Abstract. This paper deals with a degenerate diffusion Patlak-Keller-Segel system in \( n \geq 3 \) dimension. The main difference between the current work and many other recent studies on the same model is that we study the diffusion exponent \( m = \frac{2n}{n+2} \) which is smaller than the usually used exponent \( m^* = \frac{2-2/n}{n} \) in other studies. With the exponent \( m = \frac{2n}{n+2} \), the associated free energy is conformal invariant and there is a family of stationary solutions \( U_{\lambda,x_0}(x) = C(n)\left(\frac{\lambda}{|x-x_0|^{n-2}}\right)^{\frac{2+n}{n+2}} \), \( \forall \lambda > 0, x_0 \in \mathbb{R}^n \). For radially symmetric solutions, we prove that if the initial data are strictly below \( U_{\lambda,0}(x) \) for some \( \lambda \) then the solution vanishes in \( L^1_{\text{loc}} \) as \( t \to \infty \); if the initial data are strictly above \( U_{\lambda,0}(x) \) for some \( \lambda \) then the solution either blows up at a finite time or has a mass concentration at \( r = 0 \) as time goes to infinity. For general initial data, we prove that there is a global weak solution provided that the \( L^m \) norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow up of the solution if the \( L^m \) norm for initial data is larger than the \( L^m \) norm of \( U_{\lambda,x_0}(x) \), which is constant independent of \( \lambda \) and \( x_0 \), and the free energy of initial data is smaller than that of \( U_{\lambda,x_0}(x) \).

Key words. Chemotaxis, critical diffusion exponent, nonlocal aggregation, critical stationary solution, global existence, mass concentration, radially symmetric solution

AMS subject classifications. 35K65, 35B45, 35J20

1. Introduction and preliminaries. In this paper, we study Patlak-Keller-Segel model in \( n \geq 3 \) dimension with homogeneous degenerate diffusion:

\[
\begin{aligned}
\rho_t &= \Delta \rho^m - \text{div}(\rho \nabla c), & \quad x \in \mathbb{R}^n, \quad t \geq 0, \\
-\Delta c &= \rho, & \quad x \in \mathbb{R}^n, \quad t \geq 0, \\
\rho(x,0) &= \rho_0(x), & \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where diffusion exponent is taken to be \( m = \frac{2n}{n+2} \in (1, 2) \). This model is widely used to describe the collective motion of cells. Here \( \rho(x,t) \) represents the bacteria density and \( c(x,t) \) represents the chemical substance concentration.

We assume the initial data \( \rho_0(x) \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n) \), where \( L^1_+ \) denotes non-negative integrable functions. In the second equation of (1.1), \( c(x,t) \) is given by the fundamental solution,

\[
c(x,t) = \frac{1}{(n-2)n\alpha(n)} \int_{\mathbb{R}^n} \frac{\rho(y,t)}{|x-y|^{n-2}} dy, \quad \alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad (1.2)
\]

\( \alpha(n) \) is the volume of \( n \)-dimension unit ball.

The first equation in (1.1) can also be written into

\[
\rho_t = \Delta \rho^m + \rho^2 - \nabla c \cdot \nabla \rho.
\]
Thus the classical solution $\rho$ of equation (1.1) preserves non-negativity if it is initially so. Hence we are leading to study nonnegative solutions, 

$$\rho(x, t) \geq 0, \quad x \in \mathbb{R}^n, \ t \geq 0.$$  

There is a naturally associated free energy for (1.1) which is given by

$$F(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m(x, t) dx - \frac{1}{2} \int_{\mathbb{R}^n} \rho(x, t) c(x, t) dx. \quad (1.3)$$

By using the fundamental solution representation in (1.2), we are able to rewrite the free energy into

$$F(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m(x, t) dx - \frac{1}{2(n-2)n\alpha(n)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(x, t)\rho(y, t)}{|x-y|^{n-2}} dxdy. \quad (1.4)$$

The different signs in the above free energy represents the competition between diffusion and nonlocal aggregation. This is the key feature of this system.

There is a natural variational structure for (1.1). The first order variation of $F$ gives the chemical potential:

$$\mu = \frac{\delta F}{\delta \rho} = \frac{m}{m-1} \rho^{m-1} - c. \quad (1.5)$$

By defining the drift velocity $v = -\nabla \mu$, the first equation in (1.1) can be rewritten into a continuity equation:

$$\rho_t + \text{div}(\rho v) = 0, \quad (1.6)$$

or

$$\rho_t = \text{div} \left( \rho\nabla \left( \frac{m}{m-1} \rho^{m-1} - c \right) \right). \quad (1.7)$$

Moreover, by taking inner product of $\frac{\delta F}{\delta \rho}$ with (1.6), we get the following energy-dissipation relation

$$\frac{dF(\rho)}{dt} + \int_{\mathbb{R}^n} \rho |\nabla \mu|^2 dx = 0,$$

or

$$\frac{dF(\rho)}{dt} + \int_{\mathbb{R}^n} \rho \left| \nabla \left( \frac{m}{m-1} \rho^{m-1} - c \right) \right|^2 dx = 0, \quad (1.8)$$

which leads to the fact that $F(\rho(\cdot, t))$ is a monotone nonincreasing function of $t$.

Another important group of quantities is useful in our analysis. They are the $i$th-moment of $\rho$, $i = 0, 1, 2$, defined by

$$m_0(t) = \int_{\mathbb{R}^n} \rho(x, t) dx, \quad m_1(t) = \int_{\mathbb{R}^n} x\rho(x, t) dx, \quad m_2(t) = \int_{\mathbb{R}^n} |x|^2 \rho(x, t) dx.$$ 

By a direct computation, we have the following conservation relations for these moments:
Proposition 1.1. The following equations for the 0,1,2-th moments hold,

\[ m_0'(t) = \frac{d}{dt} \int_{\mathbb{R}^n} \rho(x,t) \, dx = 0, \tag{1.9} \]

\[ m_1'(t) = \frac{d}{dt} \int_{\mathbb{R}^n} x\rho(x,t) \, dx = 0, \tag{1.10} \]

\[ m_2'(t) = \frac{d}{dt} \int_{\mathbb{R}^n} |x|^2 \rho(x,t) \, dx = -4 \int_{\mathbb{R}^n} \rho^m(x,t) \, dx + 2(n-2)F(\rho(\cdot,t)). \tag{1.11} \]

The identity (1.11) will be used to show a finite time blow-up behavior when the $L^m$-norm of the initial data is larger than a critical value in Section 3.

The most commonly used version of Keller-Segel model is a system with linear diffusion in density, i.e. $m = 1$. In the case of two dimension, the nonlocal aggregation comes from the logarithmic potential which is exactly the fundamental solution of Laplacian. There were lots of papers in the literature in discussing global existence and blow up criteria including multi-dimensional case, and both for parabolic-elliptic and parabolic-parabolic systems. We will not give a detailed review in this direction, but refer the readers to the review paper [20] or Chapter 5 in [29]. A sharp bound on the critical mass, $m_c = 8\pi$, was given by Dolbeault and Perthame in [16] by using logarithmic Haddy-Littlewood-Sobolev inequality. Critical mass means that if the initial mass is less than $m_c$, solution will exist globally, otherwise there must be mass concentration. There is an interesting connection between 2D Keller-Segel system and the Navier-Stokes equations in vorticity-stream function formulation, where the Navier-Stokes equations can be obtained by replacing the $\nabla c$ drift velocity with $\nabla_\perp c$.

This connection was explored and an alternative proof of global weak solution was given in [11] based Delort’s theory on 2-D incompressible Euler equation [15]. There are in-depth analysises for the case of critical mass $m_c = 8\pi$ in [2, 4].

In space dimension $n \geq 3$, some authors worked with the classical Keller-Segel model, i.e. linear diffusion, in which case the typical space is proved to be $L^2$ [29]. Global existence, finite time blow up and large time asymptotic behavior was studied, for example in [7, 29, 35]. While in multi-dimension, there are also several modifications of the Keller-Segel model. A simple and direct way is to use logarithmic interaction kernel instead of the $1/|x|^{n-2}$ kernel from Laplacian [10].

A more physical modification in multi-dimension is an introduction of degenerate slow diffusion to balance the nonlocal aggregation, as was suggested by Hillen and Painter in [18, 19], to describe volume filling and quorum sensing in models. Many works on the degenerate diffusion, or quasilinear parabolic type, Keller-Segel system can be found by different groups of mathematicians in the last few years, [3, 13, 14, 18, 19, 21, 22, 24, 27, 28, 31, 32, 33, 34, 35]. The general model of nonlinear diffusion and nonlinear advection can be written as

\[ \rho_t = \text{div} (A(\rho)\nabla \rho - N(\rho)\nabla c). \tag{1.12} \]

Particularly, many researchers were devoted to the following special form

\[ \rho_t = \Delta \rho^m - \text{div}(\rho^{m-1}\nabla c). \tag{1.13} \]

The existence of solution, finite time blow up and large time asymptotic behavior were extensively studied, either in the whole space $\mathbb{R}^n$ or in a smooth bounded domain with
homogeneous Neumann boundary conditions, either in a parabolic-parabolic form or in a parabolic-elliptic form. A critical diffusion exponent played a key role in their analysis, \( m^* = q - 2/n \). \((1.13)\) can be recast as

\[
\rho_t = \Delta \rho^m + \rho^q - (q-1)\rho^{q-2}\nabla \rho \cdot \nabla c. \tag{1.14}
\]

For singularity of type \( \rho(x) = |x|^{-\lambda} \), the last two terms in above equation are of same order \( |x|^{-q\lambda} \). Hence \( q = m + 2/n \) analogous to the Fujita exponent \([17]\). In the case \( q = 2 \), the exponent is \( m^* = 2 - 2/n \). In 2005, Horstmann and Winkler \([21]\) studied the case \( m = 1 \) and they found a critical exponent to be \( m = 1 > q - 2/n \). When \( q < 1 + 2/n \), they proved the existence of global solutions for large initial data. When \( q > 1 + 2/n \), there are some cases such that the problem has unbounded solution. In 2006, Sugiyama \([32]\) and Sugiyama and Kunii \([34]\), studied the cases \( q = 2 \) and \( q \geq 2 \), they proved that if \( m > m^* \) (this case was refereed as subcritical), then diffusion dominates the system and there is a global solution for arbitrary \( L^1 \cap L^\infty \) initial data; If \( m < m^* \) (this case was refereed as supercritical), then aggregation dominates the system and there is a finite time blow up of solution for some large initial data;

If \( 1 \leq m \leq q - 2/n \), then there is global existence of solution with decay property for small data. Luckhaus and Sugiyama \([27]\) showed that with small initial data, the globally existed solution in the long time behaves like Barenblatt solution of the porous media solution in the case \( m < q - 2/n \). In 2006, Senba and Suzuki \([31]\) proved that if the diffusion has a positive coefficient \( A(\rho) \) which increases faster than \( \rho^{m-1} \) \(( m < 2 - 2/n) \), then there is global existence of classical solution, with \( C^{2,\alpha} \) type regularity. In 2008, Kowalczyk and Szymańska \([24]\) obtained the global existence of nonnegative weak solution in the case of \( A(\rho) > C\rho^{m-1} \), \( m > 3 - 4/n \) for \( n > 2 \).

Noted that \( 3 - 4/n \geq m^* = 2 - 2/n \). In 2009, Cieślak and Laurencot \([13]\) showed when \( A(\rho) \geq C(1 + \rho)^{m-1} \), \( m > 2 - 2/n \), there is a global classic solution for any \( L^\infty \) initial data. In 2009, Blanchet, Carrillo and Laurencot \([3]\) studied the case \( m = m^* \) and showed that there is a critical mass \( M_\epsilon \) such that if initial mass \( m_0 < M_\epsilon \) and in addition \( \rho_0 \in L^\infty \cap H^1(\mathbb{R}^n) \), then a global weak solution exists and satisfies an energy-dissipation inequality. They also proved that if \( m_0 > M_\epsilon \), \( \rho_0 \in L^\infty \cap H^1(\mathbb{R}^n) \) and the free energy is negative initially, then there is a finite time blow up for the solution in \( L^m(\mathbb{R}^n) \). For the critical mass \( m_0 = M_\epsilon \) case, they discussed the large time behavior of solution. In 2010, Sugiyama obtained some partial regularity results for the solution in the critical case \( m = q - 2/n \) in \([33]\). In 2011, Ishida and Yokota \([22]\) established the global existence of weak solution with large initial data when \( m > q - 2/n \) for parabolic-parabolic type. Recently Bedrossian, Rodríguez and Bertozzi \([1]\) studied the corresponding critical exponent for general interaction potentials.

According to the above relatively complete discussions about the nonlinear diffusion in Keller-Segel system, it seems that not much has been left for future study. The main reason to choose exponent \( m^* = 2 - 2/n \) is that under the mass-invariant scaling \( \rho_\lambda(x,t) = \lambda^\alpha \rho(\lambda x, \lambda t) \) for the system \((1.1)\), there is a balance between diffusion and potential drift. However, in our current work, we will try to understand more about the exponents in \((1.1)\) from a different point of view. There are many reasons for us to take the diffusion exponent \( m = \frac{2n}{n+2} \), which is smaller than \( m^* = 2 - 2/n \) when \( n \geq 3 \), as we will discuss below.

We first show that there is a family of positive stationary solutions to the equation \((1.1)\). In fact, by taking \( \rho = (\frac{m-1}{m})^{1/(m-1)} \) in \((1.7)\) and plugging it into \((1.1)\), we
obtain the following equation

$$-\Delta c = \left(\frac{m-1}{m}\right)^p c^p, \quad x \in \mathbb{R}^n, \quad p = \frac{1}{m-1}. \quad (1.15)$$

Solutions to the above equation are stationary solutions of (1.1). Indeed, from the energy-dissipation relation (1.8), \(c\) is given by (1.2), and \(c\) decay at infinity, one knows that positive stationary solutions are given by the above equation. However, some compactly support solutions do not satisfy this equation.

It is a well known result [6, 12] that (1.15) has critical exponent \(p_c = \frac{n+2}{n} \), or equivalently \(m = \frac{2n}{n+2} \). Whenever \(p < p_c\), or equivalently \(m > \frac{2n}{n+2} \) (for example \(m^* = 2 - 2/n > \frac{2n}{n+2} \)), the only nonnegative solution of (1.15) is 0. At \(p = p_c\), all positive solutions of (1.15) must be of the form

$$C_{\lambda,x_0}(x) = \frac{2^{\frac{n+2}{4}} n^{\frac{n}{2}}}{n-2} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}, \quad \text{for some } \lambda > 0, x_0 \in \mathbb{R}^n. \quad (1.16)$$

The corresponding stationary solutions \(\rho(x)\) of (1.1) are given by:

$$U_{\lambda,x_0}(x) = \left(\frac{m-1}{m}\right)^{\frac{n+2}{4}} C_{\lambda,x_0}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}. \quad (1.17)$$

Remark 1.2. For two dimension \(n = 2\), one has that (see [6, 12])

1. \(m = m^* = 1\), the system becomes the Keller-Segel system with linear diffusion.
2. The Lane-Emden equation (1.15) is replaced by

$$-\Delta c = e^c$$

and the corresponding positive solutions (1.17) and (1.16) are replaced by

$$U_{\lambda,x_0}(x) = 8 \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^2, \quad C_{\lambda,x_0}(x) = \log U_{\lambda,x_0}(x), \quad \lambda > 0, x_0 \in \mathbb{R}^2.$$

3. The Hardy-Littlewood-Sobolev inequality used below will be replaced by logarithmic Hardy-Littlewood-Sobolev inequality. The equality holds if and only if \(p = A U_{\lambda,x_0}(x)\) for some positive constants \(A\), \(\lambda > 0\) and \(x_0 \in \mathbb{R}^2\).

The function \(U_{\lambda,x_0}(x)\) in (1.17) is known as the Lane-Emden function in astrophysics, which has infinity second moment for \(n \geq 2\). The value \(\|U_{\lambda,x_0}\|_{L^n}\) is independent of \(\lambda\) and \(x_0\), and is given by (1.23). Particularly, when \(n = 2\) this value is \(8\pi\). Using the conservation laws (1.9)-(1.10), one can uniquely determine the parameters \(\lambda\) and \(x_0\) in the stationary solution and we state the result in the following proposition:

Proposition 1.3. If \(U_{\lambda,x_0}(x) = \lim_{t \to \infty} \rho(x,t)\), then the parameters \(\lambda > 0\) and \(x_0 \in \mathbb{R}^n\) are uniquely determined by \(m_0\) and \(m_1\) in the following relations

$$x_0 = m_1/m_0, \quad \lambda = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \alpha(n) = m_0.$$  

Now we discuss connections among \(U_{\lambda,x_0}(x)\), free energy and Hardy-Littlewood-Sobolev inequality. From (1.5) we have that \(\frac{\partial}{\partial p}(U_{\lambda,x_0}(x)) = 0\). In other words, \(U_{\lambda,x_0}(x)\) is also a family of critical points to \(F(\rho)\). Moreover, the stationary solutions
Let $\rho \in L^m(\mathbb{R}^n)$, then
\[
\int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} dxdy \leq C(n)\|\rho\|_{L^m}^2, \tag{1.18}
\]
where
\[
C(n) = \pi^{(n-2)/2} \frac{1}{\Gamma(n/2 + 1)} \left\{ \Gamma(n/2) \right\}^{-2/n}. \tag{1.19}
\]
Moreover, the equality holds if and only if $\rho(x) = AU_{\lambda,x_0}(x)$, for some constant $A$ and parameters $\lambda > 0$, $x_0 \in \mathbb{R}^n$.

Consequently, we have the following decomposition of the free energy
\[
\mathcal{F}(\rho) = \frac{1}{m-1}\|\rho\|_{L^m}^2 \left( 1 - \frac{(m-1)c_nC(n)}{2} \|\rho\|_{L^m}^{4/(n+2)} \right) + \frac{c_n}{2} \left( C(n)\|\rho\|_{L^m}^2 - \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} dxdy \right)
\]
\[
= \mathcal{F}_1(\rho) + \mathcal{F}_2(\rho), \tag{1.20}
\]
where $c_n = 1/(n(n-2)\alpha(n))$. Since $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$, it is also a critical point for $\mathcal{F}_1(\rho)$. Indeed we will show that it is a maximum point for $\mathcal{F}_1(\rho)$. This property will be used in the proof of a finite time blow up discussion in Section 3.

All the facts we listed above are reflected by the following proposition. i.e. with diffusion exponent $m = \frac{2n}{n+2}$, the free energy $\mathcal{F}(\rho)$ is invariant under translations, similarities, orthogonal transformations and inversions (Kelvin transformations).

**Proposition 1.5.** The following facts hold
1. $\mathcal{F}(\rho_x) = \mathcal{F}(\rho)$ with $\rho_x(x) := \rho(x + \bar{x})$, $\forall \bar{x} \in \mathbb{R}^n$;
2. $\mathcal{F}(\rho_\lambda) = \mathcal{F}(\rho)$ with $\rho_\lambda(x) := \lambda^{\frac{n+2}{2}}\rho(\lambda x)$, $\forall \lambda > 0$;
3. $\mathcal{F}(\rho_R) = \mathcal{F}(\rho)$ with $\rho_R(x) := \rho(R^{-1}x)$, $\forall R^*R = I$;
4. $\mathcal{F}(\rho_{\bar{x},\lambda}) = \mathcal{F}(\rho)$ with $\rho_{\bar{x},\lambda}(x) := \frac{\lambda}{|x-\bar{x}|} \rho\left( \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2} \right)$, $\forall \bar{x} \in \mathbb{R}^n$, $\lambda > 0$.

We will give a proof of this proposition in the Appendix.

Remark 1.6. By Liouville's theorem [25], any smooth conformal mapping on a domain of $\mathbb{R}^n$, $n > 2$, can be expressed as a composition of translations, similarities, orthogonal transformations and Kelvin transformations. These transformations are all Möbius transformations.

Remark 1.7. The last transformation in Proposition 1.5 is motivated by the Kelvin transformation for $c$, i.e.,
\[
c_{\bar{x},\lambda}(x) = \left( \frac{\lambda}{|x-\bar{x}|} \right)^{n-2} c\left( \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2} \right).
\]

Remark 1.8. Invariants of the translations and similarities in Proposition 1.5 allow that parameters $x_0$ and $\lambda$ in $U_{\lambda,x_0}(x)$ be free. Invariants for free energy on the orthogonal and Kelvin transformations guarantee that the profile of stationary solution is unique.
Moreover, using (1.3) with
\[ C_{\lambda,x_0}(x) = \frac{m}{m-1} U_{\lambda,x_0}^{m-1}(x) = \frac{2n}{n-1} U_{\lambda,x_0}^{m-1}(x), \] (1.21)
we get
\[ \mathcal{F}(U_{\lambda,x_0}(x)) = \frac{2}{n-2} \|U_{\lambda,x_0}(x)\|_{L^m}^m, \] (1.22)
while we can calculate the right hand side explicitly,
\[ \|U_{\lambda,x_0}(x)\|_{L^m}^m = n^n \pi^{n+1} 2^{1-\frac{n}{2}} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)}. \] (1.23)
It is exactly the fact that \( L^m \) norm of \( U_{\lambda,x_0}(x) \) is a constant independent of \( \lambda \) and \( x_0 \). And this quantity will play an important role in our discussion of global existence and finite time blow up in general initial data.

There is also an interesting connection between (1.1) and the Euler-Poisson equations for gaseous stars. In three dimension, \( m^* = 4/3 \) and \( m = 6/5 \). Both exponents also appear to be some kind of critical adiabatic exponents in Euler-Poisson equations in terms of stability and instability of spherically symmetric steady states [30, 23].

This paper is arranged as follows. In Section 2, we prove that, for radially symmetric solutions, if the initial data is strictly below \( U_{\lambda,0}(x) \) for some \( \lambda \) then the solution vanishes in \( L_{loc}^1 \) as \( t \to \infty \); if the initial data is strictly above \( U_{\lambda,0}(x) \) for some \( \lambda \) then the solution either blows up at a finite time or has a mass concentrates at \( r = 0 \) as \( t \to \infty \).

In Section 3, we prove that there is a global weak solution provided that the \( L^m \) norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow up of the solution if the \( L^m \) norm for initial data is larger than that of \( U_{\lambda,x_0}(x) \) in (1.23) and the free energy of initial data is smaller than that of \( U_{\lambda,x_0}(x) \) in (1.23).

2. Decay and blow-up for radially symmetric solution. In the section, we will study large time behavior to the radially symmetric solution of (1.1). Radially symmetric solutions \((\rho(t, r), c(t, r))\) of the system (1.1) satisfy
\[
\begin{cases}
(r^{n-1}\rho)_t = (r^{n-1}(\rho^n)' - (r^{n-1}\rho c')', & r \in (0, \infty), t \geq 0, \\
-(r^{n-1}c')' = r^{n-1}\rho, & r \in (0, \infty), t \geq 0, \\
\rho'(t, r = 0) = 0, & t \geq 0, \\
\rho(t = 0, r) = \rho_0(r), & r \in (0, \infty),
\end{cases}
\] (2.1)
where ' stands for the derivative with respect to \( r \). We will show that the stationary solution \( U_{\lambda,0}(x) \) \((x_0 = 0)\) in (1.17) is a critical profile in the following sense, if the initial data \( \rho_0 \) is strictly below a stationary solution for some \( \lambda \), then all radially symmetric solutions are vanishing in \( L_{loc}^1(\mathbb{R}^n) \) as \( t \to \infty \), if the initial data \( \rho_0 \) is strictly above a stationary solution for some \( \lambda \), then all radially symmetric solutions either has a finite time blow up or has a mass concentrate concentration at \( x = 0 \) point as \( t \to \infty \). For simplicity, we will use notation \( U_\lambda(|x|) = U_{\lambda,0}(x) \) in this section. The following theorem is our main result in this section.

**Theorem 2.1.** Assume that the initial data \( \rho_0 \geq 0 \) is radially symmetric,
1. If \( \exists \lambda_0 > 0 \) s.t.
\[
\rho_0(r) < U_{\lambda_0}(r), \quad r > 0,
\]
then any radially symmetric solution \( \rho(r,t) \) of (1.1) is vanishing in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( t \to \infty \).

2. If \( \exists \lambda_0 > 0 \) s.t.
\[
\rho_0(r) > U_{\lambda_0}(r), \quad r > 0,
\]
then any radially symmetric solution \( \rho(r,t) \) of (1.1) must blow up at a finite time \( t^* \) or has a mass concentration at \( r = 0 \) as time goes to infinity in the sense that there is \( r(t) \to 0 \) as \( t \to \infty \) and a positive constant \( C \) such that
\[
\int_{B(0,r(t))} \rho \, dx \geq C.
\]

Inspired by a similar result in two dimensional case [29], we work on the following weighted primitive variable (integral of density \( \rho \) in the ball with radius \( r \) and center at origin)
\[
M(t,r) := n\alpha(n) \int_0^r \sigma^{n-1} \rho(t,\sigma) \, d\sigma. \tag{2.2}
\]

By the second equation in (2.1), one has
\[
M(t,r) = -n\alpha(n)r^{n-1}\rho',
\]
Thus (2.1) can be reduced to a single equation for \( M(t,r) \). By integrating (2.1), we have
\[
\begin{cases}
M_t = n\alpha(n)r^{n-1} \left[ \left( \frac{M'}{n\alpha(n)r^{n-1}} \right)^m \right]' + \frac{M'M}{n\alpha(n)r^{n-1}}, & r \in (0,\infty), t \geq 0, \\
M(t,0) = 0, M(t,\infty) = m_0, & t \geq 0, \\
M(0,r) = n\alpha(n) \int_0^r \sigma^{n-1} \rho_0(\sigma) \, d\sigma, & r \in (0,\infty). \tag{2.3}
\end{cases}
\]
From (2.2), we have
\[
M' = n\alpha(n)r^{n-1}\rho, \quad \text{for all } r \in (0,\infty), t \geq 0.
\]
Thus \( M'(t,r) \geq 0 \), i.e., \( M(t,r) \) is an increasing function in \( r \).

The main advantage of using equation (2.3) instead of using (2.1) is that we can use comparison principle by constructing a super-solution for decay estimates and constructing a sub solution for mass concentration estimates.

The stationary problem of (2.3) is reduced to
\[
\begin{cases}
n\alpha(n)r^{n-1} \left[ \left( \frac{M'}{n\alpha(n)r^{n-1}} \right)^m \right]' + \frac{M'M}{n\alpha(n)r^{n-1}} = 0, & r \in (0,\infty), \\
M(0) = 0, \quad M(\infty) = m_0. \tag{2.4}
\end{cases}
\]
Recall (1.16) and (1.17), a family of stationary solutions to (1.1) are given by
\[
(U_{\lambda}(r), C_{\lambda}(r)) = \left( 2^{\frac{n+2}{n-2}} n \frac{\lambda}{\lambda^2 + r^2} \left( \frac{\lambda}{\lambda^2 + r^2} \right)^{\frac{n-2}{n+2}}, \quad 2^{\frac{n+2}{n-2}} n \frac{\lambda}{\lambda^2 + r^2} (n-2)^{-1} \left( \frac{\lambda}{\lambda^2 + r^2} \right)^{\frac{n-2}{n+2}} \right),
\]
where \( \lambda > 0 \) is a free parameter. Hence, the corresponding family of explicit solutions to (2.4) are given by

\[
\tilde{M}_\lambda(r) = n\alpha(n) \int_0^r \sigma^{n-1} U_\lambda(\sigma) d\sigma = K_\lambda(n) \frac{1}{(1 + \lambda^2 r^{-2})^{n/2}}
\] (2.5)

with \( K_\lambda(n) = \alpha(n) 2^{n/2} n^{n/2} \lambda^{-n/2} \).

Proof of Theorem 2.1. In the following two subsections we will prove 1 and 2 of the theorem in the form of Lemma 2.2 and Lemma 2.3, respectively.

\[ \square \]

2.1. Super-solution with subcritical initial data. In this subsection, we will show that the solutions of the problem (2.3) vanish as \( t \to \infty \) for any finite space interval, when initial data is dominated by a stationary solution \( \tilde{M}_{\lambda_0}(r) \) in (2.5) for some \( \lambda_0 > 0 \). More precisely, we have

**Lemma 2.2.** For \( n \geq 3 \), assume that

\[
m_0 = M(t, \infty) < K_{\lambda_0}(n), \quad M(0, r) < \tilde{M}_{\lambda_0}(r), \quad \forall r > 0,
\]

for some \( \lambda_0 > 0 \). Then the solutions of (2.3) diminish in time in the following sense

\[
M(t, r) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in any interval} \quad 0 \leq r \leq R.
\]

Thus \( \rho(t, x) \) in (1.1) vanishes in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( t \to \infty \).

**Proof.** Due to the facts that \( M(0, r) \) and \( \tilde{M}_{\lambda_0}(r) \) are bounded nondecreasing functions and \( M(0, r) < \tilde{M}_{\lambda_0}(r) \), there exist a \( \mu \in (0, 1) \) s.t. \( M(0, r) \leq \mu \tilde{M}_{\lambda_0}(r) \).

Notice that \( M(t, \infty) = m_0 < K_{\lambda_0}(n) \), without lose of generality, we can choose the same \( \mu \) such that \( M(t, \infty) = m_0 < \mu K_{\lambda_0}(n) \).

We construct a super-solution of (2.3) by modifying constant \( \lambda \) in the denominator of its stationary solution (2.5). By taking \( \lambda = \lambda(t) = (A_1 t + \lambda_0^n)^{1/n} \), for some \( A_1 > 0 \) to be determined, and cutting \( \mu \tilde{M}_\lambda(t) \) by a constant \( m_0 \) when \( r \geq R(t) \), we can construct a super-solution \( \tilde{N}(t, r) \) in the following

\[
\tilde{N}(t, r) := \min \left\{ m_0, \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda^2(t) R^{-2}(t))^{n/2}} \right\},
\]

where the cut off location \( R(t) \) is given by

\[
m_0 = \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda^2(t) R^{-2}(t))^{n/2}},
\] (2.6)

i.e.,

\[
(1 + \lambda^2(t) R^{-2}(t))^{n/2} = \frac{\mu K_{\lambda_0}(n)}{m_0},
\]

or

\[
R(t) = \left( \frac{\lambda^2(t)}{\left( \frac{\mu K_{\lambda_0}(n)}{m_0} \right)^{2/n} - 1} \right)^{1/2}.
\]
Hence $N(t, r)$ is continuous and $N(t, r) = m_0$ for $r > R(t)$, $N(t, r) = \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda(t)r^{-2})^{n/2}}$ for $r \leq R(t)$.

Now we prove that $N(t, r)$ is a supersolution to (2.3). Obviously, the constant state $m_0$ is also a super-solution. Next we only need to show that $\bar{N}(t, r) = \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda(t)r^{-2})^{n/2}}$ is a super-solution for $r \leq R(t)$, i.e. to prove

$$\text{LHS} := \bar{N}_t - n\alpha(n)r^{n-1} \left[ \left( \frac{\bar{N}'}{n\alpha(n)r^{n-1}} \right)^m \right]' - \frac{\bar{N}'N}{n\alpha(n)r^{n-1}} \geq 0. \quad (2.7)$$

A direct calculation of (2.7) term by term gives that

$$\bar{N}_t = \bar{N} \frac{-n\lambda(t)r^{-2}}{1 + \lambda^2(t)r^{-2}} \lambda'(t),$$

$$n\alpha(n)r^{n-1} \left[ \left( \frac{\bar{N}'}{n\alpha(n)r^{n-1}} \right)^m \right]' = -2\bar{N}n^2\alpha(n)^{1-m}\lambda^{2m-1}(t)\left( \frac{\mu K_{\lambda_0}(n)}{1 + \lambda^2(t)r^{-2})^{n/2}} \right)^{m-1}r^{-n},$$

$$\frac{\bar{N}'N}{n\alpha(n)r^{n-1}} = \mu K_{\lambda_0}(n) \bar{N} \frac{n\alpha(1 + \lambda^2(t)r^{-2})^{n/2}}{n\alpha(n)(1 + \lambda^2(t)r^{-2})^{n/2}}.$$

So, we have

$$LHS = \bar{N} \frac{n\lambda(t)r^{-2}}{1 + \lambda^2(t)r^{-2}} \left( -\lambda'(t) + 2n\alpha(n)^{1-m}\lambda^{2m-1}(t) \frac{\left( \frac{\mu K_{\lambda_0}(n)}{1 + \lambda^2(t)r^{-2})^{n/2}} \right)^{m-1}r^{-n}}{n\alpha(n)(1 + \lambda^2(t)r^{-2})^{n/2}} \right) - \mu K_{\lambda_0}(n) \frac{\lambda(t)r^{-n}}{n\alpha(n)(1 + \lambda^2(t)r^{-2})^{n/2}}$$

$$= \bar{N} \frac{n\lambda(t)r^{-2}}{1 + \lambda^2(t)r^{-2}} \left( -\lambda'(t) + A(t) \frac{\left( \frac{\mu K_{\lambda_0}(n)}{1 + \lambda^2(t)r^{-2})^{n/2}} \right)^{m-1}r^{-n}}{n\alpha(n)(1 + \lambda^2(t)r^{-2})^{n/2}} \right), \quad (2.8)$$

where

$$A(t) := 2n^2\alpha(n)^{2-m}\lambda^{2m-2}(t) - (\mu K_{\lambda_0}(n))^{2-m}.$$ 

Moreover, (2.8) can be simplified as

$$LHS = \bar{N} \frac{n\lambda(t)}{r^2 + \lambda^2(t)} \left( -\lambda'(t) + A(t) \frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)(r^2 + \lambda^2(t))^{n/2}} \right), \quad (2.9)$$

By using the expressions $m = \frac{2n}{n+2}$, $K_{\lambda_0}(n) = \alpha(n)2^{\frac{n+2}{n+2}}n^{\frac{n+2}{n+2}}\lambda_0^{\frac{n-2}{n+2}}$ and $\lambda(t) \geq \lambda_0$, we have

$$A(t) \geq 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1 - \mu^{2-m}) \Rightarrow A_0 > 0. \quad (2.10)$$

Therefore, (2.9) and (2.10) together imply that

$$LHS \geq \bar{N} \frac{n\lambda(t)}{r^2 + \lambda^2(t)} \left( -\lambda'(t) + A_0 \frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)(r^2 + \lambda^2(t))^{n/2}} \right)$$

$$\geq \bar{N} \frac{n\lambda(t)}{r^2 + \lambda^2(t)} \left( -\lambda'(t) + A_0 \frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)(R^2(t) + \lambda^2(t))^{n/2}} \right)$$

$$= \bar{N} \frac{n\lambda(t)}{r^2 + \lambda^2(t)} \left( -\lambda'(t) + A_1n^{-1}\lambda^{-n+1}(t) \right), \quad (2.11)$$
where we have used notation

\[ A_1 := A_0 \left( \frac{\mu K_{\lambda_0}(n)}{\alpha(n) (R^2(t) + \lambda^2(t))^{\frac{n}{2}}} \right)^m. \]

From (2.6) and simple computation, one has that \( A_1 \) is a positive constant, i.e.

\[ A_1 = A_0(\mu K_{\lambda_0}(n))^{m-2}m_0\alpha(n)^{-1} \left[ \left( \frac{\mu K_{\lambda_0}(n)}{m_0} \right)^\frac{n}{2} - 1 \right]^{n/2} > 0. \]

Now we can choose \( \lambda(t) \) such that the right hand side of (2.11) is 0, and it follows that \( LHS \geq 0 \).

The choice of \( \lambda(t) \) must satisfies

\[ \lambda'(t) = A_1 n^{-1} \lambda^{-n+1}(t), \quad \text{and} \quad \lambda(0) = \lambda_0. \]

It is easy to see that \( \lambda(t) \) is given by

\[ \lambda(t) = (A_1 t + \lambda_0^n)^{1/n}, \quad t \geq 0. \quad (2.12) \]

Furthermore \( M(0, r) \leq \bar{M}_{\lambda_0}(r) \), and

\[ M(t, 0) \leq \lim_{r \to 0} \bar{N}(t, r) = \lim_{r \to 0} \frac{\mu K_{\lambda_0}(n)}{1 + \lambda^2(t) r^{-2}} R(t)^{n/2} = 0, \]

thus \( \frac{\mu K_{\lambda_0}(n)}{1 + \lambda^2(t) r^{-2}} R(t)^{n/2} \) is a super-solution of (2.3). The minimum of two super-solutions is also a super-solution, i.e., \( \bar{N}(t, r) \) is a super-solution to (2.3).

By the comparison principle, we deduce that the solution of (2.3) satisfies \( M(t, r) \leq \bar{N}(t, r) \) in \([0, \infty) \times [0, \infty)\). By (2.6) and (2.12), we have that

\[ \lambda(t) \to \infty \quad \text{and} \quad R(t) \to \infty \quad \text{as} \quad t \to \infty. \]

Therefore, for a given interval \( r \in (0, R_0) \), it holds that

\[ M(t, r) \leq \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda^2(t) R_0^{-2})^{n/2}} \to 0, \quad \text{as} \quad t \to \infty. \]

This completes the proof of Lemma 2.2. \( \square \)

2.2. Blow-up with super-critical initial data. In this subsection, we will prove that if the initial data is above a stationary solution \( \bar{M}_{\lambda_0}(r) \) in (2.5) for some \( \lambda_0 > 0 \) and there is no finite time blow up in solution, then radially symmetric solutions must have mass concentration at \( x = 0 \) as time goes to infinity.

**Lemma 2.3.** For dimension \( n \geq 3 \). Assume that

\[ m_0 = M(t, \infty) > K_{\lambda_0}(n), \quad M(0, r) > \bar{M}_{\lambda_0}(r), \quad \text{for all} \ r > 0, \]

for some \( \lambda_0 > 0 \) and there is no finite time blow up. Then there is a positive constant \( C > 0 \) and a function \( r(t) \to 0 \) as \( t \to \infty \) such that all solutions \( M(t, r) \) satisfy

\[ M(t, r(t)) \geq C. \]
Or equivalently, radially symmetric solutions \( \rho \) to (1.1) have mass concentration at \( x = 0 \) as \( t \to \infty \), i.e.

\[
\int_{B(0,r(t))} \rho \, dx \geq C.
\]

**Proof.** We assume there is no finite time blow up. Hence the solution exists globally. We shall show that this global solution must has a mass concentration at \( x = 0 \) for large enough \( t \). In other words, we will show that there exists a positive constant \( C \) and a radius function \( r(t) > 0 \) such that as \( t \to \infty \), we have \( r(t) \to 0 \) and

\[
M(t, r(t)) \geq C > 0,
\]

i.e.,

\[
\int_{B(0,r(t))} \rho \, dx \geq C > 0.
\]

Similar to the discussion in the beginning of the proof of Lemma 2.2, we can choose \( \mu_0 > 1 \) such that \( \mu_0 K_{\lambda_0}(n) < m_0 = M(t, \infty) \) and \( M(0, r) > \mu_0 M_{\lambda_0}(r) \) for all \( r > 0 \). We construct a sub-solution \( \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda(t)^{2r-2})^{n/2}} \) of the equation (2.3) from the stationary solution \( M_{\lambda_0}(r) \) in (2.5) by taking \( \lambda = \lambda(t) = \lambda_0 e^{B_1 t} \) for some \( B_1 < 0 \). Similar to the construction of super-solution in the previous subsection, we cut off it by \( \frac{m_0}{(1 + \lambda(t)^{2r-2})^{n/2}} \) for \( r \geq R(t) \) for some \( R(t) \) which will be specified later. We take the sub-solution in the following form:

\[
N(t, r) := \max \left\{ \frac{m_0}{(1 + \lambda_0^{2r-2})^{n/2}}, \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda(t)^{2r-2})^{n/2}} \right\}.
\]

(2.14)

It can be shown that both terms on the right hand side of (2.14) are sub-solutions. For the first term, we have

\[
\begin{align*}
\frac{N_t - n\alpha(n)r^{n-1} \left[ \left( \frac{N'}{n\alpha(n)r^{n-1}} \right)^m \right]'}{\frac{n'}{n\alpha(n)r^{n-1}}} & - \frac{N'_N}{n\alpha(n)r^{n-1}} \\
& = \frac{\left[ 2\alpha(n)^2 m^2 n^2 \lambda_0^{2m} - (m_0)^2 - m^2 \lambda_0^2 \right]}{n\alpha(n)(1 + \lambda_0^{2r-2})^{n+1}} \frac{n(m_0)^{m-1} r^{-n-2}}{n\alpha(n)(1 + \lambda_0^{2r-2})^{n+1}} \\
& \leq \frac{\left[ 2\alpha(n)^2 m^2 n^2 \lambda_0^{2m} - K_{\lambda_0}(n)^2 - m\lambda_0^2 \right]}{n\alpha(n)(1 + \lambda_0^{2r-2})^{n+1}} \frac{n(m_0)^{m-1} r^{-n-2}}{n\alpha(n)(1 + \lambda_0^{2r-2})^{n+1}} \\
& = 0.
\end{align*}
\]

(2.15)

Together with the boundary conditions

\[
N(t, 0) = \lim_{r \to 0} \frac{m_0}{(1 + \lambda_0^{2r-2})^{n/2}} = 0,
\]

\[
N(t, \infty) = \lim_{r \to \infty} \frac{m_0}{(1 + \lambda_0^{2r-2})^{n/2}} \leq m_0 = M(t, \infty),
\]

and initial condition \( N(0, r) \leq M(0, r) \), we have that \( \frac{m_0}{(1 + \lambda_0^{2r-2})^{n/2}} \) is a sub-solution.

Next we show that the second term \( \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda(t)^{2r-2})^{n/2}} \) is also a sub-solution in the interval \( 0 \leq r \leq R(t) \) where \( N \) achieves its maximum by \( \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda(t)^{2r-2})^{n/2}} \). The radius
Notice that there exists constant $R_0 : r \leq R(t) \leq R_0$ such that 

$$
\frac{m_0}{(1 + \lambda_0^2 R^{-2}(t))^{n/2}} = \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda_0^2 (t) R^{-2}(t))^{n/2}}. 
$$

(2.16)

By (2.17) and (2.18), we have 

$$
\text{LHS} := N \frac{n \lambda (t) t^{-m}}{1 + \lambda(t) t^{-m}} \left( -\lambda' (t) + B(t) \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-1} \lambda(t) t^{-m} \right),
$$

where 

$$
B(t) := 2n^2 \alpha (n)^2 - (\mu_0 K_{\lambda_0}(n))^{2-m} t^{-m} \leq 2n^2 \alpha (n)^2 - (\mu_0 K_{\lambda_0}(n))^{2-m} \left( 1 - \mu_0^{2-m} \right) =: B_0 < 0. 
$$

(2.18)

By (2.17) and (2.18), we have 

$$
\text{LHS} \leq N \frac{n \lambda (t) t^{-m}}{1 + \lambda(t) t^{-m}} \left( -\lambda' (t) + B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-1} \lambda(t) t^{-m} \right)
$$

$$
\leq N \frac{n \lambda (t) t^{-m}}{1 + \lambda(t) t^{-m}} \left( -\lambda' (t) + B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-1} \lambda(t) t^{-m} \right)
$$

$$
= N \frac{n \lambda (t) t^{-m}}{1 + \lambda(t) t^{-m}} \left( -\lambda' (t) + B_2(t) \lambda(t) \right), 
$$

(2.19)

where we have used notation 

$$
B_2(t) := B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-1} \lambda(t) t^{-m}.
$$

From (2.16) and simple computations, we have 

$$
B_2(t) \leq B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-2} \frac{m_0}{(R_0^2 + \lambda_0^2 t^{-m})^{n/2}} \leq B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-2} \frac{m_0}{(R_0^2 + \lambda_0^2 t^{-m})^{n/2}} \leq B_0 \left( \frac{\mu_0 K_{\lambda_0}(n)}{\alpha (n)} \right)^{m-1} (\alpha (n))^{-1} R_0^n < 0.
$$

Denote the constant term above as $B_1$. (2.19) leads to 

$$
\text{LHS} \leq N \frac{n \lambda (t) t^{-m}}{1 + \lambda(t) t^{-m}} \left( -\lambda' (t) + B_1 \lambda(t) \right).
$$

Since $B_1$ is a negative constant, we can set $\lambda(t) = B_1 \lambda(t)$. This gives the solution 

$$
\lambda(t) = \lambda_0 e^{B_1 t} 
$$

and this is our choice of $\lambda(t)$. With this choice of $\lambda(t)$ we have shown that 

$$
\text{LHS} \leq 0.
$$
Now \( \forall t > 0 \), comparison principle shows that
\[
M(t,r) \geq N(t,r) = \frac{K_{\lambda_0}(n)}{1 + (\lambda_0 e^{B_1 t})^2 r^{-2} n/2}.
\]
We evaluate above inequality at \( r = \lambda_0 e^{B_1 t} \),
\[
M(t,r(t)) \geq N(t,r(t)) = \frac{K_{\lambda_0}(n)}{2 n/2} > 0.
\]
This completes the proof of the lemma.

### 3. Existence and blow-up with general initial data.

We will discuss the existence and blow up of the solution with more general initial data, not limited to the radially symmetric case. The main tools in this part are the entropy inequality and second moment estimates. For simplicity, we will use notations \( L^p \) to represent \( L^p(\mathbb{R}^n) \).

#### 3.1. Global existence.

In this subsection, we will prove the following theorem on global existence of weak solution of (1.1) if the initial data satisfies
\[
\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s := \left( \frac{4 m^2}{(2 m - 1)^2 C_{GNS}} \right)^{1/m}, \tag{3.1}
\]
where \( C_{GNS} \) is the universal constant appeared in Gagliardo-Nirenberg-Sobolev inequality (see (4.1) below).

**Theorem 3.1.** For initial date \( \rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n) \) and \( \|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s \), there is a global weak solution \( (\rho, c) \) to (1.1) with regularity
\[
\rho \in L^\infty(0, T; L^1_+ \cap L^m(\mathbb{R}^n)) \cap L^{m+1}(0, T; L^{m+1}(\mathbb{R}^n)),
\]
\[
\nabla \rho \in L^r(0, T; L^r(\mathbb{R}^n)), \quad r = \min \left(2, \frac{3m + 2}{n + 4} \right), \tag{3.2}
\]
\[
c \in L^\infty(0, T; L^s(\mathbb{R}^n)), \quad \frac{n}{n - 2} < s \leq \frac{2n}{n - 2},
\]
\[
\nabla c \in L^\infty(0, T; L^2(\mathbb{R}^n)).
\]
Moreover \( \|\rho(\cdot, t)\|_{L^m(\mathbb{R}^n)} \) decays algebraically in time,
\[
\|\rho(\cdot, t)\|_{L^m(\mathbb{R}^n)} \leq C t^{-\frac{1}{m(n-1)}}, \quad \text{for large } t, \tag{3.3}
\]
where \( \beta = \frac{2m^2 - 3m + 2}{m(m-1)} > 1 \).

**Proof.** We split the proof into six steps.

**Step 1.** We start with the regularized problem, for \( \varepsilon > 0 \),
\[
\begin{aligned}
\partial_t \rho_\varepsilon &= \Delta \rho_\varepsilon + \varepsilon \Delta \rho_\varepsilon - \text{div}(\rho_\varepsilon \nabla c_\varepsilon), \quad x \in \mathbb{R}^n, t \geq 0, \\
-\Delta c_\varepsilon &= J_\varepsilon \ast \rho_\varepsilon, \quad x \in \mathbb{R}^n, t \geq 0, \\
\rho(x, 0) &= \rho_0(x), \quad x \in \mathbb{R}^n.
\end{aligned} \tag{3.4}
\]
where \( J_\varepsilon \) is a mollifier with radius \( \varepsilon \) and satisfies \( \int_{\mathbb{R}^n} J_\varepsilon \, dx = 1 \). We know from parabolic theory that the above regularized problem has a global smooth nonnegative solution \( \rho_\varepsilon \) for \( t > 0 \) if the initial data is nonnegative.
Step 2. In this step, we will prove the following basic energy estimates.

\[
\|\rho_\varepsilon\|_{L^\infty(R_+;L^m)} + \|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2(R_+;L^2)} + \|\varepsilon^{\frac{1}{2}} \nabla \rho_\varepsilon^{\frac{m}{2}}\|_{L^2(R_+;L^2)} \leq C, \\
\|\rho_\varepsilon\|_{L^{m+1}(R_+;L^{m+1})} \leq C, \\
\|\nabla c_\varepsilon\|_{L^\infty(R_+;L^2)} + \|c_\varepsilon\|_{L^\infty(R_+;L^n)} \leq C, \\
\frac{n}{n-2} < s \leq \frac{2n}{n-2},
\]

(3.5) (3.6) (3.7)

where, C stands for positive constants depending only on \(\|\rho_0\|_{L^1}, \|\rho_0\|_{L^m}\) and \(n\).

Taking \(m \rho_\varepsilon^{m-1}\) as a test function in (1.1), we have

\[
\frac{d}{dt} \int_{R^n} \rho_\varepsilon^m dx + \frac{4m^2(m-1)}{(2m-1)^2} \int_{R^n} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2 dx + \varepsilon \frac{4(m-1)}{m} \int_{R^n} \left| \nabla \rho_\varepsilon^{\frac{m}{2}} \right|^2 dx \\
= (m-1) \int_{R^n} \rho_\varepsilon^m J_\varepsilon \ast \rho_\varepsilon dx.
\]

(3.8)

The right hand side of above equation can be estimated by Gagliardo-Nirenberg-Sobolev inequality

\[
\int_{R^n} \rho_\varepsilon^m J_\varepsilon \ast \rho_\varepsilon dx \leq \|\rho_\varepsilon\|^m_{L^{m+1}} \|J_\varepsilon \ast \rho_\varepsilon\|_{L^{m+1}} \leq \|\rho_\varepsilon\|^m_{L^{m+1}} \|\rho_\varepsilon\|_{L^{m+1}} \\
= \left| \rho_\varepsilon^{-\frac{1}{2}} \right|_{L^{m+\frac{m+1}{m}}} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2_{L^2} \left| \rho_\varepsilon^{\frac{m-1}{2}} \right|_{L^{2(m-1)}}^{\frac{2(2-m)}{m}} \leq C_{GNS} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2_{L^2} \left| \rho_\varepsilon^{2-m} \right|_{L^m} \leq C_{GNS} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2_{L^2} \left| \rho_\varepsilon^{2-m} \right|_{L^m}.
\]

(3.9)

If the last term of (3.8) can be strictly dominated by the second term of (3.8) which can be realized by taking initial data satisfying (3.1), then we can close the estimate. In other words, if we choose \(\rho_0\) such that

\[
(m-1) \left( \frac{4m^2}{(2m-1)^2} - C_{GNS} \|\rho_0\|^{2-m}_{L^m} \right) =: \delta > 0,
\]

then we can obtain the estimate,

\[
\frac{d}{dt} \int_{R^n} \rho_\varepsilon^m dx + \delta \int_{R^n} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2 dx + \varepsilon \frac{4(m-1)}{m} \int_{R^n} \left| \nabla \rho_\varepsilon^{\frac{m}{2}} \right|^2 dx \leq 0.
\]

(3.10)

Thus one has

\[
\|\rho_\varepsilon\|_{L^m} \leq \|\rho_0\|_{L^m} < C_s,
\]

(3.11)

and

\[
\delta \int_0^\infty \int_{R^n} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2 dx dt + \varepsilon \frac{4(m-1)}{m} \int_0^\infty \int_{R^n} \left| \nabla \rho_\varepsilon^{\frac{m}{2}} \right|^2 dx dt < C_s.
\]

(3.12)

Combining the last inequality of (3.9) with (3.11) and (3.12), we obtain that

\[
\int_0^\infty \|\rho(\cdot,t)\|_{L^{m+1}}^{m+1} dt \leq C_{GNS} C_s^{2-m} \int_0^\infty \int_{R^n} \left| \nabla \rho_\varepsilon^{m-\frac{1}{2}} \right|^2 dx dt \leq \frac{C_{GNS} C_s^{3-m}}{\delta}.
\]

Therefore, (3.6) follows. Apply the weak Young inequality \([26, \text{p.107 formula (9)}]\) to

\[
\nabla c_\varepsilon = \frac{1}{n \alpha(n)} \int_{R^n} \frac{x - y}{|x - y|^n} (J_\varepsilon \ast \rho_\varepsilon)(y) dy,
\]
one has
\[
\|\nabla c_\epsilon\|_{L^2} \leq C\|\frac{x-y}{|x-y|^n}\|_{L^q_w} \|J_\epsilon \ast \rho_\epsilon\|_{L^m} \leq C\|\rho_\epsilon\|_{L^m} \leq C, \tag{3.13}
\]
where \( q = \frac{n}{n-1} \) satisfies \( 1 + \frac{1}{q} = \frac{1}{\eta} + \frac{1}{m} \). \( L^q_w \) stands for the weak \( L^q \)-space and \(|x|^{1-n} \in L^m_w/(n-1) |x| \) see [26, p.106]. This fact was used in the second inequality above.
By the second equation of (3.4), we know
\[
c_\epsilon = \frac{1}{(n-2)n\alpha(n)} \int_\mathbb{R}^n \frac{1}{|x-y|^{n-2}}(J_\epsilon \ast \rho_\epsilon)(y)dy.
\]
Using weak Young's inequality again, \( \rho_\epsilon \in L^\infty(\mathbb{R}^+, L^1 \cap L^m) \), we can easily get
\[
\|c_\epsilon\|_{L^\infty(\mathbb{R}^+, L^s)} \leq C, \tag{3.14}
\]
where \( \frac{n}{n-2} \leq s \leq \frac{2n}{m-n} \). (3.13) and (3.14) imply that (3.7) is true.

**Step 3.** We will show the following time regularity
\[
\|\partial_t \rho_\epsilon\|_{L^2(0,T; W^{-1,p}(U))} \leq C(U,T), \tag{3.15}
\]
where \( p = \min\{ \frac{2n}{m+1}, \frac{2(m+1)}{m+3}, m \} = \frac{2(m+1)}{m+3} > 1 \) and \( U \) is any bounded open subset of \( \mathbb{R}^n \). \( C(U,T) \) stands for a positive constant dependents only on \( U, T, \|\rho_0\|_{L^m} \) and \( n \). For simplicity in presentation, we drop the dependence of \( \|\rho_0\|_{L^1}, \|\rho_0\|_{L^m} \) and \( n \) in the constant notation \( C(U,T) \).
(3.15) follows by directly using the approximate equation (3.4) and the following estimates (details will be given below),
\[
\|\nabla \rho_\epsilon^m\|_{L^2(\mathbb{R}^+, L^{2(m+1)}/\mathbb{R}^n)} \leq C, \tag{3.16}
\]
\[
\|\rho_\epsilon \nabla c_\epsilon\|_{L^{m+1}(\mathbb{R}^+, L^{2(m+1)}/\mathbb{R}^n)} \leq C, \tag{3.17}
\]
and
\[
\|\varepsilon^{1/2} \nabla \rho_\epsilon\|_{L^2(\mathbb{R}^+, L^m)} \leq C. \tag{3.18}
\]
By the weak formulation of (3.4), for any test function \( \phi(x) \in C^\infty_0(U) \), we deduce that for any \( t \),
\[
\left| \int_{\mathbb{R}^n} \partial_t \rho_\epsilon \phi \, dx \right| = \left| - \int_U \nabla \rho_\epsilon^m \cdot \nabla \phi \, dx - \varepsilon \int_U \nabla \rho_\epsilon \cdot \nabla \phi \, dx + \int_U \rho_\epsilon \nabla c_\epsilon \cdot \nabla \phi \, dx \right|
\leq \|\nabla \rho_\epsilon^m\|_{L^{2(m+1)}/\mathbb{R}^n} \|\nabla \phi\|_{L^{m+1}/\mathbb{R}^n} + \|\rho_\epsilon \nabla c_\epsilon\|_{L^{2(m+1)}/\mathbb{R}^n} \|\nabla \phi\|_{L^{2(m+1)/m-1}}
\quad + \|\varepsilon^{1/2} \nabla \rho_\epsilon\|_{L^m} \|\nabla \phi\|_{L^{2(m+1)/m-1}}
\leq C(U) \left( \|\nabla \rho_\epsilon^m\|_{L^{2(m+1)/m-1}} + \|\rho_\epsilon \nabla c_\epsilon\|_{L^{2(m+1)/m-1}} + \|\varepsilon^{1/2} \nabla \rho_\epsilon\|_{L^m} \right) \|\nabla \phi\|_{L^{2(m+1)/m-1}}.
\]
Hence,
\[
\|\partial_t \rho_\epsilon\|_{W^{-1,p}(U)} \leq C(U) \left( \|\nabla \rho_\epsilon^m\|_{L^{2(m+1)/m-1}} + \|\rho_\epsilon \nabla c_\epsilon\|_{L^{2(m+1)/m-1}} + \|\varepsilon^{1/2} \nabla \rho_\epsilon\|_{L^m} \right).
\]
Furthermore, by (3.6) and (3.7) we can get that (3.17) is true in the following estimates

\[
\int_0^T \| \partial_t \rho_c \|_{W^{-1, p(U)}}^2 \, dt \\
\leq C(U) \left( \int_0^T \| \nabla \rho_c \|_{L^{m+1}}^{2(m+1)} \, dt + \int_0^T \| \rho_c \nabla c \|_{L^{2(m+1)}}^2 \, dt + \int_0^T \| \varepsilon^{1/2} \nabla \rho_c \|_{L^m}^2 \, dt \right) \\
\leq C(U, T) \left( \int_0^T \| \nabla \rho_c \|_{L^{m+1}}^{2(m+1)} \, dt + \int_0^T \| \rho_c \nabla c \|_{L^{2(m+1)}}^{m+1} \, dt + \int_0^T \| \varepsilon^{1/2} \nabla \rho_c \|_{L^m}^2 \, dt \right) \\
\leq C(U, T).
\]

Thus, (3.15) is true.

Now we are coming back to the proof of (3.16)-(3.18). Hölder’s inequality gives that

\[
\int_{\mathbb{R}^n} |\nabla \rho_c|^2 \, dx = C \int_{\mathbb{R}^n} |\rho_c^{\frac{m}{m+1}} \nabla \rho_c^{\frac{2}{m+1}}| \, dx \\
\leq C \left( \int_{\mathbb{R}^n} \rho_c^m \, dx \right)^{1/(m+1)} \left( \int_{\mathbb{R}^n} |\nabla \rho_c|^{m/(m+1)} \, dx \right)^{2/(m+1)} \\
= C \| \rho_c \|^m \| \nabla \rho_c \|^{2m/(m+1)}.
\]

Estimates in (3.12) and the above discussion implies that (3.16) is true. More precisely, by using \( r = \frac{m+1}{m} \) we have

\[
\| \nabla \rho_c \|_{L^2(\mathbb{R}^n + \mathbb{R}^{\frac{2m}{m+1}})} = \int_0^\infty \| \nabla \rho_c \|_{L^{m+1}}^{2(m+1)} \, dt \\
\leq C \int_0^\infty \| \rho_c \|^m \| \nabla \rho_c \|_{L^2}^{2m/(m+1)} \, dt \\
\leq C \int_0^\infty \| \nabla \rho_c \|_{L^2} \, dt = C \int_0^\infty \| \nabla \rho_c \|_{L^2} \, dt \leq C.
\]

Similarly, we have

\[
\int_{\mathbb{R}^n} |\rho_c \nabla c|^{2(m+1)} \, dx \leq \left( \int_{\mathbb{R}^n} \rho_c^{m+1} \, dx \right)^{2/(m+3)} \left( \int_{\mathbb{R}^n} |\nabla c|^2 \, dx \right)^{(m+1)/(m+3)} \\
= \| \rho_c \|^{2(m+1)/(m+3)} \| \nabla c \|^{2(m+1)/(m+3)}.
\]

Moreover, by (3.6) and (3.7) we can get that (3.17) is true in the following estimates with \( r_1 = \frac{m+1}{2} \).

\[
\| \rho_c \nabla c \|_{L^{m+1} \left( \mathbb{R}^n + \mathbb{R}^{\frac{2(m+1)}{m+3}} \right)} = \int_0^\infty \| \rho_c \nabla c \|_{L^{m+1} \left( \mathbb{R}^n + \mathbb{R}^{\frac{2(m+1)}{m+3}} \right)}^{2(m+1)} \, dt \\
\leq \int_0^\infty \| \rho_c \|_{L^{m+1}}^{2r_1(m+1)/(m+3)} \| \nabla c \|_{L^2}^{2r_1(m+1)/(m+3)} \, dt \\
\leq C \int_0^\infty \| \rho_c \|_{L^{m+1}}^{m+1} \, dt = C \int_0^\infty \| \rho_c \|_{L^{m+1}}^{m+1} \, dt \leq C.
\]

Since we can write

\[
\varepsilon^{1/2} \nabla \rho_c = \frac{2}{m} \varepsilon^{1/2} \rho_c^{1-\frac{m}{2}} \nabla \rho_c^{\frac{m}{2}},
\]
the term for parabolic regularization is easily done with Hölder’s inequality and (3.5)
\[ \| \varepsilon^{1/2} \nabla \rho_\varepsilon \|_{L^2(\mathbb{R}^+;L^n)} \leq C. \]

**Step 4.** In this step we show the estimates for \( \nabla \rho_\varepsilon \):
\[ \| \nabla \rho_\varepsilon \|_{L^r(0,T;L^n(\mathbb{R}^n))} \leq C, \quad \text{with } r = \min\left(2, \frac{3n+2}{n+4}\right). \] (3.19)

The estimate will be divided into two cases: \( n < 6 \) and \( n \geq 6 \). In the case of \( n < 6 \) which is equivalently \( m - \frac{1}{2} < 1 \), we can use the above estimate (3.12) to get estimate for \( \nabla \rho_\varepsilon \), while in the case \( n \geq 6 \), (3.12) is not useful. We will use \( \rho_\varepsilon^{2-m} \) as test function and get the estimate for \( \nabla \rho_\varepsilon \) directly from diffusion term.

**For the case** \( n < 6 \): We recast \( \nabla \rho_\varepsilon \) as
\[ \nabla \rho_\varepsilon = \frac{1}{m-1/2} \rho_\varepsilon^{3/2-m} \nabla \rho_\varepsilon^{m-1/2}. \]

From the estimates we obtained before, \( \rho_\varepsilon \in L^{m+1}(\mathbb{R}^+;L^{m+1}) \) and \( \nabla \rho_\varepsilon^{m-1/2} \in L^2(\mathbb{R}^+;L^2) \), by Hölder’s inequality, we have
\[
\int_{\mathbb{R}^n} |\nabla \rho_\varepsilon|^{2(m+1)} dx = C \int_{\mathbb{R}^n} \left| \rho_\varepsilon^{3/2-m} \left| \nabla \rho_\varepsilon^{m-1/2} \right|^{2(m+1)} \right|^2 dx
\leq C \left( \int_{\mathbb{R}^n} |v|^{2(m+1)} dx \right)^{1/p} \left( \int_{\mathbb{R}^n} \left| \nabla \rho_\varepsilon^{m-1/2} \right|^{2(m+1)} dx \right)^{1/q},
\]
where \( v := \rho_\varepsilon^{3/2-m} \in L^{\frac{m+1}{m-1}}(\mathbb{R}^+;L^{\frac{m+1}{m-1}}) \), with \( \frac{m+1}{(m-2)/2} > 2 \). We choose \( p = \frac{4-m}{3-2m} > 1 \) and \( q = \frac{4-m}{m+1} > 1 \) such that \( \frac{2(m+1)}{4-m} q = 2 \). Hence we have
\[ \| \nabla \rho_\varepsilon \|_{L^{\frac{2(m+1)}{4-m}}} \leq C \| v \|_{L^{\frac{m+1}{m-1}}} \| \nabla \rho_\varepsilon^{m-1/2} \|_{L^2}. \]

Furthermore, by using Hölder’s inequality in time integral, we have that
\[
\int_0^\infty \| \nabla \rho_\varepsilon \|_{L^{\frac{2(m+1)}{4-m}}} \, dt \leq C \left( \int_0^\infty \| v \|_{L^{\frac{m+1}{m-1}}} \, dt \right)^{1/p} \left( \int_0^\infty \| \nabla \rho_\varepsilon^{m-1/2} \|_{L^2} \, dt \right)^{1/q}
\leq C \left( \int_0^\infty \| v \|_{L^{\frac{m+1}{m-1}}} \, dt \right)^{1/p} \left( \int_0^\infty \| \nabla \rho_\varepsilon^{m-1/2} \|_{L^2} \, dt \right)^{1/q},
\]
where \( p \) and \( q \) are the same as before. Thus,
\[ \nabla \rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}(\mathbb{R}^+;L^{\frac{2(m+1)}{4-m}}) \quad \text{bounded uniformly in } \varepsilon. \]

Combining with the fact that \( \rho_\varepsilon \in L^\infty(\mathbb{R}^+;L^1 \cap L^m) \cap L^{m+1}(\mathbb{R}^+;L^{m+1}) \), we deduce that
\[ \rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}(\mathbb{R}^+;W^{1,\frac{2(m+1)}{4-m}}) \quad \text{bounded uniformly in } \varepsilon. \] (3.20)
In the case \( n \geq 6 \), By using \( \rho_\varepsilon^{2-m} \) as test function in the approximate problem (3.4), one has

\[
\int_{\mathbb{R}^n} \frac{1}{3 - m} \frac{d}{dt} \rho_\varepsilon^{3-m} dx + m(2 - m) \int_{\mathbb{R}^n} |\nabla \rho_\varepsilon|^2 dx + \varepsilon(2 - m) \int_{\mathbb{R}^n} \rho_\varepsilon^{1-m} |\nabla \rho_\varepsilon|^2 dx = \int_{\mathbb{R}^n} \rho_\varepsilon \nabla c_\varepsilon \cdot \nabla \rho_\varepsilon^{2-m} dx \leq C \int_{\mathbb{R}^n} \rho_\varepsilon^{1-m} dx.
\]

Now we only need to estimate \( \int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx \). Let \( u := \rho_\varepsilon^{m-1/2} \), we will use \( u \in L^\infty(\mathbb{R}^+_+; L^{\frac{m}{m-1}}) \) which is exactly \( \rho_\varepsilon \in L^\infty(\mathbb{R}^+_+; L^m) \). From (3.12), we have \( \nabla u \in L^2(\mathbb{R}^+_+; L^2) \). By Gagliardo Nirenberg–Sobolev inequality, we have

\[
\int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx = \int_{\mathbb{R}^n} u^{\frac{4-m}{m-1}} dx \leq C \|\nabla u\| \|u\|^{(1-\theta)\frac{4-m}{m-1}} \|u\|^{\frac{1}{m-1}},
\]

where \( 0 < \theta = \frac{4(3n-2)}{(n+2)(n+4)} < 1 \). Hence for any fixed \( T > 0 \), it holds

\[
\int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx dt \leq C \int_0^T \|\nabla u\|^{\frac{16}{m-1}} \|u\|^{(1-\theta)\frac{4-m}{m-1}} dt \leq C \left( \|\nabla u\|_{L^{\infty}(\mathbb{R}^+_+; L^{\frac{m}{m-1}})}, \|u\|_{L^2(\mathbb{R}^+_+; L^2)}, T \right),
\]

where we have used \( \frac{16}{m+1} \leq 2 \) for \( n \geq 6 \). Consequently, for any fixed \( T > 0 \),

\[
\frac{1}{3 - m} \int_{\mathbb{R}^n} \rho_\varepsilon^{3-m} dx + m(2 - m) \int_0^T \int_{\mathbb{R}^n} |\nabla \rho_\varepsilon|^2 dx dt + \varepsilon(2 - m) \int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{1-m} |\nabla \rho_\varepsilon|^2 dx dt \leq \frac{1}{3 - m} \|\rho_0\|_{L^{3-m}} + C \leq C \left( \|\rho_0\|_{L^m}, \|\rho_0\|_{L^1} \right) + C.
\]

In the above discussion we have used the fact that \( 3 - m \leq m \) in the case of \( n \geq 6 \). Thus we have

\[
\int_0^T \int_{\mathbb{R}^n} |\nabla \rho_\varepsilon|^2 dx dt \leq C, \text{ for any fixed } T > 0.
\]

Combining with the fact that \( \rho_\varepsilon \in L^\infty(\mathbb{R}^+_+; L^1 \cap L^m) \cap L^{m+1}(\mathbb{R}^+_+; L^{m+1}) \), we deduce that

\[
\rho_\varepsilon \in L^2(0, T; W^{1,2}) \quad \text{bounded uniformly in } \varepsilon. \tag{3.21}
\]

From (3.20), (3.21) and fact that \( \frac{2(m+1)}{4-m} = \frac{3n+2}{n+4} \), we have proved (3.19).

**Step 5.** From bounds (3.5), (3.6), (3.15) and (3.19) there exist subsequences of \( \rho_\varepsilon \) and \( c_\varepsilon \), without relabeled for convenience, have the following weak convergence

\[
\rho_\varepsilon \rightharpoonup \rho \text{ in } L^{m+1}(0, T; L^{m+1}(\mathbb{R}^n)),
\]

\[
\rho_\varepsilon \rightharpoonup \rho \text{ in } L^\infty(0, T; L^1 \cap L^m(\mathbb{R}^n)),
\]

\[
c_\varepsilon \rightharpoonup c \text{ in } L^\infty(0, T; L^s(\mathbb{R}^n)), \quad \frac{n}{n-s} < s \leq \frac{2n}{n-2},
\]

\[
\nabla \rho_\varepsilon \rightharpoonup \nabla \rho \text{ in } L^r(0, T; L^r(\mathbb{R}^n)), \quad r = \min \left( 2, \frac{3n+2}{n+4} \right),
\]

\[
\nabla c_\varepsilon \rightharpoonup \nabla c \text{ in } L^\infty(0, T; L^2(\mathbb{R}^n)).
\]
From (3.15), (3.17), and Lions-Aubin’s lemma, for any bounded domain $\Omega \subset \mathbb{R}^n$, there exists a subsequence of $\rho_\varepsilon$, without relabeled, such that

$$\rho_\varepsilon \to \rho \quad \text{in } L^r(0, T; L^{\bar{p}}(\Omega)), $$

where $r = \min \left(2, \frac{3n+2}{n+4}\right)$ and $\bar{p} = \min \left\{ \frac{3(n+2)n}{n^2+n-2}, \frac{2n}{n-2} \right\} > 2$. Let $\{B_k\}_{k=1}^\infty \subset \mathbb{R}^n$ be a sequence of balls centered at 0 with radius $R_k$, $R_k \to \infty$. By a standard diagonal argument, we can find a subsequence of, without relabeled, $\rho_\varepsilon$, have the following uniform strong convergence

$$\rho_\varepsilon \to \rho \quad \text{in } L^r(0, T; L^{\bar{p}}(B_k)), \quad \text{for all } k.$$

This leads to existence of a global weak solution. The regularity (3.2) comes directly from (3.22).

**Step 6.** In this step, we prove that the global weak solution obtained in step 5 decays to zero as $t \to \infty$.

By Gagliardo Nirenberg Sobolev inequality,

$$\int_{\mathbb{R}^n} \rho^{m+1} dx = \left\| \rho^{m-\frac{1}{2}} \right\|_{L^\frac{m+1}{m-\frac{1}{2}}}^{\frac{m+1}{m-\frac{1}{2}}} \leq C_{GNS} \left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \left\| \rho^{m-\frac{1}{2}} \right\|_{L^\frac{2(m-\frac{1}{2})}{m-\frac{3}{2}}} = C_{GNS} \left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \left\| \rho \right\|_{L^2}^{2-m}.$$

Or equivalently,

$$\left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \geq \frac{1}{C_{GNS}\left\| \rho \right\|_{L^2}^{2-m}} \int_{\mathbb{R}^n} \rho^{m+1} dx.$$

We have the following inequality for weak solution,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho^m dx \leq -\delta \int_{\mathbb{R}^n} |\nabla \rho^{m-\frac{1}{2}}|^2 dx \leq -\frac{\delta}{C_{GNS}\left\| \rho \right\|_{L^2}^{2-m}} \int_{\mathbb{R}^n} \rho^{m+1} dx.$$

On the other hand, we have

$$\left\| \rho \right\|_{L^m} \leq \left\| \rho \right\|_{L^{m+1}}^{\theta} \left\| \rho \right\|_{L^1}^{1-\theta}, \quad \theta = \frac{m^2 - 1}{m^2}.$$

Combining with the previous inequality, we have an inequality for $\left\| \rho \right\|_{L^m}$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho^m dx \leq -\frac{\delta}{C_{GNS}} \left\| \rho \right\|_{L^m}^{m-2} \left\| \rho \right\|_{L^m}^{\frac{m^2}{m-1}} \left\| \rho_0 \right\|_{L^1} \left( \int_{\mathbb{R}^n} \rho^m dx \right)^{\frac{1}{m}},$$

where $C_n = \frac{\delta}{C_{GNS}} \left\| \rho_0 \right\|_{L^\frac{m}{m-1}}^{\frac{1}{m}}, \quad \beta = \frac{2m^2 - 3m + 2}{m(m-1)} > 1$.

Then by solving this ordinary differential inequality, we have

$$\left\| \rho(\cdot, t) \right\|_{L^m} \leq \left( \frac{1}{(\beta - 1)C_n t + \left\| \rho_0 \right\|_{L^m}^{m(1-\beta)}} \right)^{\frac{1}{m(\beta-1)}},$$

which implies that the solution decays to zero in $L^m$ norm as $t \to \infty$.

$$\left\| \rho(\cdot, t) \right\|_{L^m} \leq C t^{-\frac{1}{m(\beta-1)}}, \quad \text{for large } t.$$

This completes the proof of the theorem. □
3.2. Blow-up of general solution. In this subsection, we will discuss the blow-up of the solution when $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$.

Recall that (1.20) gives a decomposition of the free energy

$$\mathcal{F}(\rho) = \frac{1}{m-1} \|\rho\|_{L^m}^m \left(1 - \frac{(m-1)c_n C(n)}{2} \|\rho\|_{L^m}^{4/(n+2)}\right) + \frac{c_n}{2} \left(C(n) \|\rho\|_{L^m}^2 - \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} \, dx \, dy\right)$$

=: $\mathcal{F}_1(\rho) + \mathcal{F}_2(\rho)$,

where $c_n = 1/(n(n-2)\alpha(n))$ and $\mathcal{F}_2(\rho) \geq 0$ from Hardy-Littlewood-Sobolev’s inequality and

$$\mathcal{F}_1(\rho) = f(\|\rho\|_{L^m}^m), \quad f(s) = \frac{1}{m-1} s - \frac{c_n}{2} C(n) s^{\frac{2}{m}}.$$

As we have already mentioned in the introduction that $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$. Hence it is also a critical point for $\mathcal{F}_1(\rho)$. In the following lemma, we show that $\|U_{\lambda,x_0}\|_{L^m}^m$ is indeed a maximum point for $f(s)$.

**Lemma 3.2.** $f(s)$ is a strict concave function and it reaches the unique maximum point at

$$s^* := \left(\frac{2n^2 \alpha(n)}{C(n)}\right)^{\frac{1}{m}} = \|U_{\lambda,x_0}\|_{L^m}^m,$$

where $U_{\lambda,x_0}$ is any stationary solutions of the equation (1.1), constants $\alpha(n)$ and $C(n)$ are given by (1.2) and (1.19), respectively.

**Proof.** Take first and second order derivatives for $f(s)$, one has $f'(s) = \frac{1}{m-1} - \frac{c_n}{m} C(n) s^{\frac{2}{m}}$ and $f''(s) = -\frac{c_n (2-m)}{m^2} C(n) s^{\frac{2}{m}-1} < 0$ for all $s > 0$. Thus $f(s)$ attains its maximum at

$$s^* = \left(\frac{2n^2 \alpha(n)}{C(n)}\right)^{\frac{1}{m}}.$$

Now we show that $s^* = \|U_{\lambda,x_0}\|_{L^m}^m$. By the formula of free energy with $\rho = U_{\lambda,x_0}$

$$\mathcal{F}(U_{\lambda,x_0}) = \frac{1}{m-1} \|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2} \int_{\mathbb{R}^n} U_{\lambda,x_0} C_{\lambda,x_0} \, dx,$$

and the critical case of Hardy-Littlewood-Sobolev’s inequality (1.18)

$$\mathcal{F}(U_{\lambda,x_0}) = \frac{1}{m-1} \|U_{\lambda,x_0}\|_{L^m}^m \left(1 - \frac{c_n}{2} \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{U_{\lambda,x_0}(x)U_{\lambda,x_0}(y)}{|x-y|^{n-2}} \, dx \, dy\right)$$

$$= \frac{1}{m-1} \|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2(n-2)\alpha(n)} C(n) \|U_{\lambda,x_0}\|_{L^m}^2,$$

we have

$$\int_{\mathbb{R}^n} U_{\lambda,x_0} C_{\lambda,x_0} \, dx = \frac{1}{(n-2)\alpha(n)} C(n) \|U_{\lambda,x_0}\|_{L^m}^2.$$

Notice that $C_{\lambda,x_0} = \frac{2n^2}{n-2} U_{\lambda,x_0}^{m-1}$ as in (1.21) in the introduction, we have

$$2n^2 \alpha(n) \|U_{\lambda,x_0}\|_{L^m}^m = C(n) \|U_{\lambda,x_0}\|_{L^m}^2.$$
from which we have
\[ \| U_{\lambda,x_0} \|_{L^m}^n = \left( \frac{2n^2 \alpha(n)}{C(n)} \right)^{\frac{n}{n-2}} = s^*. \]

Another way to prove \( s^* = \| U_{\lambda,x_0} \|_{L^m}^n \) is direct verification of (1.23). \[ \square \]

Before stating and proving our main theorem in this section, let us first prove the following technical lemma.

**Lemma 3.3.** Assume \( F(\rho_0) < F(U_{\lambda,x_0}) \), \( \| \rho_0 \|_{L^m} > \| U_{\lambda,x_0} \|_{L^m} \) and \( \rho \) is a solution of (1.1), then there is \( \mu > 1 \) such that \( \rho \) satisfies
\[ \| \rho \|_{L^m}^n > \mu \| U_{\lambda,x_0} \|_{L^m}^n. \]

**Proof.** Since \( F(\rho_0) < F(U_{\lambda,x_0}) \), we can choose \( \delta > 0 \) such that \( F(\rho_0) < \delta F(U_{\lambda,x_0}) \). By (1.20) with Hardy-Littlewood-Sobolev's inequality (1.18) and the fact that \( F(\rho(t),t) \) is non-increasing in \( t \), we have
\[ f(|\rho|_{L^m}^n) = F_1(\rho) \leq F(\rho) \leq F(\rho_0) < \delta F(U_{\lambda,x_0}) = \delta f(s^*). \]
Then for any \( s > \| U_{\lambda,x_0} \|_{L^m}^n \), \( f(s) \) is a strictly decreasing function. So it has a strictly decreasing inverse function \( f^{-1} \). Hence if \( \| \rho_0 \|_{L^m} > \| U_{\lambda,x_0} \|_{L^m} \), we have for some \( \mu > 1 \),
\[ \| \rho \|_{L^m}^n > \mu \| U_{\lambda,x_0} \|_{L^m}^n. \]
\[ \square \]

**Theorem 3.4.** Assume \( m_2(0) = \int_{\mathbb{R}^n} |x|^2 \rho_0(x) dx < \infty, F(\rho_0) < F(U_{\lambda,x_0}) \) and \( \| \rho_0 \|_{L^m(\mathbb{R}^n)} > \| U_{\lambda,x_0} \|_{L^m(\mathbb{R}^n)} \), then there is a finite time \( t^* > 0 \) such that
\[ \| \rho(t) \|_{L^m(\mathbb{R}^n)} \to \infty \text{ as } t \to t^*. \]

**Proof.** Here we use the formula
\[ \nabla c = -\frac{1}{n\alpha(n)} \frac{x}{|x|^n} * \rho(x). \]
By Lemma 3.3, (1.11) and the monotonicity of free energy, we deduce that
\[ \frac{d}{dt} m_2(t) \leq -4\mu \| U_{\lambda,x_0} \|_{L^m}^n + 2(n-2)F(\rho_0) \leq -4\mu \| U_{\lambda,x_0} \|_{L^m}^n + 2(n-2)F(U_{\lambda,x_0}) = -4(\mu-1) \| U_{\lambda,x_0} \|_{L^m}^n < 0, \]
where we have used \( F(U_{\lambda,x_0}) = \frac{2}{n^2} \| U_{\lambda,x_0} \|_{L^m}^n \) in the third equality, see (1.22). This means that there is a \( t^* > 0 \) such that \( \lim_{t \to t^*} m_2(t) = 0 \).

On the other hand, \( \forall R > 0 \), by using Hölder inequality, we have
\[ \int_{\mathbb{R}^n} \rho(x) dx \leq \int_{B_R} \rho(x) dx + \int_{B_R^c} \rho(x) dx \leq CR^{(n-2)/2} \| \rho \|_{L^m} + \frac{1}{R^2} m_2(t). \]
Now by choosing \( R = (\frac{m_2(t)}{C \| \rho \|_{L^m}^{n-2}})^{2/(n-2)} \), we have
\[ \| \rho \|_{L^1} \leq C \| \rho \|_{L^m}^n m_2(t) \frac{n-2}{n+2}. \]
So, \( \lim_{t \to t^*} \frac{\|\rho\|_{L^m}}{t^{1/2}} \geq \lim_{t \to t^*} \frac{\|\rho\|_{L^1}}{C(n)m_2(t)^{1/2}} = \infty. \)

Remark 3.5. There is a gap between \( C_s \) and \( \|U_{\lambda,x_0}\|_{L^m} \). So there is still a space, in the case that \( C_s \leq \|U_0\|_{L^m} \leq \|U_{\lambda,x_0}\|_{L^m} \) that we don’t know the solution exists or blows up. See appendix for the comparison between \( C_s \) and \( \|U_{\lambda,x_0}\|_{L^m} \).

4. Appendix.

4.1. Proof of Proposition 1.5. The invariance of the free energy \( F(\rho) \) is obvious in the translation transformation \( \rho_{x}(x) = \rho(x + \bar{x}) \) and the orthogonal transformation \( R\rho(x) = \rho(R^{-1}x) \), since the determinant to Jacobian matrix is 1 for the these two transformations.

By the scaling transformation \( \rho_\lambda(x) = \lambda^{\frac{n+2}{2}} \rho(\lambda x) \) and direct computation, we have

\[
F(\rho_\lambda) = \int_{\mathbb{R}^n} \frac{\lambda^{(n+2)m}}{m-1} \rho^m(\lambda x) \, dx - \frac{c_n}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\lambda^{n+2} \rho(\lambda x) \rho(\lambda y)}{|x-y|^{n-2}} \, dxdy
\]

\[
= \int_{\mathbb{R}^n} \frac{\lambda^n \rho^m(x)}{m-1} \, dx - \frac{c_n}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(\lambda x) \rho(\lambda y)}{|x-y|^{n-2}} \lambda^{n-2} \lambda^{n+2} \, dxdy
\]

\[
= F(\rho).
\]

Notice that the Kelvin transformation of \( c \) is

\[
c_{\bar{x},\lambda}(x) = \left( \frac{\lambda}{|x-\bar{x}|} \right)^{n-2} c\left( \bar{x} + \lambda^2(x-\bar{x}) \right).
\]

we thus have the related transformation for \( \rho \) in the following

\[
\rho_{\bar{x},\lambda}(x) = \left( \frac{\lambda}{|x-\bar{x}|} \right)^{n+2} \rho\left( \bar{x} + \lambda^2(x-\bar{x}) \right).
\]

Therefore the free energy for \( \rho_{\bar{x},\lambda} \) is

\[
F(\rho_{\bar{x},\lambda}) = \int_{\mathbb{R}^n} \frac{1}{m-1} \left( \frac{\lambda}{|x-\bar{x}|} \right)^{(n+2)m} \left[ \rho \left( \bar{x} + \lambda^2(x-\bar{x}) \right) \right]^m \, dx
\]

\[
- \frac{c_n}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \rho\left( \bar{x} + \lambda^2(x-\bar{x}) \right) \rho\left( \bar{x} + \lambda^2(y-\bar{x}) \right) \left( \frac{\lambda^2}{|x-\bar{x}||y-\bar{x}|} \right)^{n+2} \, dxdy
\]

\[
=: I_1 - \frac{c_n}{2} I_2.
\]

Here

\[
I_1 := \int_{\mathbb{R}^n} \frac{1}{m-1} \left( \frac{\lambda}{|x-\bar{x}|} \right)^{2m} \left[ \rho \left( \bar{x} + \lambda^2(x-\bar{x}) \right) \right]^m \, dx
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{m-1} \left[ \rho \left( \bar{x} + \lambda^2(x-\bar{x}) \right) \right]^m \, d \left( \bar{x} + \lambda^2(x-\bar{x}) \right),
\]

\[
I_2 := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left( \frac{\lambda^2|y-\bar{x}|}{|x-\bar{x}||y-\bar{x}|} \right)^{2-n} \rho(z) \rho(w) \, dwdz
\]

\[
= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(z) \rho(w)}{|z-w|^{n-2}} \, dwdz,
\]
where \( z = \bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}, \) \( w = \bar{x} + \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}, \) and we have used
\[
|z - w| = \left| \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2} - \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2} \right| = \frac{\lambda^2|x-y|}{|x-\bar{x}||y-\bar{x}|}.
\]
Hence we have
\[
F(\rho_{\bar{x},\lambda}) = I_1 - \frac{c_n}{2} I_2 = \int_{\mathbb{R}^n} \frac{\rho^m(z)}{m-1} dz - \frac{c_n}{2} \int_{\mathbb{R}^n} \frac{\rho(z)\rho(w)}{|z-w|^{n-2}} dzdw = F(\rho).
\]

**4.2. Gap between \( C_s \) and \( \|U_{\lambda,x_0}\|_{L^m} \).** Since \( L^m \) norm of \( U_{\lambda,x_0} \) doesn't depend on \( \lambda \) and \( x_0 \), we will use \( \|U\|_{L^m} = \|U_{\lambda,x_0}\|_{L^m} \) in the following. Among the estimates in the proof of global existence of weak solution, Theorem 3.1, we used only an important Gagliardo-Nirenberg-Sobolev inequality which is
\[
\|v\|_{L^r} \leq \|v\|^{\theta}_{L^2} \|v\|^{1-\theta}_{L^r} \leq \tilde{C}_{GNS} \|\nabla v\|_{L^2}^{\theta} \|v\|_{L^r}^{1-\theta},
\]
where in our case, \( r = \frac{2(m+1)}{2m-1} \), \( \theta = \frac{2m-1}{m+1} \) and \( C_{GNS} = (\tilde{C}_{GNS})^{\frac{m+1}{2}} \).

From [26, p. 202], we know the best constant for Sobolev embedding
\[
\|\nabla f\|_{L^2} \geq S_n \|f\|_{L^2}, \quad S_n = \frac{n(n-2)}{4} 2^\frac{n}{2} \pi^{1+\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right)^{-\frac{1}{2}},
\]
which gives
\[
C_{GNS} = \left(S_n^{-\frac{2}{n}}\right)^\frac{m+1}{m-2}.
\]
(4.1)

We can calculate that \( C_s \) is strictly less than \( \|U\|_{L^m} \). In fact, from (3.1) and (1.23) that
\[
C_s - \|U\|_{L^m} = \left(\frac{4m^2}{(2m-1)^2 C_{GNS}}\right)^{\frac{1}{n-2}} - \left(n^{\frac{n+1}{2}} 2^{\frac{1}{2}} \pi^{1+\frac{n}{2}} \Gamma^{-1} \left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right)^{\frac{1}{n-2}}
\]
\[
= \left(\frac{4m^2}{(2m-1)^2 \frac{4}{n(n-2)} 2^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \left(\frac{n+1}{2}\right)^{\frac{1}{2}}}\right)^{\frac{n+2}{n-2}} - \left(n^{\frac{n+1}{2}} 2^{\frac{1}{2}} \pi^{1+\frac{n}{2}} \Gamma^{-1} \left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right)^{\frac{n+2}{n-2}}
\]
\[
= \left[\frac{m^2(n-2)}{(2m-1)^2}\right]^{\frac{n+2}{n-2}} - \left[\frac{n^2+2}{n-2}\right]^{\frac{n+2}{n-2}} \left(n^{\frac{n+1}{2}} 2^{\frac{1}{2}} \pi^{1+\frac{n}{2}} \Gamma^{-1} \left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right)^{\frac{n+2}{n-2}} < 0,
\]
due to \( m^2(n-2)/(2m-1)^2 < n/2 \) for all \( n \geq 3 \).

**Remark 4.1.** Although there is a big gap between these two constants, we can see that our globally existed weak solution also have algebraic decay in time when \( \|\rho_0\|_{L^m} < C_s \). While for initial data \( C_s \leq \|\rho_0\|_{L^m} \leq \|U\|_{L^m} \), we don't expect decay in time. Our conjecture is to have global existence in this case. We leave this topic to our future work.

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