Numerical tools for the stability analysis of 2D flows. Application to the two and four-sided lid-driven cavity.

J.M. Cadou¹‡, Y. Guevel¹, G. Girault¹,²
¹Laboratoire d’Ingénierie des Matériaux de Bretagne, Université Européenne de Bretagne, Université de Bretagne Sud, Rue de Saint Maudé, B.P. 92116, 56321 Lorient Cedex - France
²Centre de recherche des Écoles de Saint-Cyr Coëtquidan, Écoles de Coëtquidan, 56381, Guer cedex, France.
E-mail: jean-marc.cadou@univ-ubs.fr

Abstract. This paper deals with the numerical study of bifurcations in the two-dimensional lid-driven cavity. Two specific geometries are considered. The first geometry is the two-sided non facing (2SNF) cavity: the velocity is imposed on the upper and the left sides of the cavity. The second geometry is the four-sided (4S) cavity where all the sides have a prescribed motion. For the first time, the linear stability analysis is performed by coupling two specific algorithms. The first one is dedicated to the computation of the stationary bifurcations and the bifurcated branches. Then, a second algorithm is dedicated to the computations of Hopf bifurcations. In this study, for both problems, it is shown that the flow becomes asymmetric via a stationary bifurcation. The critical Reynolds numbers are close to 1070 and 130 respectively for the 2SNF and 4S cavity. Following the stationary bifurcated branches, supplementary results concerning the stability are found. Firstly, for both examples, a second stationary bifurcation appears on the unstable solution, for a Reynolds number equal to 1890 and 360, respectively, for the 2NSF and the 4S cavity. Secondly, a second stationary bifurcation is found on the stable solutions of the 4S lid-driven cavity for a critical Reynolds number close to 860. Nevertheless, no Hopf bifurcation has been found on this stable bifurcated branch for Reynolds numbers between 130 and 1000. Concerning the 2SNF lid-driven cavity, Hopf bifurcation points have been determined on these stable bifurcated solutions. The first bifurcation occurs for a Reynolds number close to 3000 and a Strouhal number equals to 0.47.

Keywords: Lid-driven cavity, numerical tools, stationary bifurcations, Hopf bifurcations

‡ Corresponding author: jean-marc.cadou@univ-ubs.fr
1. Introduction

The 2D one-sided lid-driven cavity is certainly among the most studied examples in numerical fluid mechanics - see, for example, a detailed review in Erturk (2009). This fundamental example is characterized by a complex flow evolution allowing one to easily evaluate specific numerical methods. In the case of the 2D one-sided lid-driven cavity, the flow is steady up to a Reynolds number close to 8000, and then becomes periodic in time. For this critical Reynolds number, a Hopf bifurcation appears -cf. Brezillon et al (2010), where a consequent bibliography is given on this topic.

Stability analysis is often performed for the problem of the two dimensional one-sided lid-driven cavity whereas other geometries such as the two-sided and four-sided lid-driven cavities are addressed less often. Studied for the first time by Kuhlmann et al (1997), the flow in the two-sided lid-driven cavity is set up by the motion of two walls facing opposite directions. One refers, for example, to de Vicente et al (2011) or Theofilis (2003) for a description of this problem. Recently, Wahba (2009) and Arumuga et al (2011), investigated the problems of the two-sided and four-sided cavities through the use of a numerical method. For the first problem, the motion is imposed by two non-facing sides. For the other problem, the motion is created by the four sides such that two parallel faces move at the same velocity in opposite directions. All their results show that a stationary bifurcation takes place and, consequently, the flow loses its symmetry beyond a critical Reynolds number. Their results show the multiplicity of steady solutions. While these authors did not perform a linear stability analysis, they provide accurate critical Reynolds numbers for which the flow becomes asymmetric.

The first objective of the present work is to confirm these critical Reynolds number values. The second objective is to compute and follow the bifurcated branches emanating from these singular points. Finally, from the bifurcated solutions, it is proposed to check if either a Hopf bifurcation or a second steady bifurcation occurs. This stability analysis is carried out by coupling two efficient numerical algorithms based on a perturbation method. The first one, introduced by Guevel et al (2011), makes it possible to compute accurate steady bifurcation points as well as the following of the resulting bifurcated branches. Presented by Brezillon et al (2010), the second algorithm automatically detects the Hopf bifurcation points on steady nonlinear solutions. For this study and for the first time, it is proposed to couple these two algorithms in order to perform a full linear stability analysis on a wide Reynolds number range.

This paper is organized in three sections. In section (2), the Navier-Stokes equations are recalled. Section (3) describes all the numerical tools used in this work to compute the two kind of bifurcations. In section (4), these numerical methods are applied to the two-sided and four-sided lid-driven cavity. The paper ends on concluding remarks.
2. Governing equations

The flow of an incompressible and Newtonian fluid is described by the following Navier-Stokes equations:

\[ u_t - \nu \nabla^2 u + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0 \quad \text{in} \quad (\Omega) \quad (1) \]

\[ \nabla \cdot u = 0 \quad \text{in} \quad (\Omega) \quad (2) \]

\[ u = \lambda u_d \quad \text{on} \quad (\Gamma) \quad (3) \]

where \( u \) and \( p \) are respectively the velocity vector and the pressure. The symbols \( \rho \) and \( \nu \) stand for the density and the kinematic viscosity of the fluid. On the boundary \( \Gamma \) of the fluid domain \( \Omega \), a velocity field \( \lambda u_d \) is imposed. The parameter \( \lambda \) can be identified as the Reynolds number by considering a geometrical reference length, \( Re = \frac{\lambda u_d L}{\nu} \). The previous equations are rewritten on the following form:

\[ M(\dot{U}) + L(U) + Q(U, U) - \lambda F = 0 \quad (4) \]

where \( U \) is a mixed unknown vector (i.e. \( U = \{ u, p \} \)). The operators \( M, L \) and \( Q \) are respectively the mass matrix, a linear and a quadratic operator. The operator \( L \) contains the pressure and the diffusion terms whereas the convective term is included in \( Q \). The load vector \( \lambda F \) is equivalent to the velocity field imposed on the boundary \( \Gamma \). A more precise description of these operators is given in Cadou et al (2001).

3. Stability analysis

3.1. Linear stability analysis

The stability of the flow is studied by considering the regular solution, \( U^S \), which is solution of the stationary Navier-Stokes equations:

\[ L(U^S) + Q(U^S, U^S) - \lambda^S F = 0 \quad (5) \]

The regular stationary solution, \( U^S \), is perturbed by a load vector, \( \Phi f \). The term \( f \) is a known and random vector. The unknown scalar \( \Phi \) represents the intensity of this load vector. This load vector introduces in the flow a velocity and a pressure fluctuation denoted by \( \Delta U \) and defined by:

\[ U = U^S + \Delta U \quad (6) \]

The previous equation is introduced into the Navier-Stokes equation (4) and, by neglecting second order terms in \( \Delta U \), the following linear stability equation is obtained:

\[ \mathcal{L}(U^S, \Delta U) = \Phi f \quad (7) \]

Finally, \( \Phi \) is our bifurcation indicator, and determining a numerical bifurcation point consists in finding the stationary solution \( U^S \) (i.e. the Reynolds number) for which the scalar \( \Phi \) is equal to zero.
Let it be noted that, as this study deals with two kinds of bifurcations (stationary and Hopf bifurcation), the fluctuation field $\Delta U$ is different for each type of bifurcations. Consequently, the linear stability problem, Equation (7), is also different for each kind of bifurcation. This point will be made clear in the following sections. Before presenting the numerical method used in this paper, the following unknown vector $X$ should be introduced. It is defined by as follows:

$$X = \{U^S, \lambda, \Delta U, \Phi\}$$

Finally, determining a bifurcation point consists in finding the unknown $X$ verifying the stationary Navier-Stokes equations (5) and the linear stability problem (7), and for which the scalar $\Phi$ is null. Moreover, as the number of unknowns is greater than the number of equations, an additional condition is added to obtain a well-posed problem. This additional condition is generally an orthogonality or normality condition and will be defined in the next section.

3.2. Numerical method

In this study, the Asymptotic Numerical Method is used to solve problems (5) and (7). This method has been initially introduced in a solid mechanics framework, see Cochelin (1994). This method consists in associating a perturbation method to a spatial discretization technique (for example the Finite Element Method). Hence, the unknown vector, $X$ is sought as a truncated power series of a parameter $\eta$:

$$X = \sum_{i=0}^{p} \eta^i X_i$$

where $p$ is the truncature order and $X_0$ is a known and regular solution. The order of truncature is generally chosen between 20 and 30. The previous relation is introduced into equations (5) and (7) and, by equating like power of $\eta$, we obtain the following set of linear equations written at the order of truncature equal to $p$:

$$L_t(U^S_0)U^S_p = \lambda^S_p F + FQ(U^S_l)$$

with $1 \leq l \leq (p - 1)$

$$L_t(U^S_0, \Delta U_0) \Delta U_p = \Phi_p f + FQ(U^S_l, \Delta U_l)$$

where subscript $t$ designates the tangent operators computed at the initial point $X_0$. The main advantage of using asymptotic expansions is the fact that they transform the initial nonlinear problem to a set of linear ones which all have the same tangent operator and only differ from the right-hand side. Hence, in Equations (10), the right-hand side $FQ(U^S_l)$ depends on the previous computed solutions, $U^S_l$. This remark is also true for the second right-hand side $FQ(U^S_l, \Delta U_l)$. Finally, only one matrix triangulation and ‘p’ backward-forward substitutions are needed to compute all the terms of the asymptotic expansions in (9). As previously mentioned, an additional condition is required to have a well posed problem (same number of equations and unknowns). For the two proposed indicators, the following condition is used:

$$\|\Delta U\|^2 = \|\Delta U^i_0\|^2$$
where $\Delta U_{0}^{ini}$ is the initial solution of the linear perturbation problem (10) at the order of truncature equal to zero. The initial value of the indicator, $\Phi_{0}^{ini}$, is in this case chosen equal to 1. As Equation (11) is quadratic in the unknown $U$, the use of a perturbation method is straightforward. Let us remark that this condition is used to avoid numerical instabilities, see Cadou et al (2006).

Once the terms $X_i$ have been computed, the polynomial expression is replaced by a rational equivalent one, called Padé approximants, Baker and Graves-Morris (1996), and defined by:

$$X_{Padé}^p = X_0 + \sum_{i=1}^{p-1} f_i(\eta)\eta^i X_i$$

(12)

where $f_i(\eta)$ are rational functions of the perturbation parameter $\eta$, see for example Elhage-Hussein et al (2000). As the validity range of Padé approximants is greater than the one of the polynomial approximations, the number of matrix triangulations required to compute the whole nonlinear solution $X$ is then lower with the former than with the latter. This validity domain of Padé approximants (i.e. the maximum value of the perturbation parameter, $\eta_{max}$) is easily determined by using the following criterion (see Elhage-Hussein et al (2000))

$$\delta = \frac{\|X_{Padé}^p(\eta_{max}) - X_{Padé}^{p-1}(\eta_{max})\|}{\|X_{Padé}(\eta_{max}) - X_0\|}$$

(13)

where $\delta$ represents a small user parameter which is generally chosen between $10^{-6}$ and $10^{-3}$. In Equation (13), the symbol $\|\bullet\|$ is the Euclidian norm of the vector $\bullet$. The maximum value of $\eta_{max}$ is introduced into the relation (12) and allows for the definition of a new starting point $X_0$ and then compute another part of the nonlinear solution branch. Hence, as introduced by Cochelin (1994), a continuation method based on asymptotic expansions and Padé approximants is defined.

3.3. Stationary bifurcation

The computation of the stationary solution and of the bifurcation indicator is carried out by solving the discrete form of equations (10) which are written at the order of truncature $p$:

$$K_0^p U^S = \lambda^p F - \sum_{r=1}^{p-1} Q(U^S_r, U^S_{p-r})$$

$$K_0^p \Delta U_p = \Phi_p f - \tilde{Q}(\Delta U_0, U^S_p) - \sum_{r=1}^{p-1} \tilde{Q}(\Delta U_r, U^S_{p-r})$$

(14)

where $K_0^p$ stands for the discrete tangent matrix computed at $U^S_0$. In the previous equations, the operator $\tilde{Q}(a, b)$ represents the convective term $Q(a, b) + Q(b, a)$. Two equations are added to the system (14). The first one is the additional condition (11) and the second one deals with the definition of the perturbation parameter $\eta$:

$$\eta = < u - u_0, u_1 >$$

(15)
where the symbol represents the velocity part of the mixed unknown vector \( \mathbf{U} \) and the operator \( < \cdot, \cdot > \) stands for the Euclidian scalar product. With Equation (15), the perturbation parameter, \( \eta \), is defined by the projection of the velocity increment, \( u - u_0 \), on the tangent velocity \( u_1 \). Finally, with the set of equations (11), (14) and (15), the stationary solution \( \mathbf{U}^S \), the bifurcation mode \( \Delta \mathbf{U} \) and the indicator \( \Phi \) can then be determined at each order \( p \) of the asymptotic expansions. These computations are carried out with a single matrix triangulation and \( 2 \times p \) forward and backward substitutions. In fact, the stationary solution and the linear stability problems have the same tangent operator, \( \mathbf{K}_0 \), see system (14), for all the truncature orders of the polynomial approximation. The computational cost to determine the indicator is therefore very low because this is performed in parallel with the stationary solution \( \mathbf{U}^S \).

Moreover, as the linear stability problem depends on the stationary solution (see the second equation of the system (14)), the latter is firstly determined and is then introduced in the second equation of (14) to determine the stationary indicator \( \Phi \). It should be noted that, the right-hand sides of the system (14) are easy to code due to the fact that these vectors are quite identical to the residual vectors in the classical iterative methods.

A stationary bifurcation is found when the indicator bifurcation, \( \Phi \), is equal to zero. So with the Padé approximants definition, Equation (12), a stationary bifurcation corresponds to a root of the numerator. Thus, one has to numerically compute these roots, denoted by \( \eta_r \), and to check if they are really solutions of the Navier-Stokes equations. This is simply realized by introducing the value \( \eta_r \) into the relation (12). The stationary part of this latter, \( \mathbf{U}^S \) and \( \lambda \), is then introduced into the stationary Navier-Stokes equations (5) to compute the classical residual vector. If the norm of this latter is lower than a given accuracy (typically \( 10^{-4} \)) then the root \( \eta_r \) corresponds to a bifurcation point.

Once the bifucation point is determined, it is proposed to compute the bifurcated stationary branches emanating from this point. This is done by using a method firstly introduced in solid mechanics framework, see Vannucci et al (1998), and applied to the Navier-Stokes equations in Guevel et al (2011). These computations are also carried out by using a perturbation method. In this case, the major difficulty is to triangulate the tangent matrix at the singular point (or bifurcation point). By using an augmented system, the bifurcated branches can be computed. Another key-point is the fact that, for these computations, the left eigenmode is required in fluid mechanics, see Seydel (1994). Finally, two matrix triangulations are needed, one for the computation of the left eigenmode and a second one for the augmented system, to compute a first part of the four bifurcated branches emanating from the singular point. The following part of the bifurcated curves is carried out by using the perturbation method as previously defined. For each branch, Hopf bifurcation points can be determined by using the numerical method defined in the next section.
3.4. Hopf bifurcation

In the case of Hopf bifurcation points, the fluctuation $\Delta U$ is defined by the following expression:

$$\Delta U(x, t) = V(x) \exp(i\omega t)$$

(16)

where $V(x)$ is a complex vector and $\omega$ represents the angular frequency. By introducing the relation (16) into the linear stability problem (7), we obtain a complex linear ones. The latter is solved with the perturbation method in which the perturbation parameter is the angular frequency $\omega$. The additional condition for the Hopf indicator is the same as the one used for the stationary bifurcation, Eq. (11). The vectors $\Delta U^{\text{ini}}_0$, $\Delta U$ are replaced respectively with $V^{\text{ini}}_0$, $V$. The vector $V^{\text{ini}}_0$ is solution of the linear stability problem where the angular frequency is equal to zero and $\Phi^{\text{ini}}_0=1$. To avoid great size problems, the stationary Navier-Stokes are firstly solved. The velocity vector $U^S$ is the resulting solution and is fixed in problem (10). Hence, the Hopf bifurcation indicator is computed for a fixed value of the Reynolds number: $\Phi(\omega)^{\text{Re fixed}}$. For each Reynolds number, an indicator curve is carried out and a bifurcation point is found for any value of the angular frequency that cancels the indicator. Nevertheless, the determination of accurate bifurcation points requires a lot of computed indicator curves. As for each curve, a great number of continuation steps (and consequently a great number of matrix triangulations) is needed, the computational time to determine accurate bifurcation points can be important. To avoid this, it has been proposed to link the indicator computations with a Newton method (Jackson (1987)), see Brezillon et al (2010). In fact, the minima of the indicator curve, at a chosen Reynolds number, are introduced as initial guesses to the Newton method. This method - referred to as 'hybrid method' - can provide several Hopf bifurcation points even if the indicator curve is computed far from the first critical Reynolds number.

4. Numerical tests

4.1. Results with the proposed algorithms

The previous methods are now applied to the example of the two-dimensional lid-driven cavity. The geometry and the boundary conditions are given in Fig. 1. Two numerical examples are considered. The first one is the two-sided non facing lid-driven cavity (denoted by 2SNF) and the second one is the four-sided lid-driven cavity (denoted by 4S). For these examples, the mixed unknown vector $U$ is defined by $U = \{u, v, p\}$. $u$ and $v$ are the velocity components in the direction $x$ and $y$ respectively and $p$ is the pressure. The spatial discretization is performed by using the finite element method. The continuity equation is modified by introducing a penalized pressure term. The finite element is a quadrilateral one with 9 nodes for the velocity (quadratic interpolation) and three nodes for the pressure (linear interpolation), see Zienkiewicz and Taylor (1991). For these examples, the Reynolds number is computed from the following relation:
Boundary condition for 2SNF:
\[ U(x, y) = (1, 0, 0) \text{ on } \Gamma_1 \]
\[ U(x, y) = (0, 0, 0) \text{ on } \Gamma_2 \text{ and } \Gamma_3 \]
\[ U(x, y) = (0, -1, 0) \text{ on } \Gamma_4 \]

Boundary condition for 4S:
\[ U(x, y) = (1, 0, 0) \text{ on } \Gamma_1 \]
\[ U(x, y) = (0, +1, 0) \text{ on } \Gamma_2 \]
\[ U(x, y) = (-1, 0, 0) \text{ on } \Gamma_3 \]
\[ U(x, y) = (0, -1, 0) \text{ on } \Gamma_4 \]

Figure 1. Geometry and boundary conditions for the lid driven cavity

Table 1. Critical Reynolds numbers for the first stationary bifurcation in the lid driven cavity (LDC).

<table>
<thead>
<tr>
<th>Author</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2SNF</td>
<td>This study</td>
<td>1073.02</td>
<td>1065.25</td>
</tr>
<tr>
<td>Wahba (2009)</td>
<td>1073</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4S</td>
<td>This study</td>
<td>130.83</td>
<td>130.41</td>
</tr>
<tr>
<td>Wahba (2009)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ Re = \frac{u_d L}{\nu} \] with \( u_d \) stands for the imposed velocity on the side \( \Gamma_1 \) of the cavity, \( L \) represents the width of the cavity and \( \nu \) is the kinetic viscosity. We also introduce the Strouhal number defined by the following expression: \( St = \frac{L \omega}{2 \pi \lambda u_d} \) where \( \omega \) designates the angular frequency. Three meshes are considered for the two geometries and the corresponding number of degrees of freedom are summarized in Table (1). In this table, we also indicate the Reynolds numbers corresponding to the first stationary bifurcation obtained with these three meshes and for both examples. For the 2SNF, the first stationary bifurcation is found for a Reynolds number between \([1060, 1070]\) according to the considered mesh. For the 4S example, this critical Reynolds number is close to 129. These two critical values agree well with the results obtained by Wahba (2009), see Table (1). In Figure (2(a)), the evolution of the bifurcation indicator, \( \Phi \), is plotted versus the Reynolds number for both examples. These two curves show that the indicator is null for the critical Reynolds summarized in Table (1). The ”sharp” behaviour of the indicator curves close to the critical Reynolds numbers is due to the additionnal condition (11). The singular points are computed by automatically determining the roots of the numerator of the Padé approximants (Eq. 12). These roots are denoted by \( \eta_c \). By introducing these values into the definition of the Padé approximants (Eq. 12),
one can obtain on the one hand accurate values for the critical Reynolds numbers and on the other hand the bifurcation mode at the singular points. For both examples and for the critical values given in Table (1), the streamlines of the stationary bifurcation mode are plotted in Fig. (3). Once these singular points are computed, one can follow the resulting bifurcating branches. This is done by using the numerical method introduced in Guevel et al (2011). Hence, in Figure (4), these nonlinear branches are plotted for both examples. With this method, one can compute the three bifurcating branches - the two asymmetric ones and the symmetric one - without introducing any perturbation parameter to jump into the branches. The velocity streamlines for some given Reynolds numbers are drawn in Fig. (5) and Fig. (6) respectively for the 2SNF and the 4F cases. These velocity streamlines correspond well to the results shown in Wahba (2009) and Arumuga Perumal and Anoop Dass (2011) for quite similar Reynolds numbers. Once these branches are computed, a new stability analysis can be performed. For example, one can search for additional steady bifurcation or Hopf bifurcation points. Hence, on
the symmetric branches of both examples, a second steady bifurcation is found, for a Reynolds number equal to 1890 and 359 respectively for the 2SNF and 4S LDC. Nevertheless, these two symmetric solutions are unstable, this fact being confirmed by an eigenvalue computation with ARPACK, Lehoucq et al (1998). From these bifurcation points, one can also compute the bifurcating branches (see Fig. 4). However, the most
Table 2. Critical Reynolds and Strouhal numbers for the Hopf bifurcation, 2SNF lid-driven cavity.

<table>
<thead>
<tr>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re&lt;sub&gt;c&lt;/sub&gt;</td>
<td>ω&lt;sub&gt;c&lt;/sub&gt;</td>
<td>St</td>
</tr>
<tr>
<td>3014</td>
<td>883</td>
<td>0.46</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3224</td>
<td>1259</td>
<td>0.62</td>
</tr>
<tr>
<td>3735</td>
<td>1821</td>
<td>0.77</td>
</tr>
<tr>
<td>4419</td>
<td>2498</td>
<td>1.06</td>
</tr>
</tbody>
</table>

interesting results are those dealing with the asymmetric branches, which are stable solutions. The two types of bifurcation have therefore been sought for these solutions. Concerning the 4S case and Mesh 3, no Hopf bifurcation points have been found for a Reynolds number between 130 and 1000. Contrariwise, a second stationary bifurcation is found for a Reynolds number equal to 867. For the 2SNF example, Hopf bifurcations have been searched for a Reynolds number in the range [1070,4000] on the asymmetric stable solutions. The first four Hopf bifurcations are given in Table (2) for the three considered meshes. Let us recall that these bifurcation points are determined by using a hybrid method introduced in Brezillon et al (2010). The minima of the indicator curve, see Fig. (2(b)), are introduced as initial guesses of a Newton iterative method, Jackson (1987). For the Mesh 1, the second Hopf bifurcation point has not been found (see Table (2)). This is confirmed through eigenvalue computation with ARPACK. In fact only three eigenvalues cross the imaginary axis for a Reynolds number close to 4000. We have also performed a stationary analysis on these two bifurcated branches (Mesh 3). Two turning points have been found on these bifurcated branches, respectively for Re=1137 and Re=1130. Between these two Reynolds numbers the asymmetric solutions are unstable.

4.2. comparisons with classical methods

This section is devoted to comparisons between our proposed algorithms and classical methods to compute stationary solutions and also to perform stability analysis. We only consider, for these comparisons, the 2NSF lid-driven cavity with the mesh 3 (42050 dof). Generally, the stationary solutions of the Navier-Stokes equations are computed by using an incremental iterative scheme. In this study, we consider the well-known Newton-Raphson method. With this method, a Reynolds increment is imposed and is chosen equal to ΔRe = 200. With this value, 6 steps are necessary to get the stationary solution up to a Reynolds number equal to 1200. As the required accuracy on the nonlinear solutions is fixed and equal to 10<sup>-4</sup>, the number of Newton’s iterations to satisfy the latter criterion is 2 or 3 by prediction step. The same accuracy is obtained with the Asymptotic Numerical Method with a parameter δ chosen equal to 10<sup>-6</sup> in the expression
Table 3. Comparisons between the presented algorithms and classical methods, 2NSF lid-driven cavity, mesh 3. \( n_1 \) is the number of steps, \( n_2 \) is the number of matrix triangulations, \( t_1 \) represents the computational time for the determination of stationary solution, \( t_2 \) stands for the computational time for the stability analysis (bifurcation indicator), \( t \) is the sum of the two previous computational times.

<table>
<thead>
<tr>
<th>Stationary bifurcation</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( t_1 ) (s)</th>
<th>( t_2 ) (s)</th>
<th>( t ) (s)</th>
<th>( \text{Re}_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method</td>
<td>6</td>
<td>6</td>
<td>6<em>9+6</em>4</td>
<td>6*4</td>
<td>102</td>
<td>1061</td>
</tr>
<tr>
<td>Newton’s method</td>
<td>6</td>
<td>20</td>
<td>20*9</td>
<td>-</td>
<td>180</td>
<td>[1000, 2000]</td>
</tr>
<tr>
<td>( \Delta \text{Re} = 200 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Hopf bifurcation</th>
<th>( n_1 )</th>
<th>( t_2 ) (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bifurcation indicator</td>
<td>order 20</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>ordre 30</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>order 40</td>
<td>40</td>
</tr>
</tbody>
</table>

| ARPACK, Lehoucq et al (1998) | 6    | 3000  |

(13). With the Newton-Raphson method, the stationary bifurcation is detected with the help of the sign of determinant of the Jacobian matrix. In Table (3), we compare the computational cost needed for each method for the 2NSF lid-driven cavity (mesh 3) and for the first stationary bifurcation (\( \text{Re}_c = 1061 \)). The order of truncation of the asymptotic expansions is \( p=20 \). This leads to 6 steps to compute the stationary solution and to determine the bifurcation point. With this method the computational time, \( t \) in Table 3, is equal to 102s. This time represents the 6 matrix triangulations (9s for each matrix triangulation) and the time to compute the quantities \( \tilde{U}_p^\delta \), \( \Delta U_p \) and \( \phi_p \) up to \( p=20 \) by solving the system (14). These times are equal to 4s. This means that, with the proposed methods (stationary and indicator computations) each step requires nearly two matrix triangulations. With the Newton-Raphson method, 20 matrix triangulations are needed (cumulated in the prediction and iteration scheme), which leads to a computation cost equal to 180s which is greater than the 102s needed with the proposed algorithms. Moreover, only an estimation of the critical point is performed with the Newton-Raphson method (between 1000 and 1200) whereas with the presented method an accurate critical Reynolds number is obtained.

For the Hopf bifurcation, we compare our results with a classical eigenvalue calculus realized with the ARPACK package (see Lehoucq et al (1998)). The Hopf bifurcation indicator is computed for an initial Reynolds number equal to 2800 (before the bifurcations summarized in Table (2)). The indicator calculi are carried out for an angular frequency from 0 to 1500 rad/s, which leads to 40 or 50 steps of the perturbation according the chosen order of truncature of the polynomial expansions. From these values, the first four bifurcation points of the Table (2) are found whatever the order of truncature chosen (\( p \) between 20 and 40). From the results summarized in Table (2), one
can conclude that the computation cost with the proposed indicator is approximately two times greater than with an ARPACK computation. Nevertheless, let us recall that with the proposed method, Hopf bifurcation points can be predicted before reaching the singular Reynolds numbers. Whereas, classical methods, such as eigenvalue computations, can only check if a bifurcation point occurs before the computed Reynolds number. So, in order to circumvent this drawback, it is envisaged to apply model reduction techniques for the calculation of the indicator, as Boumediene et al. (2011) used them successfully for the study of nonlinear vibrations of thin plates.

5. Conclusion

In this study, a full stability analysis is performed for the two-sided and four-sided lid-driven cavity by means of numerical tools based on a perturbation method. For the two cavity geometries, it is shown that the values of the fundamental steady bifurcations are the same as those of the literature: Re=1061 and Re=129 for, respectively, the two-sided and four-sided lid-driven cavity. This study adds some knowledge relative to the stability of the bifurcated branches. In the case of the four-sided lid-driven cavity, it is shown that a second steady bifurcation takes place for a Reynolds number Re=867 and no Hopf bifurcation is found. For the two-sided lid-driven cavity, no steady bifurcation is found whereas three Hopf bifurcations are computed for Re=2922, 3212 and 4078. The full stability analysis is performed by associating, for the first time, two specific numerical tools: the first one is dedicated to the stationary stability analysis, and the second one performs computations of Hopf bifurcations. Application of these tools on three-dimensional problems is envisaged. In that case, the stationary analysis can be performed without major drawback. Nevertheless, the calculation of Hopf bifurcation, as shown in the 2D case, still remains a challenge.

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