A damage-softening statistical constitutive model considering rock residual strength

Zhi-liang Wang\textsuperscript{a,}\textsuperscript{*}, Yong-chi Li\textsuperscript{a}, J.G. Wang\textsuperscript{b}

\textsuperscript{a}School of Engineering Science, University of Science and Technology of China, Anhui, Hefei 230027, China  
\textsuperscript{b}Centre for Protective Technology, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore

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Abstract

Under stress, the microcracks in rock evolve (initiation, growth and coalescence) from damage to fracture with a continuous process. In order to describe this continuous process, a damage-softening statistical constitutive model for rock was proposed based on the Weibull distribution of mesoscopic element strength. This model usually adopts the Drucker–Prager criterion as its distribution parameter of mesoscopic element strength, which may produce larger damage zone in numerical simulations. This paper mainly studies the effects of strength criteria and residual strength on the performance of this damage-softening statistical constitutive model of rock. Main works include following three aspects: Firstly, the mechanical behaviors of rock are comparatively studied when the Drucker–Prager and the Mohr–Coulomb criteria are employed, respectively, as the distribution parameter. Then, a coefficient is introduced to make this constitutive model be capable of describing the residual strength of rock. Finally, a user-defined subroutine is concisely developed for this model and checked through typical strain paths. The current work lays a good foundation for further application of this model in geotechnics and geosciences.

Keywords: Microcracks; Mesoscopic element; Weibull distribution; Strength criterion; Damage-softening; Rock residual strength

1. Introduction

The mechanism of rock failure is complicated since flaws such as cracks and voids commonly exist in rock. These discontinuities have significant influences on the damage processes and failure characteristics of rock (Grady and Kipp, 1987). It is now recognized that the process of macroscopic fracture actually involves a multiplicity of microcracks in a brittle rock. The essential nature of brittle rock lies not only in the initiation and propagation of individual cracks but also in the interaction and coalescence of crack populations (Blair and Cook, 1998; Liu et al., 2004). A great deal of work has been done to model the main features of rock damage and fragmentation so far (Wawersik and Fairhurst, 1970; Kranz, 1979; Blair and Cook, 1992).

The nucleation and growth of microcracks typically ranges from 0.01 to 1.0 mm in width, leading to a concentration of these microcracks into a narrow zone and producing a visible macroscopic fissure wider than 1.0 mm (Cerrolaza and Garcia, 1997), so the process from damage to fracture will be studied on a mesoscopic scale in the present study. A mesoscopic element is considered to be
isotropically elastic and its elastic properties are defined by the Young’s modulus and the Poisson’s ratio. The stress–strain relationship is linear-elastic until the given damage threshold is attained, and is then followed by softening (Liu et al., 2004). Thus, the macroscopic strength, the static and dynamic properties of rock depend on the statistical mechanical properties of individual mesoscopic elements. The properties can be described by a phenomenological model through statistical method.

It is well known that statistic physics is a ligament that communicates continuum mechanics, damage mechanics and material mechanics. It also plays an important role in the development of rock constitutive laws (Weibull, 1951; Krajcinovic and Silva, 1982; Weber and Saint-Lot, 1983; Xie and Gao, 2000). For example, an external load makes the pre-existing flaws and microcracks in brittle rock propagate and nucleate until the loss of rock strength. However, numerical simulation of this dynamic propagation of every microcrack is not practical in either numerical algorithms or computational cost. Instead, the effects of microcracks on the stiffness of rock can be expressed through an equivalent continuum model on a macroscopic or mesoscopic scale. This continuum model can take the damage-induced softening, residual strength and failure characteristics into account (Ma et al., 1998).

A damage-softening statistical constitutive model for rock was ever developed by Tang (1993) and Cao and Fang (1998). This model has an important parameter, namely the distribution parameter of mesoscopic element strength. Cao and Fang (1998) adopted the Drucker–Prager (D–P) criterion as this distribution parameter. However, their model produces a larger damage zone and obtains conservative results. In this study, classical Mohr–Coulomb (M–C) criterion will be introduced into the statistical constitutive model of rock to improve the model accuracy in damage-softening and residual strength. This paper is organized as follows: Firstly, the numerical performances of the statistical constitutive model are comparatively studied when the M–C and the D–P criteria are employed, respectively, as the distribution parameter of mesoscopic elemental strength. Then, a coefficient is introduced into the damage variable to make this model be able to describe the residual strength of rock. At last, a user-defined subroutine is developed for this statistical constitutive model and its performance is checked with typical strain paths of isotropic compression and $K_0$ compression.

2. Principle of statistical constitutive law

In recent years, continuum damage mechanics has become a powerful tool to model the constitutive behaviors of brittle rock. Most of damage models are based on homogeneous rock and describe the responses of rock by the degradation of its elasticity. Previous research work showed that continuum damage models can effectively simulate the elastic degradation caused by pre-existing microcracks in rock (Grady and Kipp, 1987). Although rock always exhibits anisotropy after macroscopic fissures occur, isotropic damage model is still an effective method to estimate the gross damage of rock subjected to external loading.

2.1. Evolution equation of damage variable

When the stress of a mesoscopic element satisfies its strength criterion, this element fails and a degraded Young’s modulus arises. The extent of degradation depends on the accumulation of mesoscopic elemental fragmentation. Based on the linear damage mechanics (Lemaitre, 1992), the Young’s modulus of a damaged material is expressed as

$$E_d = E(1 - D), \quad (1)$$

where $D$ is a damage variable which takes a value between 0 and 1 which corresponds to intact or undamaged as well as fully damaged states, respectively. $E_d$ is the Young’s modulus of damaged material and $E$ is the Young’s modulus for undamaged material. This paper assumes that mesoscopic element and its damage are isotropic, thus $E$, $E_d$ and $D$ are all scalar.

Before rock is modeled, following assumptions are made for simplification: (i) rock is an isotropic, homogenous, continuous and brittle material with pre-existing microcracks on a macro-scale; (ii) elastic damage constitutive law is applicable to each mesoscopic element; (iii) the damage of rock is continuously developed and is the gradual accumulation of failures in mesoscopic elements; (iv) strength of mesoscopic elements observes following Weibull distribution function (Weibull, 1951; Keats and Lawrence, 1997; Tang, 1997):

$$P(F) = \begin{cases} 
\frac{m}{F_0} \left( \frac{F}{F_0} \right)^{m-1} \exp \left[-\left( \frac{F}{F_0} \right)^m \right] & F > 0, \\
0 & F \leq 0,
\end{cases} \quad (2)$$

where $F$ is an elemental strength parameter or stress level and $F_0$ is its mean value. $m$ is the shape
parameter or a homogeneous index of material (Tang, 1997) which measures the concentration of $F$. As shown in Fig. 1, larger $m$ implies higher uniformity.

Let $N$ denote the number of all mesoscopic elements and $N_f$ denote the number of all failed mesoscopic elements. The damage variable $D$ can be directly defined as

$$D = \frac{N_f}{N}. \quad (3)$$

When the stress level $F$ increases to $F + dF$, the number of failed mesoscopic elements increases by $N P(F) dF$. If external load increases from zero to $F$, the total number of failed mesoscopic elements is

$$N_f(F) = \int_0^F N P(y) dy = N \left\{ 1 - \exp \left[ - \left( \frac{F}{F_0} \right)^m \right] \right\}. \quad (4)$$

Substituting Eq. (4) into (3) yields

$$D = 1 - \exp \left[ - \left( \frac{F}{F_0} \right)^m \right]. \quad (5)$$

This is the damage evolution equation of mesoscopic elements in the statistical constitutive model of rock. Fig. 2 shows the variations of damage variable with $F/F_0$ for different $m$. The damage variable increases with the increase of $F/F_0$. It is also noted that larger $m$ has bigger damage variable at the same stress level, thus fully damaged state can be more easily achieved.

2.2. Stress–strain relation for conventional triaxial test

In conventional triaxial test of rock sample, the components of ‘apparent stress’ $\sigma_1$, $\sigma_2$ and $\sigma_3$($\sigma_2 = \sigma_3$), can be measured directly from experiments. Let $\sigma_1^e$, $\sigma_2^e$ and $\sigma_3^e$($\sigma_2^e = \sigma_3^e$) denote the corresponding ‘effective stress’ components. According to Lemaitre (1992), the ‘apparent stress’ and ‘effective stress’ have following relationship:

$$\sigma_i^e = \sigma_i / (1 - D) \quad (i = 1, 2, 3). \quad (6)$$

Based on the Hooke’s law for linear elasticity, the axial strain $\varepsilon_1$ is expressed as

$$\varepsilon_1 = \frac{1}{E} (\sigma_1^e - 2\nu \sigma_3^e). \quad (7)$$

Substituting Eq. (6) into (7) produces

$$\frac{1}{1 - D} = \frac{E \varepsilon_1}{\sigma_1 - 2\nu \sigma_3}, \quad (8)$$

where $\nu$ is the Poisson’s ratio, whose change with damage is not considered in the present study according to Jaeger and Cook (1979).

Therefore, following stress–strain relationship is obtained from Eqs. (5) and (8):

$$\sigma_1 = E \varepsilon_1 \exp \left[ - \left( \frac{F}{F_0} \right)^m \right] + 2\nu \sigma_3. \quad (9)$$

Further, effective stress is obtained from Eqs. (8) and (6) as follows:

$$\sigma_i^e = \frac{\sigma_i E \varepsilon_1}{\sigma_1 - 2\nu \sigma_3} \quad (i = 1, 2, 3). \quad (10)$$
A set of detailed triaxial test data for argillaceous quartzite is obtained from Jaeger and Cook (1979). This rock has the Young’s modulus $E = 90.0$ GPa, the Poisson’s ratio $\nu = 0.25$ and internal friction angle $\varphi = 31.3^\circ$. Fig. 3 shows the experimental curves of $\sigma_1 \sim \varepsilon_1$ when confining pressure $\sigma_3$ varies from 3.45 to 27.6 MPa.

3. Effect of strength criteria

3.1. Determination of distribution parameter by means of strength criteria

A shear failure criterion for a rock element can be generally expressed as

$$f(\sigma) = \tau_f,$$  \hspace{1cm} (11)

where $\tau_f$ is the compression strength and $f(\sigma)$ is the stress level. The rock element fails when $f(\sigma) \geq \tau_f$.

The $f(\sigma)$ can be determined from the failure criterion of rock. Tang (1993) expressed the strength criterion of mesoscopic elements in strain space. However, the failure criterion of rock is usually expressed in stress space. For example, the D–P criterion was employed by Cao and Fang (1998)

$$F = f(\sigma) = a I_1 + \sqrt{J_2} = k,$$  \hspace{1cm} (12)

where $I_1$ is the first stress invariant, and $J_2$ is the second deviatoric stress invariant. $a$ and $k$ are material constants (Owen and Hinton, 1980) which are expressed by

$$a = \frac{\sin \varphi}{\sqrt{3(3 + \sin^2 \varphi)}}, \quad k = \frac{3c \cos \varphi}{\sqrt{3(3 + \sin^2 \varphi)}},$$  \hspace{1cm} (13)

in which $c$ is the cohesion.

3.2. Determination of model parameters

Fig. 3 shows that axial stress $\sigma_1$ has a peak value in the $\sigma_1 \sim \varepsilon_1$ curve in the triaxial tests of brittle rock. The peak point is quite near to the yield point which can be used to determine the model parameter $F_0$ and $m$. At this point, the derivative of $\sigma_1$ with respect to $\varepsilon_1$ should be equal to zero, therefore, this determination method is called as ‘Extremum Method’ (Friedman, 1956).

Differentiating Eq. (10) gets

$$\frac{\partial \sigma_1}{\partial \varepsilon_1} = \frac{E}{F_0} \exp \left[-\left(\frac{F}{F_0}\right)^m \right]$$

$$+ E\varepsilon_1 \exp \left[-\left(\frac{F}{F_0}\right)^m \right] (-m) \times \left(\frac{F}{F_0}\right)^{m-1} \frac{1}{F_0} \left\{ \frac{\partial F}{\partial \varepsilon_1} \right\} = 0.$$  \hspace{1cm} (16)

Rearranging of Eq. (16) gives

$$1 + (-m)\varepsilon_1 \left(\frac{F}{F_0}\right)^{m-1} \frac{1}{F_0} \left\{ \frac{\partial F}{\partial \varepsilon_1} \right\} = 0.$$  \hspace{1cm} (17)

For either the D–P or the M–C criterion, one can deduce following equation from Eqs. (9), (12), (14) and (15):

$$\varepsilon_1 \left\{ \frac{\partial F}{\partial \varepsilon_1} \right\} = F.$$  \hspace{1cm} (18)

Therefore, Eq. (17) can be re-written as

$$1 - m \left(\frac{F}{F_0}\right)^m = 0.$$  \hspace{1cm} (19)
Substitution of Eq. (19) into (10) yields
\[
\begin{align*}
  m &= \frac{1}{C_0} \ln \left( \frac{s_1}{C_0}^2 v^3 \right) + E/C_{15}^1 \left( \frac{1}{C_0} \right) \sin j F_0/C_{18}/C_{19} \left( \frac{s_1}{C_3} + s_3 \right),
  \\
  F_0 &= \sqrt[m]{m F}.
\end{align*}
\]

Hence, \( m \) and \( F_0 \) can be calculated when the values of \( s_1, \varepsilon_1 \) and \( s_3 \) at the peak point, along with \( E, v \) and \( \phi \) are known.

Above method is suitable for both the D–P and the M–C criteria, but the two criteria have a little different numerical performance. Fig. 4 compares the effects on the damage-softening behaviors of rock where hollow dots indicate experimental data. This figure shows that the broken line for the M–C criterion is in better agreement with the test data while the continuous line for the D–P criterion has larger error. In addition, Xu and Zheng (1985) also found that the plastic zone was rather larger when the D–P criterion was adopted in the stability analysis of an earth slope. The results are more conservative from the D–P criterion than from the M–C criterion.

4. Consideration of rock residual strength

Qian and Yin (1996) considered that the residual strength of geomedia plays an important role in the stability of geotechnical systems such as slope and foundation. The factor of safety may be lower if the residual strength of geomedium is ignored. In previous section, the influences of strength criteria on the performances of the rock damage-softening statistical constitutive model were discussed. However, neither is able to consider the effects of rock residual strength because their \( s_1/C_2/C_2 \) curves take on linear decrease during softening phase.

If the damage variable \( D \) in Eq. (6) is multiplied by a coefficient \( C_n \) (Xu and Wei, 2002), the stress–strain relationship for conventional triaxial tests is
\[
\sigma_1 = E\varepsilon_1 \left\{ (1 - C_n) + C_n \exp \left[ -\left( \frac{F}{F_0} \right)^m \right] \right\} + 2\sigma_3.
\]

The \( C_n \) is taken from 0 to 1. Clearly, Eq. (21) can retrogress to Eq. (9) when \( C_n \) is equal to 1.0.

Substitution of Eq. (15) into (21) gets
\[
\sigma_1 = E\varepsilon_1 \left\{ (1 - C_n) + C_n \exp \left[ -\left( \frac{(\sigma_1^1 - \sigma_3^1) - (\sigma_1^*_1 + \sigma_3^*) \sin \phi}{F_0} \right)^m \right] \right\} + 2\sigma_3.
\]

Because Eq. (22) has the term \( C_n \), the above ‘Extremum Method’ may be onerous in determination of \( m \) and \( F_0 \). Herein, the ‘Line Fitting Method’ (Friedman, 1956; Cao and Fang, 1998) is employed to determine these parameters. In Eq. (22), by moving the term \( 2\sigma_3 \) to the left and simultaneously dividing with \( \varepsilon_1 \) on both sides, one has
\[
\frac{\sigma_1 - 2\varepsilon_3}{E\varepsilon_1} + C_n - 1 = C_n \exp \left[ -\left( \frac{(\sigma_1^1 - \sigma_3^1) - (\sigma_1^*_1 + \sigma_3^*) \sin \phi}{F_0} \right)^m \right].
\]

After taking logarithms twice on Eq. (23), following equation is obtained:
\[
\ln \left[ -\ln \left( \frac{\sigma_1 - 2\varepsilon_3}{E\varepsilon_1} + C_n - 1 \right) + \ln (C_n) \right] \]
\[ m \ln((\sigma_1^f - \sigma_2^f) - (\sigma_1^f + \sigma_2^f) \sin \varphi) - m \ln(F_0). \]

(24)

This is a linear equation with slope coefficient of \( m \) and intercept \(-m \ln(F_0)\) if \( C_n \) is pre-determined. Therefore, parameters \( m \) and \( F_0 \) can be easily determined through a linear regression analysis on a set of triaxial test data of rock samples. For example, when \( \sigma_3 = 6.9 \) MPa, the final fitting equation is

\[
\sigma_1 = Ee1 \{0.10 + 0.90 \exp \left[ - \left( \frac{(\sigma_1^f - \sigma_2^f) - (\sigma_1^f + \sigma_2^f) \sin \varphi}{183.0} \right)^{3.516} \right] \} + 2v\sigma_3.
\]

(25)

and when \( \sigma_3 = 13.8 \) MPa, the stress–strain relationship is

\[
\sigma_1 = Ee1 \{0.10 + 0.90 \exp \left[ - \left( \frac{(\sigma_1^f - \sigma_2^f) - (\sigma_1^f + \sigma_2^f) \sin \varphi}{197.7} \right)^{2.890} \right] \} + 2v\sigma_3.
\]

(26)

Fig. 5 compares the \( \sigma_1 \sim \epsilon_1 \) curves with experimental data when \( C_n = 1.0 \) and 0.9. It is clear that the results of Eq. (22) can agree favorably well with those triaxial test data particularly at the residual strength of rock.

It is worth noting that both \( m \) and \( F_0 \) vary with confining pressures \( \sigma_3 \). The experimental data in Fig. 3 are used to determine their dependence on confining pressures \( \sigma_3 \). The experimental data are linearly fitted and the results are illustrated in Fig. 6. Clearly, \( m \) reduces with the increase of \( \sigma_3 \) while \( F_0 \) rises with \( \sigma_3 \). The fitted relations are

\[
\ln m = 1.591 - 0.036\sigma_3, \quad (27)
\]

\[
F_0 = 166.96 + 1.829\sigma_3. \quad (28)
\]

5. Development of constitutive subroutine

A constitutive subroutine is necessary before the above statistical damage model is implemented into one finite element code. This subroutine can make the constitutive model applicable in numerical analysis. This section will develop a user-defined subroutine for the above damage-softening statistical constitutive model.

As well known, the commercial software such as LS-DYNA or ABAQUS usually offers an interface for user-defined material model. In the user-defined subroutine, strain increments \( \Delta e_{ij} \) are usually as input and corresponding return components are stresses \( \sigma_{ij} \). According to the Hooke’s law, the above statistical constitutive model is written in K–G form as

\[
\Delta \sigma_{ij} = K_d \Delta e_{kk} \delta_{ij} + 2G_d \Delta e_{ij}, \quad (29)
\]

where \( \delta_{ij} = 0 \) for \( i \neq j \) and \( \delta_{ij} = 1 \) for \( i = j \). \( \sigma_{ij} \) and \( e_{ij} \) denote the stress tensor and the deviatoric strain tensor, respectively, and \( e_{kk} \) is the volumetric strain. The degraded moduli \( K_d \) and \( G_d \) are then calculated from the degraded Young’s modulus \( E_d \) and original Poisson’s ratio \( v \).
The flow for the user-defined subroutine is developed through following procedures:

(i) calculate strain components $\varepsilon_{ij}$ for given strain increments $\Delta \varepsilon_{ij}$;
(ii) calculate current volumetric strain $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$;
(iii) evaluate effective stress components $\sigma_{ij}^*$ based on Eq. (6);
(iv) compute distribution parameter $F$ of strength, and then evaluate damage variable $D$;
(v) let $D = 1.0$ if $D > 1.0$. Calculate degraded moduli $E_d$, $K_d$ and $G_d$;
(vi) compute stress components $\sigma_{ij}$ by use of Eq. (29).

The developed user-defined subroutine is applied to some typical strain-path tests to check its feasibility and validity. The parameters $m$ and $F_0$ are evaluated according to Eqs. (27) and (28) and other material constants ($E$, $v$ and $\phi$) are the same as mentioned before.

Fig. 7 shows the numerical results under isotropic strain compression: (a) stress–strain curves; (b) damage evolution for $C_n = 0.95$ case.

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K\textsubscript{0} compression (\(\varepsilon_2 = \varepsilon_3 = 0\)) is also simulated with this subroutine. In this situation, the increment \(\Delta\varepsilon_1\) is still taken as 1.0 \(\times\) 10\(^{-6}\). Fig. 8(a) shows that the residual strength of rock can be described with a proper \(C_n\). Lower \(C_n\) has a higher residual strength. As shown in Fig. 8(b), complete damage turns up at \(\varepsilon_v = 0.0096\). The damage variable monotonically increases with volumetric strain before complete damage.

6. Conclusions

A statistical constitutive model was proposed in the present study for rock to describe its damage-softening and residual strength. This model used the M–C criterion rather than the D–P criterion as its distribution parameter. It also introduced a coefficient \(C_n\) to improve its capability in the description of residual strength. From this work, following conclusions can be drawn:

First, the current damage-softening statistical constitutive model is effective to describe the mechanical properties of brittle rock. It can correlate stress–strain property, damage effect and residual strength of rock. Furthermore, this model has fewer parameters and is more convenient for application.

Second, the selection of distribution parameter of mesoscopic elemental strength has some influences on the performance of the current model. The use of D–P criterion normally produces larger damage zone. The classical M–C criterion is more suitable for the current model.

Third, the user-defined subroutine for this damage-softening statistical constitutive model is not only succinct but also conveniently implemented into certain large-scale commercial software. It should have wide application prospect in geotechnical engineering.

It should be stressed that the rock for the present model is assumed to remain continuous during simulation. The terminology ‘microcrack’ is extensively used, but the microcracks are not treated explicitly. On account of this reason, it might be more accurate to use the terminology ‘fictitious cracks’ as proposed by Bawden et al. (1993).

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