Abstract—This paper considers two sub-optimal transmission schemes for a family of parallel Gaussian vector broadcast channels. One of the schemes is based on the QR precoding of [2]. In QR precoding, the maximum achievable throughput depends on the order in which users are encoded. This scheme is a new algorithm for obtaining the best user ordering and channel-input covariance matrix that maximizes the total channel throughput. The proposed algorithm has linear complexity in the number of multi-carrier frequencies.

The simplicity of a linear transmitter-and-receiver architecture is attractive for a transmission scheme. The design of a linear transmitter and multiple linear receivers that maximize the total throughput is studied. It is shown that under discrete bit loading assumption for each user, this problem reduces to solving a series of Second Order Cone Programming problems.

I. INTRODUCTION

A broadcast channel is a communication model in which a single transmitter sends independent messages to multiple receivers at the same time. The capacity region of a general non-degraded broadcast channel is still an open problem, however for the special case of Gaussian vector broadcast channel, the capacity region has been determined recently. Caire and Shamai [3] have established an achievable rate region for the vector broadcast channel by applying the “dirty paper” precoding scheme at the transmitter. It was also shown in [4], [5] that the sum capacity of vector broadcast channel is equal to the maximum sum rate of this achievable region.

Finally in [9] it was shown that the capacity region and the “dirty paper” achievable region are indeed the same. Although the “dirty paper” precoding scheme achieves the sum capacity, it is too complicated to be implemented in practice. This paper investigates for transmission schemes that can be viewed as “dirty paper” precoding practical implementations and perform closely to the optimal scheme in the sum-rate sense. One such method is QR precoding introduced in [2], [8]. In [2] it is shown that the total throughput achieved by QR precoding is very close to the “dirty paper” sum-rate for DSL channel model, and in [8] the nearly optimum performance of QR precoding is proved for high SNR regime. In this method, the maximum achievable throughput depends on the order in which users are being encoded; hence, this necessitates the joint optimization over the user ordering and transmit covariance matrix to maximize the total data rate. Section II introduces an algorithm for obtaining the best user ordering and channel-input covariance matrix in order to maximize the total channel throughput.

In parallel Gaussian broadcast channels, optimal ordering can be found by an exhaustive search over all possible user orderings on different sub-channels. The complexity of this search increases exponentially with the number of parallel sub-channels. By using the dual decomposition method, the search can be executed more efficiently and the complexity can be reduced to that of linear case.

Precoding schemes require the messages for different users to be encoded consecutively. Hence, compared to encoding all users at once, precoding will introduce an additional latency and delay to the transmission process. Therefore, linear transmitter and receivers are simpler and have instantaneous-encoding low delay. The design of linear transmitter and receivers for the special case of broadcast channels with common information was first considered in [6]. This paper generalizes the previously known results about this model and shows that under a discrete bit-loading assumption [7] for each user, the optimization problem reduces to solving a series of Second Order Cone Programming problems.

The remainder of this paper is organized as follows: Section II describes the broadcast channel model which is used throughout this paper. It introduces two sub-optimal transmission schemes and devises solutions to underlying rate-maximization problems. A comparison between these two methods and the optimal one based on simulation results is included in section III. Section IV concludes this paper.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Frequency Selective Gaussian Multiple-Input Multiple-Output broadcast channels can be decomposed into a group of independent parallel Gaussian vector broadcast channels by using OFDM techniques. Each of these sub-channels can be employed to transmit independent information streams. At each time instant on sub-channel \( n = 1, 2, \ldots, N \), channel output can be expresses as:

\[
y_n = H_n x_n + z_n,
\]

where \( x_n \in \mathbb{C}^t \) and \( y_n \in \mathbb{C}^{Kr} \) are the channel input and output vectors on sub-channel \( n \) respectively. \( t \) is the number of transmit spatial dimensions and \( r \) is the number of such dimensions per receiver. \( K \) is the number of users and \( H_n \in \mathbb{C}^{Kr \times t} \).
\( \mathbf{C}^{K \times t} \) is the matrix of channel coefficients on sub-channel \( n \). \( N \) is the total number of sub-channels. \( \mathbf{z}_n \in \mathbb{C}^{K \times t} \) represents additive Gaussian noise. Without loss of generality, Gaussian noise can be assumed white, \( \mathbf{z}_n \sim \mathcal{N}(0, I) \), given that the capacity region of a vector broadcast channel does not depend on noise cross-correlation coefficients. \( \mathbf{z}_n \)’s are assumed to be independent on different tones.

For Gaussian broadcast channels there is a total transmit power constraint over all sub-channels, which can be formulated as

\[
\sum_{n=1}^{N} \text{Tr}(\mathbb{E}\{\mathbf{x}_n\mathbf{x}_n^\dagger\}) \leq KP,
\]

where \( P \) indicates the maximum transmit power per user.

This channel model corresponds to downstream transmission in DSL systems when coordination is allowed at the central office. In this scenario \( r = 1 \) and \( t = K \) respectively. Downlink transmissions in a wireless system with a base station equipped with \( t \) transmit antennas and \( K \) users with \( r \) receive antennas each, is another example of such channel model. Throughout this paper, bold small letters denote on vectors and capital letters denote on matrices. For a vector \( \mathbf{v} \), \( [\mathbf{v}]_i \) is the \( i \)th vector element and for a matrix \( \mathbf{M} \), \( [\mathbf{M}]_{i,j} \) is the element in position \((i, j)\). \( \mathbf{(.)}^\dagger \) and \( \mathbf{(.)}^T \) denote on Hermitian and Transposition operations. Finally \( r = 1 \) receive dimension per user is assumed for all analysis.

This paper considers two sub-optimal methods. They are sub-optimal in the sense that they could not achieve sum rates sufficiently close to the channel-capacity limit. The first scheme employs the precoding concept, i.e., symbols for each user \( k \) are encoded after the encoding process of previous users \( 1, 2, \ldots, k - 1 \) has been completed. It is shown that this process is not linear and therefore it can be hard to implement for many cases. In the second scheme, the transmission and reception schemes are restricted to linear operations.

A. Precoding and Ordering Problem

First scheme employs the precoding scheme introduced in [2]. This scheme is applicable only when the channel matrix is a square matrix. In other words the number of available spatial dimensions at the transmitter is equal to total number of spatial dimensions available to all the users. In wireless downlink transmission, this model corresponds to a base station with \( K \) transmit antennas and \( K \) users each with one receive antennas. This model also applies to DSL channels for downstream transmission when coordination is possible on the transmitters’ side. QR precoding can be viewed as a practical implementation of “dirty paper” precoding.

On each tone consider the QR factorization of the channel \( H_n^\dagger = Q_nR_n^\dagger D_n^\dagger \), where \( Q_n \) is a unitary matrix, \( R_n^\dagger \) is an upper triangular matrix with diagonal entries equal to one and \( D_n \) is a diagonal matrix. So the choice of \( \mathbf{x}_n = Q_nR_n^{-1}\tilde{x}_n \) will result in a cross-talk free reception, \( \mathbf{y}_n = D_n\tilde{x}_n + \mathbf{z}_n \) on each tone \( n \). However, since \( R_n^{-1} \) is not a unitary matrix, this operation may result in an undesirable transmitter power increase. As is shown in [2], this increase in transmission power can be avoided if Tomlinson-Harashima precoding is performed at the transmitter side. By using this precoder, one can show that \( \text{Tr}(\mathbb{E}\{\mathbf{x}_n\mathbf{x}_n^\dagger\}) \approx \text{Tr}(\mathbb{E}\{\tilde{x}_n\tilde{x}_n^\dagger\}) \) and there is no significant increase in transmitter power. Thus, performing QR factorization and precoding results in a set of \( K \) cross-talk free channels on each tone. Note that \( \tilde{x}_n \) is the precoder input vector and \( [\tilde{x}_n]_k \) is the symbol to be transmitted to the \( k \)th user on the \( n \)th tone. Define \( p_{nk} \) and \( g_{nk} \) to be the transmitter power and normalized channel gain of user \( k \) on tone \( n \) respectively, i.e., \( p_{nk} = \mathbb{E}\{|[\tilde{x}_n]_k|^2\} \) and \( g_{nk} = |[D_n]_{k,k}|^2 \), then the channel for user \( k \) on tone \( n \) has SNR value equal to \( p_{nk}g_{nk} \) and a data rate equal to \( \frac{1}{2}\log(1 + p_{nk}g_{nk}) \) can be achieved for this channel.

In Tomlinson-Harashima precoding, symbols in vector \( \tilde{x}_n \) are precoded in an increasing order from top to bottom, i.e., the symbol for the first user is precoded first, then the symbol for the second user and so on. The precoding order can be altered by performing Tomlinson precoding on vector \( J\tilde{x}_n \) instead of \( \tilde{x}_n \) over an equivalent channel \( JH_n = D_nR_n^\dagger Q_n^\dagger \) instead of \( H_n \), where \( J \) is a permutation matrix(\( J^\dagger J = I \)). This follows directly from the channel expression for permuted inputs and outputs: \( J\mathbf{y}_n = JH_n\mathbf{x}_n + J\mathbf{z}_n = D_nJ\tilde{x}_n + J\mathbf{z}_n \).

Depending on the ordering for which precoding is performed on tone \( n \), normalized channel gains \( g_{nk} \), which are diagonal entries of upper triangular matrix in QR factorization of \( H_n^\dagger J^\dagger \), correspond to different set of values. For a specific user ordering on tone \( n \), let \( g_{nk}(\pi_n) \) denote the normalized channel gain for user \( k \) on tone \( n \) where, \( \pi_n \) is a permutation on \( \{1, 2, \ldots, K\} \) indicating the precoding order.

For a given set of precoding orders on each tone, \( (\pi_1, \pi_2, \ldots, \pi_N) \), and a given power allocation for each user on each tone \( p_{nk} \), the sum of rates of all users on tone \( n \) and the total throughput can be expressed as

\[
r_n(\pi_n)p_{n1},p_{n2},\ldots,p_{nK} = \sum_{k=1}^{K} \frac{1}{2}\log(1 + g_{nk}(\pi_n)p_{nk}) \quad \text{and} \quad \sum_{n=1}^{N} r_n(\pi_n) \quad \text{respectively, where} \quad (\pi_n) \quad \text{illustrates the dependency of the sum rate on user ordering. Therefore, the problem of throughput maximization can be expressed as the following optimization problem:}
\]

\[
\max_{\pi_1,\pi_2,\ldots,\pi_N} \left\{ \sum_{n=1}^{N} r_n(\pi_n)(p_{n1},\ldots,p_{nK}) \right\} \quad (1)
\]

Subject to

\[
\sum_{n=1}^{N} \sum_{k=1}^{K} p_{nk} \leq KP \quad (2)
\]

where the variables are \( p_{nk} \), for \( k = 1, \ldots, K \) and \( n = 1, \ldots, N \). Clearly, this problem is not a concave problem, because the cost function, which is a maximum of a set of concave functions, is not necessarily concave. Therefore, employing convex optimization techniques directly cannot solve this problem. By exploring the cost function in (1), it is found that for a given set of orderings on tones, the data rate maximization problem subject to the total power constraint has a simple water-filling solution, thus a brute
The optimal solution of optimization problem $f_{\lambda} = 0$ maximizes the total rate for that set of orderings, subject to the total power constraint. Here, $()^+$ stands for $\max(.,0)$, and $\gamma$ is a constant that satisfies the power constraint $\sum_{k=1}^{K} \sum_{n=1}^{N} p_{nk} = KP$. Let $R(\pi_1, \pi_2, \ldots, \pi_N)$ denote the maximum sum rate for given set of orders $(\pi_1, \ldots, \pi_N)$, using this notation the maximum throughput can be represented as $\max_{\pi_1, \pi_2, \ldots, \pi_N} R(\pi_1, \pi_2, \ldots, \pi_N)$.

Since there are $K!$ possible orderings on each tone, in order to solve this optimization problem, one needs to search among all $(K!)^N$ possible orderings to find the one with maximum total rate. The complexity of such brute force search method is $O((K!)^N)$.

The Dual Decomposition approach can be used to solve the problem in (1) more efficiently with complexity of $O(N(K!))$ rather than $O((K!)^N)$. The total power constraint in (2) can be written as:

$$\sum_{k=1}^{K} p_{nk} \leq P_n \quad n = 1, 2, \ldots, N \tag{3}$$

$$\sum_{n=1}^{N} P_n \leq KP. \tag{4}$$

Let $\mathcal{L}(p_{nk}, P_1, \ldots, P_N, \lambda)$ be the Lagrangian associated with problem (1) with only explicit inequalities given in (4). Consider the inequality in (3) implicitly by defining the Lagrangian over the domain $\mathcal{D} = \{p_{nk}, P_n : 0 \leq p_{nk}, \sum_{k=1}^{K} p_{nk} \leq P_n \quad 1 \leq n \leq N\}$. Let $f(\lambda)$ for $\lambda \geq 0$ denote the Lagrange dual function of problem in (1) as is defined in [1]

$$f(\lambda) = \sup_{p_{nk}, P_n \in \mathcal{D}} \mathcal{L}(p_{nk}, P_1, \ldots, P_N, \lambda). \tag{5}$$

$$\max_{\pi_1, \pi_2, \ldots, \pi_N} \left( \sum_{n=1}^{N} r_n(\pi_n)(p_{n1}, \ldots, p_{nK}) - \lambda P_n \right) + \lambda KP. \tag{6}$$

By definition, $f(\lambda)$ is a convex function of $\lambda$ and can be used to obtain an upper bound on maximum data rate. Fortunately the dual function for a fixed $\lambda$ can be calculated efficiently by considering the fact that power constraints on each tone are decoupled and there is no complicating variable between different tones. Since the ordering is also independent on each tone, the dual function calculation can be transformed into $N$ simple maximization problems as follows

$$f(\lambda) = \sum_{n=1}^{N} f_n(\lambda) + \lambda KP \tag{7}$$

where for $n = 1, \ldots, N$:

$$f_n(\lambda) = \sup_{p_{nk}, P_n \in \mathcal{D}_n} \max_{\pi_n} \left\{ \sum_{k=1}^{K} \frac{1}{2} \log(1 + g_{nk}(\pi_n)p_{nk}) - \lambda P_n \right\}, \tag{8}$$

Here, the maximization is taken over the domain $\mathcal{D}_n = \{p_{nk}, P_n : 0 \leq p_{nk}, \sum_{k=1}^{K} p_{nk} \leq P_n \}$. Again each maximization problem in (7) is not a concave problem; However each one can be solved by examining all possible orderings on each specific tone. Note that the constraint set $\mathcal{D}_n$ is closed and compact and the maximum of finite number of logarithmic functions is a continuous function, thus the supremum is achieved and $\sup$ can be replaced by $\max$; consequently the order of maximization over $\mathcal{D}_n$ and $\pi_n$ can be interchanged. For a given ordering $\pi_n$ on tone $n$, define $f_n(\pi_n)(\lambda)$ as the optimal value of the optimization problem given below:

$$\max_{\pi_n} \left\{ \sum_{k=1}^{K} \frac{1}{2} \log(1 + g_{nk}(\pi_n)p_{nk}) - \lambda P_n \right\} \tag{9}$$

where $p_{nk}$, $P_n \geq 0$ for $k = 1, 2, \ldots, K$, where the optimization variables are $p_{n1}, p_{n2}, \ldots, p_{nK}, P_n$. This problem has a simple solution known as water filling with constant power level.

**Theorem 1:** The optimal solution of optimization problem given in (8) is of the form:

$$p_{nk} = \left( \frac{1}{2\lambda} - \frac{1}{g_{nk}(\pi_n)} \right)^+, \quad P_n = \sum_{k=1}^{K} g_{nk} \tag{10}$$

**Proof:** To verify the optimality of given solution, it is sufficient to show that they will satisfy the KKT conditions [1] for problem in (8). Let $\nu$ and $\delta_k$, $k = 1, 2, \ldots, K$ denote dual variables associated with inequalities $\sum_{k=1}^{K} p_{nk} \leq P_n$ and $p_{nk} \geq 0$ respectively. The Lagrangian of (8) can be written as

$$\mathcal{L}(p_{nk}, P_n, \nu, \delta_k) = \sum_{k=1}^{K} \frac{1}{2} \log(1 + g_{nk}(\pi_n)p_{nk})$$

$$- \sum_{k=1}^{K} (\nu p_{nk} - \delta_k P_n) + (\nu - \lambda)P_n.$$  

Derivatives of Lagrangian with respect to the prime variables for optimal solution should be zero. These conditions together with complementary slackness conditions [1], form the KKT conditions of the problem in (8):

$$\frac{\partial}{\partial p_{nk}} \mathcal{L} = -\frac{g_{nk}(\pi_n)}{2(1 + g_{nk}(\pi_n)p_{nk})} + \delta_k - \nu = 0$$

$$\frac{\partial}{\partial P_n} \mathcal{L} = \nu - \lambda = 0$$

$$\sum_{k=1}^{K} p_{nk} \leq P_n, \quad \nu \geq 0, \quad \nu \left( \sum_{k=1}^{K} p_{nk} - P_n \right) = 0$$

$p_{nk} \geq 0, \quad \delta_k \geq 0, \quad p_{nk}\delta_k = 0 \quad k = 1, \ldots, K$
It is easy to verify that (9) satisfies the above KKT conditions. Since the problem in (8) is concave and also Slater’s condition is satisfied, it is concluded that the solution given in (9) is optimal [1].

For a given \( \lambda \), each \( f_n(\lambda) \) can be computed by searching the maximum value of \( f_n(\pi_n)(\lambda) \) over all possible orderings \( \pi_n \), equivalently \( f_n(\lambda) = \max_{\pi_n} f_n(\pi_n)(\lambda) \), which states that \( f(\lambda) \) can be evaluated by \( N(K!) \) runs of the simple optimization problem given in (8) for each \( \lambda \).

Next, consider the dual of problem (1), i.e., Minimize \( f(\lambda) \) Subject to \( \lambda \geq 0 \). At the end of this subsection it will be shown that the dual optimal value is equal to the prime optimal value and the duality gap is zero for prime and dual problems, therefore, dual problem can be solved to obtain the solution to the original problem. One way to solve the dual problem is to use the Sub-gradient method as explained in [1]. The dual problem has only one variable \( \lambda \) and can easily be solved by bisection method. Bisection method requires a sub-gradient of \( f(\delta) \) at a given point \( \delta \). Assuming that \( p_{n,k}^* \) and \( P_n^* \) achieve \( f(\delta) \) at \( \delta \), or \( f(\delta) = \mathcal{L}(p_{n,k}^*, P_n^*, \ldots, P_N^*, \lambda) \), it is easy to show that \( -\sum_{n=1}^{N} P_n^* \) is a sub-gradient of \( f(\delta) \) at \( \delta \), or for any \( \delta \geq 0 \)

\[
\frac{f(\delta) - f(\delta) \cdot \sum_{n=1}^{N} P_n^* + KP}{(\delta - \lambda) \cdot \sum_{n=1}^{N} P_n^* + KP} \geq 0
\]

Having a sub-gradient in hand, the following simple bisection algorithm can be used to solve the dual problem:

**Given** an interval \([l, u]\) containing \( \lambda^* \); a required tolerance \( \epsilon > 0 \)

**Repeat**

\( \lambda := (l + u) / 2 \)

solve the problem in (5) and return \( f(\lambda), p_{n,k}, \pi_n \) and \( P_n \)

If \( \sum_{n=1}^{N} P_n \geq KP \), \( l := \lambda \) otherwise \( u := \lambda \)

**Until** \( u - l \leq \epsilon \)

The total transmit power, \( \sum_{n=1}^{N} P_n \), varies continuously as \( \lambda \) varies. This implies that at \( \lambda = \lambda^* \), total available power is used or \( \sum_{n=1}^{N} P_n = KP \). With this condition, from equation (6), it is easy to check that the value of prime and dual functions are equal for \( p_{n,k}^* \) and optimal ordering \( \pi_n \) obtained from dual problem. In other words the duality gap is zero.

**B. Linear Pre-Filtering**

This subsection restricts the transmission and reception schemes to only linear pre/post-filtering operations. In the successive precoding scheme, each user is encoded after the encoding process for previous users is completed and the symbols are in hand. This will introduce a delay into the system which is not desirable. Another drawback of precoding scheme that makes it complicated is the non-linear quantization operation performed in Tomlinson-Harashima encoder. The simplicity of linear pre-filtering and the fact that it does not increase the system delay, makes it attractive. In linear pre-filtering, the symbol to be transmitted for user \( k \) on tone \( n \) is chosen to be a linear combination of symbols for the intended user and other users on that tone, i.e., \( x_n = F_n \tilde{x}_n = \sum_{k=1}^{K} f_{n,k}[x_{n,k}] \). Here \( f_{n,k} \in \mathbb{C}^K \) is the \( k^{th} \) column of \( F_n \) and \( [x_{n,k}] \) is the symbol to be transmitted to user \( k \). Note that \( f_{n,k}[x_{n,k}] \) can be viewed as a beam forming operation performed on the symbol for user \( k \) before it is passed through the channel. Assume \( \mathbb{E}[\{x_n, x_{n,k}^*\}] = I \) or independent symbols are to be transmitted to each user, then the total transmit power constraint can be expressed as \( \sum_{n=1}^{N} \text{Tr}(F_n F_n^*) \leq KP \). Let \( t_{n,k} \) denote the \( k^{th} \) row of the matrix \( H_n \), or \( H_n^T = [t_{n,k}^T \ldots t_{n,K}^T] \), then by using MMSE receiver for each user, the data rate equal to

\[
\frac{1}{2} \log(1 + \frac{\| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2}{1 + \sum_{k \neq k} \| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2})
\]

can be achieved for user \( k \) on tone \( n \). Therefore, throughput maximization problem can be formulated as the following optimization problem:

Maximize \( \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{1}{2} \log(1 + \frac{\| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2}{1 + \sum_{k \neq k} \| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2}) \)

Subject to \( \sum_{n=1}^{N} \text{Tr}(F_n F_n^*) \leq KP \) \hspace{1cm} (10)

where the variables are \( f_{n,k} \in \mathbb{C}^K \). This problem is not a convex problem since \( f_{n,k} \)'s appear both in the numerator and denominator of fractions in logarithmic functions. However several modifications can be made to solve this problem approximately. One reasonable modification is to restrict the rates for each user on each tone to be integer values, since in practice only integer values can be chosen as data rate of uncoded symbols. Let \( R_{n,k} = \lfloor \frac{1}{2} \log(1 + \frac{\| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2}{1 + \sum_{k \neq k} \| t_{n,k}f_{n,k}f_{n,k}^* t_{n,k}^* \|^2}) \rfloor \), then the problem in (10) will be

Maximize \( \sum_{n=1}^{N} \sum_{k=1}^{K} R_{n,k} \)

Subject to \( \sum_{n=1}^{N} \text{Tr}(F_n F_n^*) \leq KP \) \hspace{1cm} (11)

Since the rates are restricted to be integers, to find a solution, one can search over all possible integer rates and select the one with maximum sum among those which satisfy the power constraint.

The following algorithm can be used to solve this problem:

1) Choose \( R_{n,k} \in \{0, 1, \ldots, r_{max}\} \) for all \( n \) and \( k \)
2) Find \( f_{n,k} \)'s that satisfy \( \forall n, k, (2^{2R_{n,k}} - 1)(1 + \sum_{i \neq k} \| t_{n,k}f_{n,i} \|^2) \)

and minimize

\[
\sum_{n=1}^{N} \text{Tr}(F_n F_n^*) = \sum_{n=1}^{N} \sum_{k=1}^{K} \| f_{n,k} \|^2
\]
3) If \( \sum_{n=1}^{N} \sum_{k=1}^{K} f_{nk}^2 \leq KP \) mark \( R_{nk} \) and \( F_n \) as the ones that satisfy power constraint, otherwise discard them.
4) Search for all possible values for \( R_{nk} \) that satisfy the power constraint.
5) Among all integer rates, choose the one with maximum sum.

\( f_{nk} \)'s in the second step can be found by solving an optimization problem. This problem can be cast as a Second Order Cone Programming (SOCP) [1]. To see this, note that if \( f_{nk} \)'s satisfy (12) then so do \( e^{\theta_{nk}} f_{nk} \) for any arbitrary \( \theta_{nk} \) and the total transmit power is maintained, thus without loss of generality assume that \( t_{nk} f_{nk} \in \mathbb{R} \). Using this assumption the problem can be reformulated to the following SOCP:

\[
\text{Minimize } t \\
\text{Subject to } \\
\|v_{nk}\| \leq \frac{1}{\sqrt{(2^{2R_{nk}} - 1)}} t_{nk} f_{nk} \quad \forall k, n \\
\exists (t_{nk} f_{nk}) = 0 \\
1 \leq \|f_{nk}^T \cdots f_{K,k}^T f_{nk}^T \cdots f_{N,k}^T f_{K,k}^T f_{nk}^T \| \leq t \\
v_{nk} = [t_{nk} f_{n1} \cdots t_{nk} f_{n(k-1)} \cdots t_{nk} f_{nK}]
\]

Therefore, the optimization problem in (10) can be solved by solving a series of SOCPs as suggested in the algorithm above. But this requires to search over \((r_{max} + 1)^{NK}\) possible values for rates and solve the corresponding SOCP in order to check the power constraint. As can be seen, the complexity of the algorithm increases exponentially with the number of users and the number of tones and makes the algorithm intractable for large values of \( N \) or \( K \). Next a dual decomposition approach is followed to decrease the complexity from \( O((1 + r_{max})^{NK}) \) to \( O((1 + r_{max})^K) \). This will make the problem tractable for moderate values of \( K \) and large values of \( N \), since the complexity grows linearly with \( N \). However, the complexity still grows exponentially with number of users \( K \).

The power constraint in (10) can be reformulated as

\[
\text{Tr}(F_n F_n^†) \leq P_n \quad \forall n = 1, \ldots, N \\
\sum_{n=1}^{N} P_n \leq KP.
\]

Let \( \mathcal{L}(f_{nk}, P_1, \ldots, P_N, \lambda) \) be the Lagrangian associated with problem (11) with only explicit inequality given in (14) and \( f(\lambda) \) for \( \lambda \geq 0 \) denote the underlying Lagrange dual function:

\[
\mathcal{L}(f_{nk}, P_1, \ldots, P_N, \lambda) = \sum_{n=1}^{N} \left( \sum_{k=1}^{K} R_{nk} - \lambda P_n \right) + \lambda KP.
\]

where \( R_{nk} = \left[ \frac{1}{K} \log(1 + \frac{\|f_{nk}\|^2}{\lambda}) \right] \) is restricted to \( R_{nk} \in \{1, 2, \ldots, r_{max}\} \) and corresponding Lagrange dual function:

\[
f(\lambda) = \sup_{f_{nk}, P_n, \text{Tr}(F_n F_n^†) \leq P_n} \mathcal{L}(f_{nk}, P_1, \ldots, P_N, \lambda).
\]

Lagrange dual function is convex by construction and can be used to calculate an upper bound on maximum throughput. For the Lagrange dual function, the constraints are independent on each tone and dual function calculation can be decomposed into \( N \) independent maximization problems:

\[
f(\lambda) = \sum_{n=1}^{N} f_n(\lambda) + \lambda KP
\]

where for \( n = 1, \ldots, N \):

\[
f_n(\lambda) = \max_{f_{nk}, P_n, \text{Tr}(F_n F_n^†) \leq P_n} \sum_{k=1}^{K} R_{nk} - \lambda P_n.
\]

Since rates belongs to a finite set and the constraint set is closed and compact, sup is replaced by max in (17). This problem can be solved using similar search approach introduced earlier to solve problem (11); For all possible values of \( R_{nk} \in \{0, 1, \ldots, r_{max}\} \), \( k = 1, 2, \ldots, K \), maximize the cost function subject to given constraints and choose the rate tuple corresponding to maximum value of cost function. It is easy to verify that for a given set of integer values for \( R_{nk}, k = 1, 2, \ldots, K \), the cost function in (17) will be maximized for \( P_n = \text{Tr}(F_n F_n^†) = \sum_{k=1}^{K} ||f_{nk}||^2 \) and \( f_{nk} \)'s, that are solutions to the following SOCP problem:

\[
\text{Minimize } t \\
\text{Subject to } \\
||v_{nk}|| \leq \frac{t_{nk} f_{nk}}{\sqrt{(2^{2R_{nk}} - 1)}} \quad \forall 1 \leq k \leq K \\
||f_{n1}^T \cdots f_{K,k}^T f_{nk}^T \cdots f_{N,k}^T f_{K,k}^T f_{nk}^T || \leq t \\
v_{nk} = [t_{nk} f_{n1} \cdots t_{nk} f_{n(k-1)} \cdots t_{nk} f_{nK}]
\]

\( f_{nk} \)'s obtained from above SOCP will minimize \( \sum_{k=1}^{K} ||f_{nk}||^2 \), equivalently they will maximize the cost function \( \sum_{k=1}^{K} R_{nk} - \lambda \sum_{k=1}^{K} ||f_{nk}||^2 \), they also satisfy the equality constraints \( ||f_{nk}||^2 = (2^{2R_{nk}} - 1)(1 + \sum_{i\neq k} ||f_{nk}||^2) \), \( k = 1, \ldots, K \), which are required for integer rate constraints. Thus the solution to problem (17) can be computed by finding the maximum cost for all possible integer rates and choosing the best one; in other words solving SOCP (18) for all \((r_{max} + 1)^K\) possible rates and choosing the maximum provides the optimal solution to (17). Thus the Lagrange dual function in (16) can be calculated by solving \( N(r_{max} + 1)^K \) SOCPs independently on different tones. Also note that the SOCP in (13) has \( NK^2 \) variables and \( NK + 1 \) inequality constraints while the one in (18) has \( K^2 \) variables with \( K + 1 \) inequality constraints, therefore it is much easier to solve compare to problem (13).

The dual problem, Minimize \( f(\lambda) \) Subject to \( \lambda \geq 0 \), can be solved to obtain an upper bound to optimal value of problem (11). Note that this problem is a convex problem and can be solved by bi-section method. Let \( f^*_n \) and \( P^*_n \) achieve \( f(\lambda) \) at \( \lambda \), then \( -\sum_{n=1}^{N} P^*_n - KP \) is a sub-gradient of \( f(\lambda) \) at \( \lambda \). Hence the bi-section algorithm introduced earlier can be employed.
to solve the dual problem using the provided sub-gradient. However, in this case $\sum_{n=1}^{N} P_n$ can only take values on a discrete set of numbers, therefore near optimal $\lambda^*, \sum_{n=1}^{N} P_n - K P$ alternates between two neighboring positive and negative discrete values and will not be zero. This phenomena may cause a problem in the bi-section algorithms; when the bi-section algorithm is terminated, $\sum_{n=1}^{N} P_n$ may be greater than $K P$. But this problem can be easily fixed by terminating the algorithm whenever both conditions hold, i.e. $\sum_{n=1}^{N} P_n \leq K P$ and $u - l \leq \epsilon$. Although the duality gap is not zero in this case, the dual problem provides an upper bound on the prime one and optimal variables $F_{nk}$ serve as an approximate optimal solution to the prime problem within accuracy of $\lambda^* \left( K P - \sum_{n=1}^{N} F_n F_n^\dagger \right)$.

**Remarks:** Proposed linear pre-filtering scheme with discrete rate constraints can be employed to maximize any weighted sum of the rates, $\sum_{k} \mu_k R_k$ with $\mu \succeq 0$ to characterize an achievable rate region for parallel gaussian vector broadcast channels using only linear transmitter and receivers.

III. SIMULATION RESULTS

The Simulation part considers a 2 by 2 Frequency selective Rayleigh fading channel corresponding to two transmit antennas and $K = 2$ users, each with one receive antenna. An exponential distribution is assumed for Multi-path Intensity profile with coherence bandwidth about 0.1 times the signal bandwidth. $\nu = 16$ taps of the channel are considered for simulations. $N = 64$ input and output samples is chosen for FFT and IFFT size. The throughput obtained by optimal ordering is compared to data rate of fixed ordering in Figure 1. In this figure the data rate is plotted versus SNR values. As can be seen, significant increase in data rate is obtained by using optimal ordering over fixed ones.

Figure 2 illustrates the performance of Linear pre-filtering scheme compared to performance of QR precoding with optimal ordering and also optimal “dirty paper” precoding. This figure shows that QR precoding with optimal ordering performs almost as well as optimal “dirty paper” coding. Although the linear pre-filtering scheme is simple, this figure illustrates that it performs nearly close to optimal and even it outperforms the QR precoding with the fixed ordering.

IV. SUMMARY

Two sub-optimal transmission schemes for a family of parallel Gaussian vector broadcast channels are considered in this paper. An efficient algorithm for obtaining the best user ordering and channel input covariance matrix, maximizing the total channel throughput, is proposed. It is shown that this method has linear complexity in terms of the number of sub-carriers.

The design of a linear transmitter and receivers architecture that maximizes the total throughput is studied. It is shown that under integer bit-loading assumption for each user, this problem reduces to solving a series of Second Order Cone Programming problems. Using the dual decomposition method, the complexity of this scheme can be reduced significantly.

**REFERENCES**