Novel PMEPR Control Approach for 64- and 256-QAM Coded OFDM Systems

Scott C.-H. Huang, Member, IEEE, Hsiao-Chun Wu, Senior Member, IEEE, and John M. Cioffi, Fellow, IEEE

Abstract—Orthogonal frequency division multiplexing (OFDM) is a prevalent telecommunication technology to mitigate multipath distortion with high-order modulations such as quadrature amplitude modulation (QAM). However, uncoded OFDM systems also have a serious drawback of high peak-to-mean envelope power ratio (PMEPR). On the other hand, coded OFDM systems can reduce the PMEPR problem but often lead to low code rates. There is thus a tradeoff between PMEPR and code rate in the design of OFDM systems. In this paper, PMEPR reduction for OFDM 64- and 256-QAM sequences is comprehensively studied. Four new families of 64-QAM sequences and seven new families of 256-QAM sequences are proposed to achieve the lowest PMEPR, the highest code rate, or the tradeoffs between these two metrics. Through the comparison with all other OFDM 16- or 64-QAM sequences, these new families of OFDM sequences can facilitate higher code rates. Furthermore, many of these new sequences have lower PMEPR than other OFDM sequences. Adjustment of the tradeoff between PMEPR and code rate can be made to meet the stringent demand in low PMEPR or the need for high code rate subject to various system requirements. Moreover, the construction method of the proposed new sequences is quite simple.

Index Terms—Peak-to-mean envelope power ratio (PMEPR), orthogonal frequency division multiplexing (OFDM), quadrature amplitude modulation (QAM), Golay sequences.

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is a multi-carrier modulation scheme, in which a large number of closely-spaced orthogonal sub-carriers are used to carry data. OFDM is an outstanding telecommunication technology for mitigating the effects of the multi-path delay spread [1]. Hence, it has been adopted for many kinds of applications in wireless systems, such as wireless local-area networks (WLAN) [2] and digital video broadcasting [3]. OFDM is widely used along with quadrature amplitude modulation (QAM) (i.e. each sub-carrier is modulated with QAM symbols), and therefore information data can be represented as sequences of complex numbers, called OFDM QAM sequences.

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promising for the future OFDM transmission systems.

The rest of paper is organized as follows. The problem statements of PMEPR and code rates for OFDM are introduced in Section II. The mathematical descriptions about Golay sequences and Golay complementary pairs are given in Section III. The new OFDM 64-QAM sequences and OFDM 256-QAM sequences are presented in Sections IV and V, respectively. The corresponding code rate analysis is given in Section VI. Final concluding remarks will be drawn in Section VII. Due to the page limit of IEEE Globecom Conference, all proofs of the theorems and lemmas are omitted in this paper.

II. PROBLEM FORMULATION

Throughout this paper, the sets of natural numbers, integers, real numbers, as well as complex numbers are denoted by \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \), respectively. Besides, \( i \equiv \sqrt{-1} \) and \( g^* \) represents the complex conjugate of \( g \). Suppose that an OFDM system has \( n \) sub-carriers, where the transmitted information symbols are represented as a vector in an \( n \)-dimensional complex vector space \( \mathbb{C}^n \). Given \( \vec{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n \), the conjugate vector \( \vec{v}^* \) is defined as \( \vec{v}^* = (v_1^*, \ldots, v_n^*) \). Let \( \vec{u}, \vec{v} \in \mathbb{C}^n \) and \( \vec{u} = (u_1, \ldots, u_n), \vec{v} = (v_1, \ldots, v_n) \). The Euclidean inner product \( \vec{u} \cdot \vec{v} \) of \( \vec{u} \) and \( \vec{v} \) is defined as \( \vec{u} \cdot \vec{v} = u_1v_1^* + u_2v_2^* + \ldots + u_nv_n^* \). Vector space \( \mathbb{C}^n \) along with the Euclidean inner product satisfy the requirement of a Complex Inner Product Space, or sometimes called Unitary Space, described as follows. Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n \) and \( s \in \mathbb{C} \). Then the following properties hold: \( \vec{u} \cdot \vec{v} = (\vec{v} \cdot \vec{u})^* \), \( (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \), \( (s\vec{u}) \cdot \vec{v} = s(\vec{u} \cdot \vec{v}) \), \( \vec{u} \cdot (s\vec{v}) = s^* (\vec{u} \cdot \vec{v}) \). \( \vec{u} \cdot \vec{u} = 0 \) if and only if \( \vec{u} = 0 \). The Euclidean norm (or length) \( \| \vec{u} \| \) of \( \vec{u} \in \mathbb{C}^n \) is defined as \( \| \vec{u} \| = \sqrt{\vec{u} \cdot \vec{u}} \). The Euclidean distance between \( \vec{u}, \vec{v} \in \mathbb{C}^n \) is \( \| \vec{u} - \vec{v} \| \).

Each vector can also be regarded as a codeword. Let the set of all viable code-word vectors \( S \) be a subset in \( \mathbb{C}^n \). Throughout this paper, the following naming convention is used for all vectors and their corresponding coefficients: Given a vector \( \vec{c} \in S \), \( c_k \) always denotes the projection of its \( (k+1) \)-th dimension, for \( 0 \leq k < n \). In other words, \( \vec{c} = (c_0, \ldots, c_{n-1}) \). Now the complex-valued transmitted OFDM signal \( S_c(t) \) is given by

\[
S_c(t) = \sum_{k=0}^{n-1} c_k e^{j2\pi(f_k t + kf_s) t}, \quad \text{for} \quad 0 \leq t \leq \frac{1}{f_s},
\]

where \( f_r \) is the carrier frequency and \( f_s \) is the bandwidth of each tone. Its peak envelope power (PEP) is defined as

\[
\text{PEP}(\vec{c}) = \max_{0 \leq t \leq 1/f_s} |S_c(t)|^2 .
\]

The peak envelope power for the subset \( S \) is defined as

\[
\text{PEP}(S) \equiv \max_{\vec{c} \in S} \text{PEP}(\vec{c}).
\]

Assume that codewords are uniformly transmitted over \( S \) (all codewords are equally probable) and the probability for transmitting any codeword is \( 1/|S| \) (\( |S| \) represents the number of codewords in \( S \)). Thus, the mean envelope power \( P_{av}(S) \) can be calculated as

\[
P_{av}(S) \equiv \mathbb{E}_{\vec{c} \in S} \{ |\vec{c}|^2 \} = \frac{1}{|S|} \sum_{\vec{c} \in S} |\vec{c}|^2,
\]

where \( \mathbb{E} \{ \} \) denotes the statistical expectation. The PMEPR for the subset \( S \) is defined as

\[
\text{PMEPR}(S) \equiv \frac{\text{PEP}(S)}{P_{av}(S)} .
\]

PMEPR is a very important metric to determine the implementation difficulty for the corresponding OFDM system. However, the existing schemes mitigating the PMEPR problem would lower the code rates as a trade-off. In this paper, this code rate concern is addressed too. PMEPR and code rate are the two metrics used to evaluate OFDM systems. However, the term code rate may have slightly different definitions in different contexts. In [5], the code rate (referred to as \( CR(1) \) here) was defined as

\[
CR(1)(S) \equiv \log_2 |S| \frac{1}{n} .
\]

Nevertheless, in other coding theory texts such as [8], [9], the code rate (referred to as \( CR(2) \) here) is often defined as

\[
CR(2)(S) \equiv \log_2 |S| \frac{1}{n} ,
\]

where \( Q \) is the size of the QAM constellation (i.e. for 64-QAM, \( Q = 64 \)). Note that \( CR(2)(S) \) also represents the ratio of the number of message symbols to the total number of symbols. According to this definition, the number of message symbols cannot exceed the total number of symbols, so \( 0 \leq CR(2)(S) \leq 1, \forall S \). If \( CR(2)(S) = 1 \), then there is no redundancy. Since \( n \) is fixed, \( CR(1)(S) \) and \( CR(2)(S) \) depend solely on the number of codewords \( |S| \). This metric \( |S| \) turns out to be one of the most important metrics in evaluating the OFDM system performance.

Now, our problem is stated as follows. Given \( n, f_r, f_s \), the objective is to design an appropriate subset \( S \subset \mathbb{C}^n \) such that PMEPR \((S) \) is reduced and \( |S| \) is enlarged compared to other existing schemes.

III. GOLAY SEQUENCES AND GOLAY COMPLEMENTARY PAIRS

To facilitate the background for our proposed new OFDM sequences, in this section, a brief introduction to Golay Sequences (GSs) and Golay Complementary Pairs (GCPs) over \( I \) is provided, where \( I \equiv \{1, -1, i, -i\} \). The underlying concepts using the symbolic system given in Section II will also be expressed. A sequence of \( n \) complex numbers can be deemed as a vector in \( \mathbb{C}^n \). From now on, sequences and vectors are regarded as synonyms. A sequence (or a vector) is called a GS if it is a member of GCP (i.e. if there exists another sequence such that together they form a GCP) [10]. To simplify later mathematical expressions regarding GS and GCP, the notations shown in Table I will be used extensively hereafter.

**Lemma 1:** Suppose \( \vec{u}, \vec{v} \in \mathbb{C}^n \) and \( \vec{\omega} = (1, \omega, \ldots, \omega^{n-1}) \in \mathbb{C}^n \) with \( |\omega| = 1 \). Then \( \langle \vec{u}, \vec{v} \rangle \in \mathbb{GCP} \) if and only if

\[
|\vec{u} \cdot \vec{\omega}^*|^2 + |\vec{v} \cdot \vec{\omega}^*|^2 = 2n, \quad \forall \vec{\omega} .
\]
Although Golay sequences and Golay complementary pairs possess important mathematical properties, the existence for such GSs of an arbitrary length $n$ in general is still unknown [11]. Hence, an arbitrary-length GS is hard to construct thereby. Recently, Davis and Jedwab found that GSs and GCPs of length $2^n$ for any $m \in \mathbb{N}$ can be constructed as second-order Reed-Muller codes [10]. The work in [10] provides a concrete construction method for GSs and GCPs of length $2^n$, which leads to some mathematical properties for our later use. Nevertheless, this method does not guarantee that all GSs or GCPs of length $2^n$ can be constructed. In fact, Li et al showed that there exist some Golay sequences of length $2^n$ for some $m$ which cannot be constructed using Davis and Jedwab’s method [12], [13]. In other words, Davis and Jedwab’s sequences/complementary pairs form a subset of GSs/GCPs, and they will be called Golay-Davis-Jedwab sequences/complementary pairs (GDJS/GDJCPs) hereafter.

**Lemma 2:** Suppose $\vec{u} \in \mathcal{G}_S$. Let $\vec{\omega} = (1, \omega, \ldots, \omega^{n-1}) \in \mathbb{C}^n$ with $|\omega| = 1$. Then

$$|\vec{u} \cdot \vec{\omega}|^2 \leq 2n.$$  

**Lemma 3:**

$$\sum_{\vec{u} \in \mathcal{G}_S} \vec{u} = \vec{0}. \quad (10)$$

**Lemma 4:**

$$\sum_{(\vec{u}, \vec{v}) \in \mathcal{G}_C} \vec{u} \cdot \vec{v} = 0. \quad (11)$$

**IV. OFDM 64-QAM SEQUENCE DESIGN**

The 64-QAM constellation set $\mathcal{G}_6 \subset \mathbb{C}$ is defined as

$$\mathcal{G}_6 = \left\{ e^{i\pi/4} (4a + 2b + p) \bigg| a, b, p \in \{0, 1\} \right\},$$

in which the subscript ‘6’ represents the exponent for a radix-2 number $2^6 = 64$. Throughout this paper, only the sequence length $n = 2^6$ is considered due to the availability of the GS construction method. Two new families of OFDM 64-QAM sequences, $\mathcal{G}_6^P$ and $\mathcal{G}_6^R$, are proposed, in which the superscripts ‘P’ and ‘R’ represent optimal ‘PMEPR’ and optimal code ‘rate’, respectively. Note that $\mathcal{G}_6^P$ as well as the associated preliminary results were already introduced in our earlier work [14]. There is trade-off between low PMEPR and high code rate in the design of OFDM sequences. Note that for the QAM constellation size $2^6$, 2-QAM Type-P sequences are the sequences in $\mathcal{G}_6^P$, which have the lowest PMEPR over $\mathcal{G}_6$, while 2-QAM Type-R sequences are those in $\mathcal{G}_6^R$, which have the highest code rate over $\mathcal{G}_6$. In addition to these two main families, there are still other types of OFDM 64-QAM sequences that balance PMEPR and code rate. Due to the page limit, only the construction methods and the PMEPR/code-rate measures for other types (Type-M) of OFDM 64-QAM sequences are presented.

**A. 64-QAM Type-P Sequences: Minimizing PMEPR**

Type-P sequences are designed to minimize PMEPR. According to Table I, the family of OFDM 64-QAM Type-P sequences $\mathcal{G}_6^P$ is defined as follows:

$$\mathcal{G}_6^P \sim \left\{ \vec{G}^{\sqrt{21}}_6 = \left\{ e^{i\pi/4} (4a + 2b + p) \bigg| a, b, p \in \{0, 1\} \right\} \right\},$$

in which the subscript ‘6’ represents the Cartesian product $\mathcal{G}_6 \times \ldots \times \mathcal{G}_6$. Its corresponding PMEPR upper-bound is presented in the following theorem.

**Theorem 1:**

$$\text{PMEPR}(\mathcal{G}_6^P) \leq 2 + \frac{8}{21} \sqrt{5} \simeq 2.85.$$  

The proof of this theorem can be found in Corollary 1 of [14].
B. 64-QAM Type-R Sequences: Maximizing Code Rate

The family of Type-R sequences $G^R_6$, which can maximize the code rate, is defined as follows:

$$G^R_6 \overset{\text{def}}{=} \{ \bar{c} \in G^n_{64} \mid \bar{c} = e^{j\pi/4} \left( 4\bar{a} + 2\bar{b} + \bar{p} \right), \bar{a}, \bar{b}, \bar{p} \in G^S \}.$$  

(19)

In order to analyze the PMEPR of $G^R_6$, its PEP should be analyzed first. The PEP is characterized by the following lemma.

Lemma 9:

$$\text{PEP}(G^R_6) \leq \frac{14}{3} n \simeq 4.67n.$$  

(20)

Regarding the mean envelope power $P_{av}(G^R_6)$ defined by (4), the following lemma facilitates its exact value.

Lemma 10: The mean envelope power of $G^R_6$ is $P_{av}(G^R_6) = n$.

According to Lemma 9 and Lemma 10, the following corollary about the PMEPR of $G^R_6$ is obtained:

Corollary 1:

$$\text{PMEPR}(G^R_6) \leq \frac{14}{3} \simeq 4.67.$$  

(21)

C. Other Miscellaneous Types of 64-QAM Sequences

In addition to Type-P and Type-R sequences, many sequences of other types can be constructed using similar techniques. Although these miscellaneous sequences have different PMEPRs and code rates, due to the page limit, only their construction methods (the code rates will be given in the subsequent section) are presented. They are called Type-M sequences for naming convenience. OFDM 64-QAM Type-M sequences can be categorized into the following two families.

$$G^M_6 \overset{\text{def}}{=} \{ \bar{c} \in G^n_6 \mid \bar{c} = e^{j\pi/4} \left( 4\bar{a} + 2\bar{b} + \bar{p} \right), (\bar{a}, \bar{p}) \in G^C \cup G^S \},$$  

(22)

$$G^M_6 \overset{\text{def}}{=} \{ \bar{c} \in G^n_6 \mid \bar{c} = e^{j\pi/4} \left( 4\bar{a} + 2\bar{b} + \bar{p} \right), \bar{a} \in G^S, (\bar{b}, \bar{p}) \in G^C \}.$$  

(23)

The PMEPRs for these Type-M sequences are given by Table II.

V. OFDM 256-QAM SEQUENCE DESIGN

The 256-QAM constellation set $G_8 \subset \mathbb{C}$ is defined by

$$G_8 \overset{\text{def}}{=} \left\{ e^{j\pi/4} \left( 8a + 4b + 2p + q \right) \mid a, b, p, q \in \mathbb{I} \right\}.$$  

(24)

### TABLE II: PMEPRs of Different OFDM QAM Sequences

<table>
<thead>
<tr>
<th>Constellation Type</th>
<th>Sequence</th>
<th>PMEPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>16-QAM</td>
<td>[4]</td>
<td>3.60</td>
</tr>
<tr>
<td>16-QAM</td>
<td>[5]</td>
<td>3.60</td>
</tr>
<tr>
<td>16-QAM</td>
<td>[7]</td>
<td>3.60</td>
</tr>
<tr>
<td>64-QAM</td>
<td>[6]</td>
<td>4.66</td>
</tr>
<tr>
<td>64-QAM</td>
<td>[7]</td>
<td>4.66</td>
</tr>
<tr>
<td>64-QAM</td>
<td>$G^P_8$</td>
<td>3.60</td>
</tr>
<tr>
<td>64-QAM</td>
<td>$G^M_8$</td>
<td>3.70</td>
</tr>
<tr>
<td>256-QAM</td>
<td>$G^P_8$</td>
<td>2.94</td>
</tr>
<tr>
<td>256-QAM</td>
<td>$G^M_8$</td>
<td>5.88</td>
</tr>
<tr>
<td>256-QAM</td>
<td>$G^N_8$</td>
<td>4.45</td>
</tr>
<tr>
<td>256-QAM</td>
<td>$G^S_8$</td>
<td>4.66</td>
</tr>
<tr>
<td>256-QAM</td>
<td>$G^M_8$</td>
<td>4.87</td>
</tr>
</tbody>
</table>

A. 256-QAM Type-P Sequences: Minimizing PMEPR

According to Table I, the collection of these 256-QAM Type-P sequences, denoted by $G^P_8$, is defined as follows:

$$G^P_8 \overset{\text{def}}{=} \left\{ \bar{c} \in G^n_8 \mid \bar{c} = e^{j\pi/4} \left( 8\bar{a} + 4\bar{b} + 2\bar{p} + \bar{q} \right), \bar{a}, \bar{b}, \bar{p}, \bar{q} \in G^C \right\}.$$  

(25)

in which $G^C$ represents the Cartesian product $G^C \times \ldots \times G^C$. According to (25), the corresponding PMEPR performance and code rate are analyzed in this section. The following lemma facilitates the PEP upper-bound for $G^P_8$ given by (25).

Lemma 11:

$$\text{PEP}(G^P_8) \leq \frac{50n}{17} \simeq 2.94n.$$  

(26)

The mean envelope power of $G^P_8$ defined by (4) is studied here. According to (25), the determination of the corresponding $P_{av}(G^P_8)$ is non-trivial. As a matter of fact, in the sequence construction, $\bar{a}$ and $\bar{b}$ are not necessarily statistically independent of each other, nor are $\bar{p}$ and $\bar{q}$. The following lemma shows that, despite the lack of statistical independence, it still results in the same mean envelope power as if they were statistically independent.

Lemma 12: The mean envelope power of $G^P_8$ is $P_{av}(G^P_8) = n$.

Since Lemma 11 provides the PEP upper-bound, the corresponding PMEPR can also be bounded thereby. The following corollary facilitates an upper bound for the PMEPR of the 256-QAM sequence family $G^P_8$.

Corollary 2:

$$\text{PMEPR}(G^P_8) \leq \frac{50}{17} \simeq 2.94.$$  

(27)
TABLE III: Number of Codewords for Different OFDM QAM Sequences

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Const. Type</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
<th>General $m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[4]</td>
<td>16-QAM</td>
<td>1.41</td>
<td>0.42</td>
<td>2.21</td>
<td>0.70</td>
<td>0.19</td>
<td>not available</td>
</tr>
<tr>
<td>[4]</td>
<td>64-QAM</td>
<td>3.50</td>
<td>2.21</td>
<td>1.82</td>
<td>2.18</td>
<td>1.18</td>
<td>0.73</td>
</tr>
<tr>
<td>[7]</td>
<td>256-QAM</td>
<td>5.00</td>
<td>3.84</td>
<td>2.65</td>
<td>5.94</td>
<td>5.94</td>
<td>0.77</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>4.00</td>
<td>2.89</td>
<td>1.95</td>
<td>2.07</td>
<td>1.82</td>
<td>0.99</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>5.00</td>
<td>3.84</td>
<td>2.65</td>
<td>5.94</td>
<td>5.94</td>
<td>0.77</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>4.50</td>
<td>3.59</td>
<td>2.54</td>
<td>2.07</td>
<td>2.07</td>
<td>0.77</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>5.00</td>
<td>3.84</td>
<td>2.65</td>
<td>5.94</td>
<td>5.94</td>
<td>0.77</td>
</tr>
</tbody>
</table>

TABLE IV: Code Rate $CR_1$ for Different OFDM QAM Schemes

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Const. Type</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[5]</td>
<td>16-QAM</td>
<td>0.70</td>
<td>0.48</td>
<td>0.31</td>
<td>0.19</td>
<td>0.11</td>
</tr>
<tr>
<td>[6]</td>
<td>64-QAM</td>
<td>0.62</td>
<td>0.40</td>
<td>0.24</td>
<td>not available</td>
<td></td>
</tr>
<tr>
<td>[7]</td>
<td>256-QAM</td>
<td>0.65</td>
<td>0.41</td>
<td>0.25</td>
<td>0.15</td>
<td>0.09</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>64-QAM</td>
<td>0.38</td>
<td>0.44</td>
<td>0.30</td>
<td>0.20</td>
<td>0.12</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>64-QAM</td>
<td>0.75</td>
<td>0.60</td>
<td>0.42</td>
<td>0.28</td>
<td>0.18</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.50</td>
<td>0.36</td>
<td>0.24</td>
<td>0.16</td>
<td>0.10</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.63</td>
<td>0.48</td>
<td>0.33</td>
<td>0.22</td>
<td>0.14</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.75</td>
<td>0.60</td>
<td>0.42</td>
<td>0.28</td>
<td>0.18</td>
</tr>
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</table>

TABLE V: Code Rate $CR_2$ for Different OFDM QAM Schemes

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Const. Type</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[5]</td>
<td>16-QAM</td>
<td>0.70</td>
<td>0.48</td>
<td>0.31</td>
<td>0.19</td>
<td>0.11</td>
</tr>
<tr>
<td>[6]</td>
<td>64-QAM</td>
<td>0.62</td>
<td>0.40</td>
<td>0.24</td>
<td>not available</td>
<td></td>
</tr>
<tr>
<td>[7]</td>
<td>256-QAM</td>
<td>0.65</td>
<td>0.41</td>
<td>0.25</td>
<td>0.15</td>
<td>0.09</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>64-QAM</td>
<td>0.38</td>
<td>0.44</td>
<td>0.30</td>
<td>0.20</td>
<td>0.12</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>64-QAM</td>
<td>0.75</td>
<td>0.60</td>
<td>0.42</td>
<td>0.28</td>
<td>0.18</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.50</td>
<td>0.36</td>
<td>0.24</td>
<td>0.16</td>
<td>0.10</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.63</td>
<td>0.48</td>
<td>0.33</td>
<td>0.22</td>
<td>0.14</td>
</tr>
<tr>
<td>$G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$, $G_{S}^{M_{1}}$</td>
<td>256-QAM</td>
<td>0.75</td>
<td>0.60</td>
<td>0.42</td>
<td>0.28</td>
<td>0.18</td>
</tr>
</tbody>
</table>

B. 256-QAM Type-R Sequences: Maximizing Code Rate

According to Table I, the collection of the 256-QAM Type-R sequences, denoted by $G_{R}^{8}$, is defined as follows:

$$G_{R}^{8} = \left\{ \hat{c} \in G_{8}^{8} \mid \hat{c} = \frac{e^{i\pi/4}}{\sqrt{8}} \left( 8\bar{a} + 4\bar{b} + 2\bar{p} + \bar{q} \right) \right\}. \tag{28}$$

According to (28), the corresponding PMEPR performance (the code rates are given in Section VI) is analyzed. The following theorem facilitates the upper bound of PEP for $G_{R}^{8}$ defined by (28).

The following lemma facilitates the upper bound of PEP for $G_{R}^{8}$ defined by (28).

**Lemma 13:**

$$\text{PEP}(G_{R}^{8}) \leq \frac{100m}{17} \simeq 5.88n. \tag{29}$$

The mean envelope power of $G_{R}^{8}$ as defined by (4) is presented in the following lemma.

**Lemma 14:** The mean envelope power of $G_{R}^{8}$ is $P_{av}(G_{R}^{8}) = n$. Since the proof for Lemma 14 is very similar to that for Lemma 10, the proof is omitted due to the page limit. According to Lemma 13 and Lemma 14, the following corollary about the PMEPR of $G_{R}^{8}$ is obtained.

**Corollary 3:**

$$\text{PMEPR}(G_{R}^{8}) \leq \frac{1000}{17} \simeq 5.88. \tag{30}$$

For comparison, the PMEPR upper-bounds for different OFDM QAM sequences are listed in Table II.

C. Other Miscellaneous Types of 256-QAM Sequences

Similar to the 64-QAM Type-M sequences, the 256-QAM Type-M sequence families are presented here. They can be categorized these sequences into five families. Due to the page limit, only the corresponding construction methods are listed.
as follows:

\[ \mathcal{G}_8^{M1} \triangleq \left\{ \vec{c} \in \mathcal{G}_8^n \mid \vec{c} = \frac{e^{j\pi/4}}{\sqrt{85}} \left( 8\vec{a} + 4\vec{b} + 2\vec{p} + \vec{q} \right) \right\}, \]

\[ (\vec{a}, \vec{p}), (\vec{b}, \vec{q}) \in \mathcal{GC\tilde{P}} \], \hspace{1cm} (31) \]

\[ \mathcal{G}_8^{M2} \triangleq \left\{ \vec{c} \in \mathcal{G}_8^n \mid \vec{c} = \frac{e^{j\pi/4}}{\sqrt{85}} \left( 8\vec{a} + 4\vec{b} + 2\vec{p} + \vec{q} \right) \right\}, \]

\[ (\vec{a}, \vec{q}), (\vec{b}, \vec{p}) \in \mathcal{GC\tilde{P}} \}, \hspace{1cm} (32) \]

\[ \mathcal{G}_8^{M3} \triangleq \left\{ \vec{c} \in \mathcal{G}_8^n \mid \vec{c} = \frac{e^{j\pi/4}}{\sqrt{85}} \left( 8\vec{a} + 4\vec{b} + 2\vec{p} + \vec{q} \right) \right\}, \]

\[ (\vec{a}, \vec{b}) \in \mathcal{GC\tilde{P}}, \vec{p}, \vec{q} \in \mathcal{GS} \}, \hspace{1cm} (33) \]

\[ \mathcal{G}_8^{M4} \triangleq \left\{ \vec{c} \in \mathcal{G}_8^n \mid \vec{c} = \frac{e^{j\pi/4}}{\sqrt{85}} \left( 8\vec{a} + 4\vec{b} + 2\vec{p} + \vec{q} \right) \right\}, \]

\[ (\vec{a}, \vec{p}) \in \mathcal{GC\tilde{P}}, \vec{b}, \vec{q} \in \mathcal{GS} \}, \hspace{1cm} (34) \]

\[ \mathcal{G}_8^{M5} \triangleq \left\{ \vec{c} \in \mathcal{G}_8^n \mid \vec{c} = \frac{e^{j\pi/4}}{\sqrt{85}} \left( 8\vec{a} + 4\vec{b} + 2\vec{p} + \vec{q} \right) \right\}, \]

\[ (\vec{a}, \vec{q}) \in \mathcal{GC\tilde{P}}, \vec{b}, \vec{p} \in \mathcal{GS} \}, \hspace{1cm} (35) \]

VI. CODE RATES AND TOTAL NUMBER OF CODEWORDS

For each type of the aforementioned sequence in this paper, its code rate can be easily calculated as long as the total number of codewords is available, which depends on the available GCPs and GSs. Nevertheless, how to generate a complete set of GCPs/GSs for a given length \( n = 2^m \) is still an open problem. The only practical alternative is to generate GDJCPs/GDJSSs instead. Since the exact number of GDJCPs/GDJSSs can be determined, they obviously serve as a lower bound for the total number of GCPs/GSs. The following two theorems are listed here.

**Theorem 2:** For any length \( 2^m \), there are exactly \( 2^{2m+1} m! \) GDJCPs.

**Theorem 3:** For any length \( 2^m \), there are exactly \( 2^{2m+3} m! \) GDJSSs.

Theorems 2 and 3 are special cases of Corollaries 4 and 5 in [10], respectively. According to these two theorems, the number of codewords can be easily determined. For example, one GCP and one GS are used to construct the 64-QAM sequences in \( \mathcal{G}_8^P \). Therefore, the total number of codewords for \( \mathcal{G}_8^P \) is \( 2^{2m+3} m! \times 2^{2m+1} m! = 2^{4m+4}(m!)^2 \). Note that, according to Theorems 2 and 3, each GS or each GCP contributes to \( O(2^m m!) \) number of sequences. For this reason, if two independent GSs are used (instead of one GCP), the code rate will be higher. Therefore, the more GSs adopted in our sequence construction methods, the higher the code rates. The total number of codewords for each type of sequences is shown in Table III. Since the numbers of codewords can be determined, it is trivial to calculate the corresponding code rates. The code rates \( CR_1 \) and \( CR_2 \) for each type of sequences are shown in Tables IV and V, respectively.

VII. CONCLUSION

In this paper, comprehensive studies of Golay-coded OFDM 64- and 256-QAM sequences were carried out. Golay sequences and Golay complementary pairs were used as building blocks to construct new OFDM 64-/256-QAM sequences. There is tradeoff between PMEPR and code rate for these sequences. Besides, the more Golay sequences adopted in the OFDM sequence design, the higher the corresponding code rates. Extensive comparison with all other Golay-coded 16-/64-QAM sequences has shown that these novel families of OFDM sequences achieve the highest code rates, and one can adjust the level of tradeoff between PMEPR and code rate to either decrease the PMEPR or increase the code rate subject to the system requirements.

REFERENCES


