}) = \gamma(\{u, y\}) \) otherwise.

If \( F_{P_1}(z), F_{P_2}(z), F_{P_3}(z) \) have been computed, by induction on \( |G| \), then we set:
\[
Br_{\mu}(P) = (P_1, P_2, -P_3) \quad \text{and} \quad F_P(z) = F_{P_1}(z) + F_{P_2}(z) - F_{P_3}(z).
\]

This is then the Brylawski-decomposition-step in the algorithm for the computation of the representation polynomial constructed for \( ws \)-posets with associated mean process \( \mu \). For further information on the Brylawski-decomposition, see [5,8].

If we denote by \( Br_{\mu}^{(2)}(P) \), the process
\[
Br_{\mu}(Br_{\mu}(P_1), Br_{\mu}(P_2), Br_{\mu}(-P_3)) = (P_{11}, P_{12} - P_{13}, P_{21}, P_{22} - P_{23}, -P_{31}, -P_{32} + P_{33}),
\]
and by \( Br_{\mu}^{(n)}(P) \) the process taken down to \( |G| = 0 \) when one obtains \( ws \)-chains, the resulting decomposition is the Brylawski-decomposition of the \( ws \)-poset \( P \) into \( ws \)-chains, each of which may then be turned into Hasse-\( ws \)-chains for which we may write out the representation functions. We thus can make sense out of weighted ordinal sums \( P = X \oplus (q \gamma) \) obtained by introducing arrows \( \alpha(x, y) = q \) between maximal elements of \( X \) and minimal elements of \( Y \), with \( \alpha(x, y) = 0 \) otherwise and no new free pairs. The result in the Brylawski-decompositions is that all the action is either in \( X \) or in \( Y \), with the \( ws \)-chains in \( Br_{\mu}^{(n)}(X) \) and \( Br_{\mu}^{(n)}(Y) \) freely coupled, but with an added factor \( \left( \frac{1+q}{z} \right)^q \) always added to the chain so that we obtain:
\[
F_P(z) = F_{X \oplus (q \gamma)}(z) = \left[ 1 + \frac{z}{z} \right]^q F_X(z) F_Y(z)
\]
as expected.

Similarly, \( X + (q \gamma) \) is obtained by introducing gaps \( \gamma(\{x, y\}) = q \) between elements of \( X \) and those of \( Y \). Thus, e.g.,

```
\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (0.5,1) {3};
  \node (q) at (-0.5,0.5) {q};
  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (2) -- (3);
\end{tikzpicture}
\end{center}
```

\begin{center}
\begin{itemize}
  \item \text{can be thought of as a} q-antichain \((1 + (q)1) + (q)1\), with \( w(1) = w(2) = w(3) = 1 \) for example.
\end{itemize}
\end{center}

There are interesting classes of \( ws \)-posets which can now be obtained easily. For example, suppose \( \text{Im } w, \text{Im } \alpha, \text{Im } \gamma \subseteq [0, 1] \). Then we may consider \( P = (P; w, \alpha, \gamma) \) to be a species of fuzzy poset (\( ws-fuzzy \) posets).

Notice that for any mean process if \((0, \ldots, 0) \leq (a_1, \ldots, a_n) \leq (1, \ldots, 1) \) then also \( 0 \leq \mu(a_1, \ldots, a_n) \leq 1 \), and thus if \( Y \) is a \( ws \)-fuzzy poset then \( Br_{\mu}(P) \) is also a \( ws \)-fuzzy poset, whence if \( |G| < \infty \), then \( Br_{\mu}^{(n)}(P) \) consists of signed fuzzy-\( ws \)-chains which have the same representation functions as signed fuzzy-Hasse-\( ws \)-chains. Obviously, if \( X \) and \( Y \) are \( ws \)-fuzzy-\( ws \)-posets and if \( 0 \leq q \leq 1 \), so are \( X \oplus (q \gamma) \) and \( X + (q \gamma) \).

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If $X$ is a finite poset taking $w \equiv 1$, $\alpha \equiv 1$, $\gamma \equiv 1$ results in a representation polynomial $P_X(z)$ (see [2–4]). The original Neggers–Stanley Conjecture supposed that this polynomial had only real roots (between $-1$ and $0$) (see [5]). This conjecture in its full generality has been disproven (see [1,9]) although very large classes of posets satisfy the condition (see [10]). The problem now is to determine which posets satisfy the Neggers–Stanley condition, i.e., for which posets does $P_X(z)$ have only real roots. Obviously the previous results on the conjecture in its various forms become results contributing to this problem as well.

Returning to the situation we are describing here, we make several observations. Even for Hasse–ws-chains, if \( \sum w(u) < \sum \alpha(u, v) \), then we obtain a singularity at $z = 0$, while if \( \sum w(u) > \sum \alpha(u, v) \), $F_P(0) = 0$. If \( \sum w(u) = \sum \alpha(u, v) \), then $F_P(z) = (1 + z)^{\sum \alpha(u, v)}$ so that $P$ has a zero at $z = -1$ and nowhere else.

Since $F_P(z)$ can be defined in terms of $Br_\mu^\infty(P)$ as a sum:

$$F_P(z) = \sum_{i=1}^{N} \epsilon_i z^{a_i} \left[ \frac{1 + z}{z} \right]^{b_i}, \quad \epsilon_i = \pm 1,$$

if $|G| < \infty$, we may consider the following question of interest.

**Question:** Given an expression:

$$F(z) = \sum_{i=1}^{N} \epsilon_i z^{a_i} \left[ \frac{1 + z}{z} \right]^{b_i}$$

what conditions on $e = (\epsilon_1, \ldots, \epsilon_N)$, $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$ guarantee that $F(z)$ has real roots only?

By liberating the problem from the representation polynomial question, it may provide additional insights into the general question and potential answers.

**Acknowledgement**

The authors are grateful to the referee for several valuable suggestions and help.

**References**


