The Complementary Error Matrix Function and Its Role Solving Coupled Diffusion Mathematical Models

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Abstract—In this paper, the error function and the complementary error function of a matrix are introduced. Basic properties of these matrix functions are studied and applied to the inverse Laplace transform of a matrix function and to solve coupled diffusion models in a semi-infinite medium.

Keywords—Coupled diffusion model, Error function.

1. INTRODUCTION

It is well known that error function and the complementary error function both play an important role in the construction of solutions of partial differential systems using integral transforms, [1]. Coupled partial differential systems are frequent modelling several different problems [2–6] and dealing with its matrix formulation some matrix operation calculus has been developed, [7,8]. Using Laplace transform, the solution of the scalar heat conduction equation in a semi-infinite medium

\[ u_t = k^2 u_{xx}, \quad x > 0, \quad t > 0, \]

with the initial and boundary conditions,

\[ u(x, 0) = 0, \quad x > 0, \]
\[ u(0, t) = T_0, \quad t > 0, \]
\[ u(x, t) \to 0, \quad \text{as} \quad x \to \infty, \quad t > 0, \]

where the thermal conductivity \( k^2 \) and \( T_0 \) are constants, is given by

\[ u(x, t) = T_0 \text{erfc} \left( \frac{x}{2k\sqrt{t}} \right), \quad (1.1) \]
see [1, p. 148]. In this paper, we consider the coupled diffusion problem

\begin{align}
  u_t &= A^2 u_{xx}, \quad x > 0, \quad t > 0, \\
  u(x, 0) &= 0, \quad x > 0, \\
  u(0, t) &= u_0, \quad t > 0, \\
  u(x, t) &\to 0, \quad \text{as } x \to \infty, \quad t > 0,
\end{align}

where $u_0$ and $u(x, t)$ are vectors in $\mathbb{C}^p$ and $A$ is a matrix in $\mathbb{C}^{p \times p}$ with properties to be determined.

The aim of this paper is to obtain an expression of the solution of problem (1.2)-(1.4) of form (1.1) and it is organized as follows. Section 2 deals with the definition of the complementary error function and the error function of a matrix as well as with the proof of some basic properties. Section 3 is concerned with some matrix operational results as well as with the study of the behavior of some matrix integrals on complex paths. In Section 4, we apply the results of Section 3 to the evaluation of the inverse Laplace transform of the matrix function $e^{-Az^2/v}$ and its relationship with the complementary error function of certain matrix. Finally, in Section 5 an expression for the solution of problem (1.2)-(1.5) of form (1.1) is obtained.

If $B$ is a matrix in $\mathbb{C}^{p \times p}$ its two-norm, denoted by $\|B\|$ is defined as

$$
\|B\| = \sup_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_2},
$$

where for a vector $x$ in $\mathbb{C}^p$, $\|x\|_2 = (x^H x)^{1/2}$ is the usual Euclidean norm of the vector $x$, see [9, p. 56].

If $A$ is a matrix in $\mathbb{C}^{p \times p}$, then

$$
\|e^{tA}\| \leq e^{\mu(A)}, \quad t \geq 0,
$$

where $\mu(A)$ is the logarithmic norm of $A$, defined by

$$
\mu(A) = \max \left\{ \lambda, \lambda \in \sigma \left( \frac{A + A^H}{2} \right) \right\},
$$

see [10, p. 110; 11]. If $A \in \mathbb{C}^{p \times p}$, $\text{Im}(A) = (A - A^H)/2i$ and $\text{Re}(A) = (A + A^H)/2$ denote the imaginary part of $A$ and the real part of $A$, respectively. We denote by $R_1(A)$, $I_1(A)$, and $I_2(A)$ the real numbers defined by

$$
R_1(A) = \min \{ \text{Re}(\lambda); \lambda \in \sigma(A) \}, \quad I_1(A) = \min \{ \text{Im}(\lambda); \lambda \in \sigma(A) \},
$$

$$
I_2(A) = \max \{ \text{Im}(\lambda); \lambda \in \sigma(A) \},
$$

then by Bendixon's theorem [12, p. 359], it follows that

$$
I_1(A) \leq \min \{ w; w \in \sigma(\text{Im}(A)) \}; \quad \max \{ w; w \in \sigma(\text{Im}(A)) \} \leq I_2(A).
$$

Finally, we recall that by Theorem 2.3 of [7], we have that if $A$ is a matrix in $\mathbb{C}^{p \times p}$ such that

$$
|\text{Re}(z)| > |\text{Im}(z)|, \quad \forall z \in \sigma(A),
$$

then

$$
\left( \int_0^\infty e^{-A^2 v^2} dv \right)^2 = \frac{\pi}{4} A^{-2}. \tag{1.12}
$$

If apart from (1.11), we assume that $\text{Re}(z) > 0$ for all $z \in \sigma(A)$, i.e.,

$$
\text{Re}(z) > |\text{Im}(z)|, \quad \forall z \in \sigma(A), \tag{1.13}
$$
then
\[ \int_{0}^{\infty} e^{-A^2u^2} \, du = \frac{\sqrt{\pi}}{2} A^{-1}. \] (1.14)

If \( \gamma : [a, b] \to \mathbb{C} \) is a path in the complex plane, we denote by \( \gamma^0 : [a, b] \to \mathbb{C} \) the path defined by \( \gamma^0(t) = \gamma(a + b - t) \), so that \( \gamma^0(a) = \gamma(b) \) and \( \gamma^0(b) = \gamma(a) \).

### 2. THE COMPLEMENTARY ERROR MATRIX FUNCTION

If \( x \) is a real number, then the error function \( \text{erf}(x) \) and the complementary error function \( \text{erfc}(x) \) are defined by
\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} \, du, \quad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} \, du, \] (2.1)
and satisfy
\[
\begin{align*}
\text{erf}(x) + \text{erfc}(x) &= 1, \\
\text{erf}(0) &= 0, \\
\text{erf}(-x) &= -\text{erf}(x), \\
\text{erfc}(-x) &= 1 - \text{erf}(-x).
\end{align*}
\] (2.2) (2.3) (2.4) (2.5)
see [1, p. 56]. These definitions and properties are extended to the case where \( x \) is a complex number by regarding integral on path in the complex plane, see [13, p. 16], however, it is unclear how to extend the above definition to the matrix case, i.e., when \( x \) is a matrix in \( \mathbb{C}^{p \times p} \). In order to motivate our definition, note that making the substitution \( u = vx \) in both integrals of (2.1) one gets
\[
\begin{align*}
\text{erf}(x) &= \frac{2x}{\sqrt{\pi}} \int_{0}^{1} e^{-(xv)^2} \, dv, \\
\text{erfc}(x) &= \frac{2x}{\sqrt{\pi}} \int_{1}^{\infty} e^{-(xv)^2} \, dv,
\end{align*}
\] (2.6) and this motivates the following definition.

**Definition 2.1.** Let \( A \) be a matrix in \( \mathbb{C}^{p \times p} \) such that
\[ |\text{Re}(z)| > |\text{Im}(z)|, \quad z \in \sigma(A). \] (2.7)
Then, we define the complementary error function of \( A \) by the expression
\[ \text{erfc}(A) = \frac{2A}{\sqrt{\pi}} \int_{1}^{\infty} e^{-(Av)^2} \, dv. \] (2.8)
We prove that (2.8) is well defined showing that integral (2.8) is absolutely convergent. Note that
\[ \sigma(-A^2) = \{ -(a_1 + ia_2)^2 = a_1^2 - a_2^2 - 2ia_1a_2; \, a = a_1 + ia_2 \in \sigma(A) \}, \]
and by hypothesis (2.7), it follows that
\[ \max\{\text{Re}(z); \, z \in \sigma(-A^2)\} = -\min\{a_1^2 - a_2^2; \, a = a_1 + ia_2 \in \sigma(A)\} = -R_1(A^2), \] (2.9)
where
\[ \min\{\text{Re}(z); \, z \in \sigma(A^2)\} = R_1(A^2) > 0. \] (2.10)
By (2.9), (2.10), and (1.6), it follows that
\[ \left\| e^{(-A^2)v^2} \right\| \leq e^{-R_1(A^2)v^2}; \quad \int_{1}^{\infty} e^{-(Av)^2} \, dv < +\infty, \] and thus, (2.8) is well defined.

**Remark 2.1.** Note that Hermitian invertible matrices satisfy condition (2.7) as well as any matrix similar to an Hermitian invertible matrix.
DEFINITION 2.2. If $A \in \mathbb{C}^{p \times p}$, we define the error function of matrix $A$ by the expression
\[
\text{erf}(A) = \frac{2A}{\sqrt{\pi}} \int_0^1 e^{-(Av)^2} \, dv.
\] (2.11)

THEOREM 2.1. Let $A$ be a matrix satisfying (2.7). Then,

(i) $\text{erf}(-A) = -\text{erf}(A)$.

(ii) $\text{erf}(A) + \text{erfc}(A) = I$, if $\text{Re}(z) > |\text{Im}(z)|$, $\forall z \in \sigma(A)$.

(iii) $\text{erfc}(-A) = I - \text{erf}(-A)$, if $(-\text{Re}(z)) > |\text{Im}(z)|$, $\forall z \in \sigma(A)$.

(iv) $\|\text{erfc}(A)\| \leq \|A\|/\sqrt{R_1(A^2)} \text{erfc}(\sqrt{R_1(A^2)})$, if $\text{Re}(z) > |\text{Im}(z)|$, $\forall z \in \sigma(A)$.

PROOF. Part (i) is a direct consequence of Definition 2.2. Let us consider (ii). By Theroem 2.3 of [8], see (1.14), we have that
\[
\int_0^\infty e^{-(Av)^2} \, dv = \frac{\sqrt{\pi}}{2} A^{-1}.
\] (2.12)

By (2.8), (2.11), and (2.12), it follows that
\[
\text{erfc}(A) + \text{erf}(A) = \frac{2A}{\sqrt{\pi}} \int_0^\infty e^{-(Av)^2} \, dv = I.
\]

This proves (ii). Part (iii) follows by applying Part (ii) to the matrix $-A$ that also satisfies (2.7).

Let us consider (iv). By (2.8),(2.10), it follows that
\[
\|\text{erfc}(A)\| \leq \frac{2\|A\|}{\sqrt{\pi}} \int_1^\infty \|e^{-v^2A^2}\| \, dv \leq \frac{2\|A\|}{\sqrt{\pi}} \int_1^\infty e^{-R_1(A^2)v^2} \, dv
\]
\[
= - \frac{\|A\|}{\sqrt{R_1(A^2)}} \frac{2}{\sqrt{\pi}} \int_1^\infty e^{-v^2} \, dv = - \frac{\|A\|}{\sqrt{R_1(A^2)}} \text{erfc} \left( \sqrt{R_1(A^2)} \right).
\]

Let us consider the series expansion of the matrix exponential arising in (2.10), then
\[
\text{erf}(A) = \frac{2A}{\sqrt{\pi}} \int_0^1 \left( \sum_{n \geq 0} \frac{(-1)^n A^{2n}}{n!} v^{2n} \right) \, dv = \frac{2}{\pi} \sum_{n \geq 0} \frac{(-1)^n A^{2n+1}}{n!(2n+1)}.
\] (2.13)

Hence, if $A$ satisfies (2.7), by Property (ii) of Theorem 2.1 and (2.13), one gets the series expansion of $\text{erfc}(A)$ given by
\[
\text{erfc}(A) = I - \text{erf}(A) = I - \frac{2}{\sqrt{\pi}} \sum_{n \geq 0} \frac{(-1)^n A^{2n+1}}{n!(2n+1)}.
\] (2.14)

Formula (2.14) suggests an extension of the definition of the complementary error matrix function. In fact, consider the entire function in the complex plane defined by
\[
f(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.
\] (2.15)

It is easy to show that power series $f(z)$ is absolutely convergent for all $z$ in the complex plane. Then, by the holomorphic matrix functional calculus, for any matrix $A \in \mathbb{C}^{p \times p}$, it is well defined
\[
f(A) = I - \frac{2}{\sqrt{\pi}} \sum_{n \geq 0} \frac{(-1)^n A^{2n+1}}{n!(2n+1)},
\] (2.16)

and note that if $A$ satisfies (2.7), then (2.16) coincides with (2.8). Thus, we can extend Definition 2.1 in the following sense.
DEFINITION 2.3. Let $A$ be a matrix in $\mathbb{C}^{p \times p}$, then the complementary error function of $A$ denoted by $\text{erfc}(A)$ is defined by

$$
erfc(A) = I - \frac{2}{\sqrt{\pi}} \sum_{n \geq 0} \frac{(-1)^n A^{2n+1}}{n!(2n + 1)}.
$$

If $A$ satisfies (2.7), then (2.17) coincides with the integral expression (2.3).

REMARK 2.2. It is clear that the integral expression (2.8) is divergent if $A$ does not satisfy (2.7). For instance, take $A = iI$, where $i$ is the imaginary unit, $i^2 = -1$. Then,

$$
\int_1^\infty e^{-(Ax)^2} \, dv = I \int_1^\infty e^{v^2} \, dv
$$

that is clearly divergent.

3. MATRIX OPERATIONAL CALCULUS

In this section, we are concerned with the evaluation of some improper matrix integrals as well as in the study of the behavior of certain matrix integrals on paths in the complex plane.

LEMMA 3.1. If $A$ satisfies (2.7), then

$$
\frac{1}{2} \int_0^\infty e^{-x^2} \frac{\sin(2Ax)}{x} \, dx = \frac{\pi}{4} \text{erf} \left( \frac{A}{\sqrt{t}} \right),
$$

$$
\int_0^\infty e^{-x^2t} \cos(2Ax) \, dx = \frac{\sqrt{\pi}}{2\sqrt{t}} \exp \left( -\frac{A^2}{t} \right).
$$

PROOF. Note that if $a \in \Omega$, $a = a_1 + ia_2$, with $|a_1| > |a_2|$, then

$$
\sin(2ax) = \sin(2x(a_1 + ia_2)) = \sin(2a_1 x) \cosh(2a_2 x) - i \cos(2a_1 x) \sinh(2a_2 x),
$$

$$
\cos(2ax) = \cos(2x(a_1 + ia_2)) = \cos(2a_1 x) \cosh(2a_2 x) + i \sin(2a_1 x) \sinh(2a_2 x).
$$

Thus,

$$
|\sin(2ax)| \text{ behaves like } e^{2|x|a_2|}, \quad \text{as } x \to \infty,
$$

$$
|\cos(2ax)| \text{ behaves like } e^{2|x|a_2|}, \quad \text{as } x \to \infty.
$$

In particular, for $t > 0$ fixed, the functions $v: \Omega \to \mathbb{C}$, $w: \Omega \to \mathbb{C}$, defined by

$$
v(a) = \frac{1}{2} \int_0^\infty e^{-x^2} \frac{\sin(2ax)}{x} \, dx,
$$

$$
w(a) = \int_0^\infty e^{-x^2t} \cos(2ax) \, dx,
$$

are holomorphic functions in $\Omega$, see [14, p. 107;15, pp. 232–233], because

$$
\int_0^\infty e^{-x^2t} 2x \sin(2ax) \, dx, \quad \int_0^\infty e^{-x^2t} \cos(2ax) \, dx,
$$

are uniformly convergent in a neighborhood of any $a \in \Omega$. By [1, p. 106], we know that

$$
\frac{1}{2} \int_0^\infty e^{-x^2} \frac{\sin(2ax)}{x} \, dx = \frac{\pi}{4} \text{erf} \left( \frac{a}{\sqrt{t}} \right), \quad a \in \mathbb{R},
$$

$$
\int_0^\infty e^{-x^2t} \cos(2ax) \, dx = \frac{\sqrt{\pi}}{2\sqrt{t}} \exp \left( -\frac{a^2}{t} \right), \quad t > 0, \quad a \in \mathbb{R}.
$$
By (3.3), (3.4) and since \( v, w, \) and erf are analytic functions in \( \Omega \), identities (3.5), (3.6) hold true for all \( a \in \Omega \) (by working separately in the convex parts of \( \Omega \) and by the identity theorem for holomorphic functions). Hence,

\[
v(a) = \frac{1}{2} \int_0^\infty e^{-ax^2} \sin (2ax) \frac{dx}{x} = \frac{\pi}{4} \text{erf} \left( \frac{a}{\sqrt{t}} \right), \quad a \in \Omega, \quad (3.7)
\]
and

\[
w(a) = \int_0^\infty e^{-ax^2} \cos (2ax) \frac{dx}{x} = \frac{\sqrt{\pi}}{2\sqrt{t}} \exp \left( -\frac{A^2}{t} \right), \quad a \in \Omega. \quad (3.8)
\]

By identities (3.7), (3.8), and by application of the holomorphic matrix functional calculus, acting on the matrix \( A \), satisfying (2.7), it follows that

\[
v(A) = \frac{1}{2} \int_0^\infty e^{-x^2t} \sin (2Ax) \frac{dx}{x} = \frac{\pi}{4} \text{erf} (A), \quad (3.9)
\]
and

\[
w(A) = \int_0^\infty e^{-x^2t} \cos (2Ax) \frac{dx}{x} = \sqrt{\pi} \exp \left( -\frac{A^2}{t} \right). \quad (3.10)
\]

**Lemma 3.2.** Let \( A \) be a matrix in \( \mathbb{C}^{p \times p} \) such that \( R_1(A) > 0, I_2(A) < 0, \) and \( R_1(A) = |R_1(A)| > |I_1(A)| \). Let \( \gamma(\theta) = \text{Re}^{i\theta} \) with \( \theta \in [-\pi, -\pi/2) \cup [\pi/2, \pi] \) and \( R > 0 \). Then,

\[
\lim_{R \to \infty} \int_{\gamma} e^{sA} ds = 0. \quad (3.11)
\]

**Proof.** First of all, note that by (1.7) and (1.10) one gets

\[
\mu(iA) = \max \left\{ \lambda : \lambda \in \sigma \left( \frac{(iA) + (iA)^H}{2} \right) \right\} = \max \left\{ \lambda : \lambda \in \sigma \left( -\left( \frac{A - A^H}{2i} \right) \right) \right\} \quad (3.12)
\]

\[
= -\min \{ \lambda : \lambda \in \sigma(\text{Im}(A)) \} = -\lambda_{\text{min}}(\text{Im}(A)) \leq -I_1(A),
\]

\[
\|e^{iAx}\| \leq e^{x\mu(iA)} \leq e^{-xI_1(A)}, \quad x \geq 0. \quad (3.13)
\]

By (1.7) and (1.10), we also have

\[
\|e^{-Ax}\| \leq e^{x\mu(-A)}, \quad x \geq 0,
\]

\[
\mu(-A) = \max \left\{ \lambda : \lambda \in \sigma \left( -\frac{(A) + (A)^H}{2} \right) \right\}
\]

\[
= -\min \left\{ \lambda : \lambda \in \sigma \left( \frac{A + A^H}{2} \right) \right\} \leq -R_1(A),
\]

\[
\|e^{-Ax}\| \leq e^{-xR_1(A)}, \quad x \geq 0, \quad (3.14)
\]

\[
\mu(-iA) = \max \left\{ \lambda : \lambda \in \sigma \left( \frac{(-iA) + (-iA)^H}{2} \right) \right\} = \max \{ \lambda : \lambda \in \sigma(\text{Im}(A)) \} \leq I_2(A),
\]

\[
\|e^{-iAx}\| \leq e^{x\mu(-iA)} \leq e^{xI_2(A)}, \quad x \geq 0. \quad (3.15)
\]
Let $s = R e^{i\theta} = R (\cos \theta + i \sin \theta)$; $\sqrt{s} = R^{1/2} e^{i\theta/2} = \sqrt{R} (\cos (\theta/2) + i \sin (\theta/2))$, $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$. Then,

$$
\left\| e^s e^{-A \sqrt{s}} \right\| = \left\| e^{R (\cos \theta + i \sin \theta)} e^{-A \sqrt{R} (\cos (\theta/2) + i \sin (\theta/2))} \right\| 
\leq e^{i R \cos \theta} \left\| e^{-A \sqrt{R} \cos (\theta/2)} \right\| \left\| e^{-i A \sqrt{R} \sin (\theta/2)} \right\|. 
$$

(3.14)

For all $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$ one gets $\cos (\theta/2) \geq 0$ and by (3.12), it follows that

$$
\left\| e^{-A \sqrt{R} \cos (\theta/2)} \right\| \leq e^{-R_1(A) \sqrt{R} \cos (\theta/2)}, 
$$

(3.15)

and by (3.13) one gets

$$
\left\| e^{-i A \sqrt{R} \sin (\theta/2)} \right\| \leq e^{i_2(A) \sqrt{R} \sin (\theta/2)}, \quad \theta \in \left[ \frac{\pi}{2}, \pi \right]. 
$$

(3.16)

If $\theta \in [-\pi, -\pi/2]$, then $\sin (\theta/2) = -|\sin (\theta/2)|$ and by (3.11), it follows that

$$
\left\| e^{-i A \sqrt{R} \sin (\theta/2)} \right\| = \left\| e^{i A \sqrt{R} \sin (\theta/2)} \right\| \leq e^{-I_1(A) \sqrt{R} \sin (\theta/2)} 
\leq e^{I_1(A) \sqrt{R} \sin (\theta/2)}, \quad \theta \in \left[ -\pi, -\frac{\pi}{2} \right]. 
$$

(3.17)

By (3.14)–(3.17), it follows that

$$
\left\| e^s e^{-A \sqrt{s}} \right\| \leq \exp \left( \sqrt{R} g(\theta, R) \right), \quad \theta \in \left[ -\pi, -\frac{\pi}{2} \right] \cup \left[ \frac{\pi}{2}, \pi \right], 
$$

(3.18)

where

$$
g(\theta, R) = \begin{cases} 
\nu(\theta) + I_1(A) \sin \left( \frac{\theta}{2} \right), & \theta \in \left[ -\pi, -\frac{\pi}{2} \right], \\
\nu(\theta) + I_2(A) \sin \left( \frac{\theta}{2} \right), & \theta \in \left[ \frac{\pi}{2}, \pi \right], 
\end{cases} 
$$

(3.19)

and

$$
\nu(\theta) = t \sqrt{R} \cos \theta - R_1(A) \cos \left( \frac{\theta}{2} \right) \leq 0. 
$$

(3.20)

Note that $g(\theta, R)$ is a continuous function of $\theta$ and that by definition $g(\theta, R) < 0$ for all $\theta \in [\pi/2, \pi]$. Furthermore, by hypothesis $R_1(A) > -I_1(A)$, and thus,

$$
\max_{\theta \in \left[ -\pi, -\frac{\pi}{2} \right]} g(\theta, R) \leq \max \left\{ \nu(\theta); \theta \in \left[ -\pi, -\frac{\pi}{2} \right] \right\} + \max \left\{ I_1(A) \sin \left( \frac{\theta}{2} \right); \theta \in \left[ -\pi, -\frac{\pi}{2} \right] \right\} 
= \nu \left( \frac{\pi}{2} \right) + I_1(A) \sin \left( \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2} (R_1(A) + I_1(A)) < 0. 
$$

(3.21)

Hence,

$$
\max \left\{ g(\theta, R); \theta \in \left[ -\pi, -\frac{\pi}{2} \right] \cup \left[ \frac{\pi}{2}, \pi \right] \right\} < 0, 
$$

(3.22)

and by (3.22), the result is established because $\lim_{R \to \infty} g(\theta, R) = 0$, uniformly for $\theta$ in the set $\left[ -\pi, -\frac{\pi}{2} \right] \cup \left[ \frac{\pi}{2}, \pi \right]$.
LEMMA 3.3. Let $A$ be a matrix in $\mathbb{C}^{n \times p}$ satisfying $R_1(A) > 0$, $I_2(A) < 0$, and $R_1(A) = |R_1(A)| > |I_1(A)|$. Let $\gamma(\theta) = Re^{i\theta}$, $\theta \in [\alpha, \pi/2] \cup [-\pi/2, -\alpha]$, with $0 < \alpha < \pi/2$, $R \cos \alpha = c > 0$. Then,
\[
\lim_{R \to \infty} \int_{\gamma} \frac{e^{st}}{s} e^{-\sqrt{\gamma} \, ds} = 0. \tag{3.23}
\]

PROOF. Note that if $s = Re^{i\theta}$, then for $\theta \in [\alpha, \pi/2] \cup [-\pi/2, -\alpha]$ one gets $R \cos \theta \leq c$, and thus,
\[
\left\| e^{st} e^{-\sqrt{R} \, Re^{i\theta}} \right\| = \left\| e^{tR \cos \theta + i \sin \theta} e^{-\sqrt{R} \cos (\theta/2) + i \sin (\theta/2)} \right\| \leq e^{tR \cos \theta} \left\| e^{-\sqrt{R} \cos (\theta/2)} \right\| \left\| e^{-\sqrt{R} \sin (\theta/2)} \right\| \leq e^{tc} \left\| e^{-\sqrt{R} \cos (\theta/2)} \right\| \left\| e^{-\sqrt{R} \sin (\theta/2)} \right\|. \tag{3.24}
\]

Note that for all $\theta \in [\alpha, \pi/2] \cup [-\pi/2, -\alpha]$, by (3.12) one gets
\[
\left\| e^{-\sqrt{R} \cos (\theta/2)} \right\| \leq e^{-R_1(A) \sqrt{R} \cos (\theta/2)}. \tag{3.25}
\]

If $\theta \in [\alpha, \pi/2]$, then $\sin (\theta/2) > 0$ and by (3.13), it follows that
\[
\left\| e^{-\sqrt{R} \sin (\theta/2)} \right\| \leq e^{I_2(A) \sqrt{R} \sin (\theta/2)}. \tag{3.26}
\]

If $\theta \in [-\pi/2, -\alpha]$, then $\sin (\theta/2) < 0$ and by (3.11), it follows that
\[
\left\| e^{-\sqrt{R} \sin (\theta/2)} \right\| = \left\| e^{I_2(A) \sqrt{R} \sin (\theta/2)} \right\| \leq e^{-R_1(A) \sqrt{R} \sin (\theta/2)} = e^{I_1(A) \sqrt{R} \sin (\theta/2)}. \tag{3.27}
\]

Let
\[
g_1(\theta) = -R_1(A) \cos \left(\frac{\theta}{2}\right) + I_1(A) \sin \left(\frac{\theta}{2}\right), \quad \text{for } \theta \in \left[-\frac{\pi}{2}, -\alpha\right],
\]
and
\[
g_2(\theta) = -R_1(A) \cos \left(\frac{\theta}{2}\right) + I_2(A) \sin \left(\frac{\theta}{2}\right), \quad \text{for } \theta \in \left[\alpha, \frac{\pi}{2}\right].
\]

Then, by (3.25)–(3.27), it follows that
\[
e^{tc} \left\| e^{-\sqrt{R} \cos (\theta/2)} \right\| \left\| e^{-\sqrt{R} \sin (\theta/2)} \right\| \leq e^{tc} e^{\sqrt{R} g_1(\theta)}, \quad \theta \in \left[-\frac{\pi}{2}, -\alpha\right], \tag{3.28}
\]
\[
e^{tc} \left\| e^{-\sqrt{R} \cos (\theta/2)} \right\| \left\| e^{-\sqrt{R} \sin (\theta/2)} \right\| \leq e^{tc} e^{\sqrt{R} g_2(\theta)}, \quad \theta \in \left[\alpha, \frac{\pi}{2}\right]. \tag{3.29}
\]

By (3.24), (3.28), and (3.29), conclusion (3.23) holds true if
\[
\max \left\{ g_1(\theta), \theta \in \left[-\frac{\pi}{2}, -\alpha\right] \right\} < 0 \tag{3.30}
\]
and
\[
\max \left\{ g_2(\theta), \theta \in \left[\alpha, \frac{\pi}{2}\right] \right\} < 0. \tag{3.31}
\]

Note that by hypothesis $R_1(A) > 0$ and $I_1(A) < I_2(A) < 0$. Thus, $g_2(\theta) < 0$ for all $\theta \in [\alpha, \pi/2]$ and since $g_2(\theta)$ is continuous, the maximum of $g_2(\theta)$ in $[\alpha, \pi/2]$ is achieved in some $\theta_0 \in [\alpha, \pi/2]$ and, thus, (3.31) holds.

If $\theta \in [-\pi/2, -\alpha]$, then by hypothesis $R_1(A) > |I_1(A)|$, one gets
\[
g_1(\theta) \leq -\frac{\sqrt{\gamma}}{2} I_1(A) - R_1(A) \cos \left(\frac{\theta}{2}\right) \leq \frac{\sqrt{\gamma}}{2} (-R_1(A) - I_1(A)) < 0,
\]
and thus, (3.30) holds. Hence, the result is established.
4. THE INVERSE MATRIX LAPLACE TRANSFORM

The aim of this section is to obtain a matrix version of Example 3.7.15 of [1, p. 105]. We begin with a definition of the matrix Laplace transform that is equivalent to the componentwise definition.

DEFINITION 4.1. Let \( f : [0, \infty[ \rightarrow \mathbb{C}^{p \times p} \) be locally integrable in \([0, \infty[\) and of exponential order \( e^{at}\) as \( t \rightarrow \infty\), for some \( a > 0\), i.e.,

\[
\lim_{t \to \infty} e^{-at\|f(t)\|} = 0, \quad (4.1)
\]

then the Laplace transform of the matrix function \( f(t) \) is defined by

\[
\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt; \quad \text{Re}\, s > a. \quad (4.2)
\]

It is easy to check that (4.2) is well defined because \( \|f(t)e^{-st}\| \) is integrable in \([0, \infty[\).

EXAMPLE 4.1. Note that if \( A \) is a matrix satisfying (2.7), then by Theorem 2.1(iv) one gets

\[
erfc \left( \frac{A}{2\sqrt{t}} \right) < \left\| A \right\| \text{erfc} \left( \frac{A}{2\sqrt{t}} \right) \quad (4.3)
\]

and as the scalar function \( \text{erfc}(A/(2\sqrt{t})) \) admits the Laplace transform, see [1, p. 86], then \( \text{erfc}(A/(2\sqrt{t})) \) admits the Laplace transform.

THEOREM 4.1. Let \( A \) be a matrix in \( \mathbb{C}^{p \times p} \) such that

\[
\text{Re}(z) > |\text{Im}(z)|, \quad \text{for all } z \in \sigma(A), \quad (4.4)
\]

\[
R_1(A) > 0, \quad I_2(A) < 0. \quad (4.5)
\]

Then,

\[
f(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-A\sqrt{s}}}{s} \right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} \exp(stI - AV\sqrt{s}) \, ds = \text{erfc} \left( \frac{A}{2\sqrt{t}} \right). \quad (4.6)
\]

PROOF. The integrand of (4.6) has a branch point at \( s = 0 \). Using the contour of integration as shown in Figure 3.4b of [1, p. 103] and applying the Cauchy fundamental theorem for matrix valued functions, see [16], it follows that

\[
\frac{1}{2\pi i} \left[ \int_{L_1} + \int_{\Gamma} + \int_{L_2} + \int_{\gamma} \right] \exp(stI - AV\sqrt{s}) \, ds = 0. \quad (4.7)
\]

Note that hypotheses (4.4) and (4.5) imply conditions involved in hypotheses of Lemmas 3.2 and 3.3. By Lemmas 3.2 and 3.3, the integral on the contour \( \Gamma \) tends to zero as \( R \to \infty \). The integral on \( L \) gives the Bromwich integral

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} \exp(stI - AV\sqrt{s}) \, ds.
\]

Now we evaluate the remaining three integrals in (4.7). On \( L_1 \), we have \( s = re^{i\pi} = -r \) and

\[
\int_{L_1} \exp(stI - AV\sqrt{s}) \, ds = \int_{-\infty}^{0} \exp(stI - AV\sqrt{s}) \, ds = -\int_{0}^{\infty} \exp(- (rtI + iAV\sqrt{r})) \, dr. \quad (4.8)
\]
On $L_2$, $s = re^{-i\pi} = -r$, and
\[
\int_{L_2} \exp \left( stI - A\sqrt{s} \right) \frac{ds}{s} = \int_0^\infty \exp \left( -rtI + iA\sqrt{r} \right) \frac{dr}{r}. \tag{4.9}
\]
Thus, the integrals along $L_1$ and $L_2$ combined yield
\[
-2i \int_0^\infty e^{-rt} \sin \left( A\sqrt{r} \right) \frac{dr}{r} = -4i \int_0^\infty e^{-x^2/2} \frac{sin(Ax)}{x} \frac{dx}{x}. \tag{4.10}
\]
By Lemma 3.1 and (4.10), one gets
\[
-4i \int_0^\infty e^{-x^2/2} \frac{sin(Ax)}{x} \frac{dx}{x} = -2\pi i \text{erf} \left( \frac{A}{2\sqrt{t}} \right). \tag{4.11}
\]
Finally, on $\gamma$, with parametrization $\gamma(\theta) = s = re^{i\theta}; \ \theta \in [-\pi, \pi]$, we have
\[
\int_{\gamma} \exp \left( stI - A\sqrt{s} \right) \frac{ds}{s} = \int_{-\pi}^\pi \exp \left( re^{i\theta}t - A\sqrt{r}e^{i\theta/2} \right) i \, d\theta. \tag{4.12}
\]
As $r \to 0$, we have
\[
\lim_{r \to 0} re^{i\theta} = 0, \quad \lim_{r \to 0} A\sqrt{r}e^{i\theta/2} = 0, \quad \text{uniformly for } \theta \in [-\pi, \pi], \tag{4.13}
\]
and by (4.12) and (4.13), and the continuity of the matrix exponential it follows that
\[
\lim_{r \to 0} \int_{\gamma} \exp \left( stI - A\sqrt{s} \right) \frac{ds}{s} = i \int_{-\pi}^\pi I \, d\theta = 2\pi i I. \tag{4.14}
\]
From (4.7)–(4.9), (4.11), and (4.14) and taking into account that $\int_0^\infty F = -\int_0^\infty F$, one gets
\[
\mathcal{L}^{-1} \left\{ \frac{e^{-A\sqrt{s}}}{s} \right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left( stI - A\sqrt{s} \right) \frac{ds}{s} = I - \text{erf} \left( \frac{A}{2\sqrt{t}} \right). \tag{4.15}
\]
Now by Theorem 2.1(ii), it follows that
\[
I - \text{erf} \left( \frac{A}{2\sqrt{t}} \right) = \text{erfc} \left( \frac{A}{2\sqrt{t}} \right), \quad t > 0, \tag{4.16}
\]
and by (4.15) and (4.16), one gets (4.6). Thus, the result is established.

5. SOLVING THE COUPLED DIFFUSION PROBLEM

Consider the coupled diffusion problem (1.2)–(1.5) where $A$ is a matrix in $\mathbb{C}^{p \times p}$ satisfying (2.7). Then, by [7], $A^{-1}$ also satisfies (2.7) and by (2.8) one gets
\[
\text{erfc} \left( A^{-1} \right) = \frac{2A^{-1}}{\sqrt{\pi}} \int_1^\infty e^{-\left( vA^{-1} \right)^2} \, dv. \tag{5.1}
\]
If $x > 0$, $t > 0$, then the matrix $A^{-1}x/2\sqrt{t}$ also satisfies property (2.7) and by (2.8), it follows that
\[
V(x, t) = \text{erfc} \left( \frac{A^{-1}x}{2\sqrt{t}} \right) = \frac{A^{-1}x}{\sqrt{\pi} \sqrt{t}} \int_0^1 \exp \left[ -\left( \frac{A^{-1}xv}{2\sqrt{t}} \right)^2 \right] \, dv. \tag{5.2}
\]
Note that considering the change of variables \( xv/2\sqrt{t} = s \), one gets
\[
V(x,t) = \frac{2A^{-1}}{\sqrt{\pi t}} \int_0^{x/\sqrt{t}} e^{-(A^{-1}s)^2} ds
\]
(5.3)

\[
= \frac{A^{-1}}{\sqrt{\pi \sqrt{t}}} \int_0^x \exp \left[ - \left( \frac{A^{-1}w}{2\sqrt{t}} \right)^2 \right] dw
\]
(5.4)

\[
= \frac{2xA^{-1}}{\sqrt{\pi}} \int_0^{1/2\sqrt{t}} \exp \left[ - \left( A^{-1}wx \right)^2 \right] dw.
\]
(5.5)

By (5.4), theorem of [17] and the commutativity of a matrix with its exponential, it follows that
\[
V_x(x,t) = \frac{A^{-1}}{\sqrt{\pi \sqrt{t}}} \exp \left[ - \left( \frac{x^2A^{-2}}{4t} \right) \right].
\]
(5.6)

\[
V_{xx}(x,t) = \frac{A^{-1}}{\sqrt{\pi \sqrt{t}}} \exp \left[ - \left( \frac{x^2A^{-2}}{4t} \right) \right] \left( - \frac{x^2A^{-2}}{2t} \right) = \frac{xA^{-3}}{2\sqrt{\pi \sqrt{t}}} \exp \left[ - \left( \frac{x^2A^{-2}}{4t} \right) \right].
\]
(5.7)

Let \( u_0 \) be the vector appearing in (1.4) and let \( u(x,t) \) be defined by
\[
u(x,t) = \begin{cases} u_0, & x = 0, \quad t \geq 0, \\ \text{erfc} \left( \frac{A^{-1}x}{2\sqrt{t}} \right) u_0, & x > 0, \quad t > 0, \\ 0, & x > 0, \quad t = 0. \end{cases}
\]
(5.8)

By (5.6), it follows that
\[
A^2u_{xx}(x,t) = -\frac{xA^{-1}}{2\sqrt{\pi \sqrt{t}}} \exp \left[ - \left( \frac{x^2A^{-2}}{4t} \right) \right] u_0, \quad x > 0, \quad t > 0.
\]
(5.9)

Let \( t > 0 \) and note that considering the series expansion of \( \exp[-A^{-2}w^2x^2] \), it follows that
\[
W(x,t) = \int_0^{1/2\sqrt{t}} \exp \left[ -A^{-2}w^2x^2 \right] dw = \int_0^{1/2\sqrt{t}} \left( \sum_{n\geq0} (-1)^n A^{-2n}x^{2n} \frac{w^{2n}}{n!} \right) dw
\]
(5.10)

\[
= \sum_{n\geq0} \frac{(-1)^n A^{-2n}x^{2n}}{n!} \int_0^{1/2\sqrt{t}} w^{2n} dw
\]
\[
= \sum_{n\geq0} \frac{(-1)^n A^{-2n}x^{2n}}{n!(2n + 1)} \frac{1}{(2t)^{n+1/2}}.
\]
(5.11)

By the differentiation theorem of series and (5.10), it follows that
\[
W_t(x,t) = -\sum_{n\geq0} \frac{(-1)^n A^{-2n}x^{2n}}{n!(2n + 1)} \frac{n + 1/2}{2^{n+1}t^{n+1+1/2}}
\]
\[
= -\frac{1}{4} \left( \sum_{n\geq0} \frac{(-1)^n A^{-2n}x^{2n}}{n!2^{2n}t^n} \right) \frac{1}{t^{3/2}}
\]
(5.12)

\[
= -\frac{1}{4t\sqrt{t}} \exp \left[ - \left( \frac{A^{-1}x}{2\sqrt{t}} \right)^2 \right], \quad t > 0, \quad x > 0.
\]
By (5.7),(5.9)-(5.11), it follows that
\[ u_t(x,t) = A^2 u_{xx}(x,t), \quad x > 0, \quad t > 0, \]
and \( u(x,t) \) satisfies (1.2)-(1.4). If apart from (2.7), matrix \( A \) satisfies
\[ \text{Re} (z) > 0, \quad \text{for all} \quad z \in \sigma(A), \quad (5.12) \]
then \( A^{-1}x/2\sqrt{t} \) also satisfies (2.7) and (5.12) and by Theorem 2.1(iv), it follows that \( R_1(A^{-2}) > 0 \) and
\[ \left\| \text{erfc} \left( \frac{A^{-1}x}{2\sqrt{t}} \right) \right\| \leq \frac{\|A^{-1}\|}{\sqrt{R_1(A^{-2})}} \text{erfc} \left( \frac{x}{2\sqrt{t}} \sqrt{R_1(A^{-2})} \right), \quad x > 0, \quad t > 0. \quad (5.13) \]
By (5.7),(5.13), it follows that (1.5) holds true. Summarizing, the following result has been established.

**Theorem 5.1.** Let \( A \) be a matrix in \( \mathbb{C}^{p\times p} \) satisfying
\[ \text{Re} \, z > |\text{Im} \, z|, \quad \text{for all} \quad z \in \sigma(A), \]
then \( u(x,t) \) defined by (5.7) is a solution of problem (1.2)-(1.5).

**References**