ISOMORPHISMS OF SOME CYCLIC ABELIAN
COVERS OF SYMMETRIC DIGRAPHS

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Abstract. Let $D$ be a connected symmetric digraph, $\mathbb{Z}_p$ a cyclic group of prime
order $p (> 2)$ and $\Gamma$ a group of automorphisms of $D$. We enumerate the number of
$\Gamma$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p \times \mathbb{Z}_p$-covers of $D$ for any nonunit $g \in \mathbb{Z}_p \times \mathbb{Z}_p$.

1. Introduction

Graphs and digraphs treated here are finite and simple.

Let $D$ be a symmetric digraph and $A$ a finite group. A function $\alpha : A(D) \rightarrow A$
is called alternating if $\alpha(y, x) = \alpha(x, y)^{-1}$ for each $(x, y) \in A(D)$. For $g \in A$, a
g-cyclic $A$-cover (or $g$-cyclic cover) $D_g(\alpha)$ of $D$ is the digraph as follows:

\[ V(D_g(\alpha)) = V(D) \times A, \text{ and } ((u, h), (v, k)) \in A(D_g(\alpha)) \text{ if and only if} \]
\[ (u, v) \in A(D) \text{ and } k^{-1}h\alpha(u, v) = g. \]

The natural projection $\pi : D_g(\alpha) \rightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$
which erases the second coordinates. A digraph $D'$ is called a cyclic $A$-cover of
$D$ if $D'$ is a $g$-cyclic $A$-cover of $D$ for some $g \in A$. In the case that $A$ is abelian,
then $D_g(\alpha)$ is simply called a cyclic abelian cover.

Let $\alpha$ and $\beta$ be two alternating functions from $A(D)$ into $A$, and let $\Gamma$ be
a subgroup of the automorphism group $\text{Aut } D$ of $D$, denoted $\Gamma \leq \text{Aut } D$. Let
$g, h \in A$. Then two cyclic $A$-covers $D_g(\alpha)$ and $D_h(\beta)$ are called $\Gamma$-isomorphic,
denoted $D_g(\alpha) \cong_{\Gamma} D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \rightarrow D_h(\beta)$ and
a $\gamma \in \Gamma$ such that $\pi \Phi = \gamma \pi$, i.e., the diagram

\[ \begin{array}{ccc}
D_g(\alpha) & \overset{\Phi}{\longrightarrow} & D_h(\beta) \\
\pi \downarrow & & \downarrow \pi \\
D & \overset{\gamma}{\longrightarrow} & D
\end{array} \]

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commutes. Let $I = \{1\}$ be the trivial subgroup of automorphisms.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\mathbb{Z}_3$-covers) of a complete symmetric digraph. Mizuno and Sato [16] gave a formula for the characteristic polynomial of a cyclic $\alpha$-cover of a symmetric digraph $D$, for any finite group $\alpha$. Mizuno and Sato [15,17] enumerated the number of $I$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p^n$-covers and $g$-cyclic $\mathbb{Z}_p^\ast$-covers, and $I$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p$-covers of $D$ for any prime $p (> 2)$. Furthermore, Mizuno, Lee and Sato [14] gave a formula for the the number of $I$-isomorphism classes of connected $g$-cyclic $\mathbb{Z}_p^n$-covers and connected $g$-cyclic $\mathbb{Z}_p^\ast$-covers of $D$ for any prime $p (> 2)$.

A graph $H$ is called a covering of a graph $G$ with projection $\pi : H \to G$ if there is a surjection $\pi : V(H) \to V(G)$ such that $\pi|_{N(v')} : N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \to G$ is an $n$-fold covering of $G$ if $\pi$ is $n$-to-one. A covering $\pi : H \to G$ is said to be regular if there is a subgroup $B$ of the automorphism group Aut $H$ of $H$ acting freely on $H$ such that the quotient graph $H/B$ is isomorphic to $G$.

Let $G$ be a graph and $\alpha$ a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to $G$. Then a mapping $\alpha : D(G) \to \alpha$ is called an ordinary voltage assignment if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The ordinary derived graph $G^\alpha$ derived from an ordinary voltage assignment $\alpha$ is defined as follows:

$$V(G^\alpha) = V(G) \times A,$$

and

$$((u, h), (v, k)) \in D(G^\alpha) \text{ if and only if } (u, v) \in D(G) \text{ and } k = ha(u, v).$$

The graph $G^\alpha$ is called an $A$-covering of $G$. The $A$-covering $G^\alpha$ is an $|A|$-fold regular covering of $G$. Every regular covering of $G$ is an $A$-covering of $G$ for some group $A$ (see [3]). Furthermore the 1-cyclic $A$-cover $D_1(\alpha)$ of a symmetric digraph $D$ can be considered as the $A$-covering $\tilde{D}^\alpha$ of the underlying graph $\tilde{D}$ of $D$.

A general theory of graph coverings is developed in [4]. $\mathbb{Z}_2$-coverings (double coverings) of graphs were dealt in [5] and [19]. Hofmeister [6] and, independently, Kwak and Lee [11] enumerated the $I$-isomorphism classes of $n$-fold coverings of a graph, for any $n \in \mathbb{N}$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The $I$-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of $I$-isomorphism classes of $\mathbb{Z}_n$-coverings, $\mathbb{Z}_p \oplus \mathbb{Z}_p$-coverings and $D_n$-coverings, $n$: odd, of graphs, respectively.

In the case of connected coverings, Kwak and Lee [13] enumerated the $I$-
isomorphism classes of connected $n$-fold coverings of a graph $G$. Furthermore, Kwak, Chun and Lee \cite{[12]} gave some formulas for the number of $I$-isomorphism classes of connected $A$-coverings of a graph $G$ when $A$ is a finite abelian group or $D_n$.

We present the number of $\Gamma$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p \times \mathbb{Z}_p$-covers of connected symmetric digraphs for any element $g \neq 0 \in \mathbb{Z}_p \times \mathbb{Z}_p$, where 0 is the unit of $\mathbb{Z}_p \times \mathbb{Z}_p$.

2. Isomorphisms of cyclic $\mathbb{Z}_p \times \mathbb{Z}_p$-covers

Let $D$ be a connected symmetric digraph and $A$ a finite abelian group. The group $\Gamma$ of automorphisms of $D$ acts on the set $C(D)$ of alternating functions from $A(D)$ into $A$ as follows:

$$\alpha^\gamma(x, y) = \alpha(\gamma(x), \gamma(y)) \text{ for all } (x, y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$.

Let $G$ be the underlying graph of $D$. The set of ordinary voltage assignments of $G$ with voltages in $A$ is denoted by $C^1(G; A)$. Note that $C(D) = C^1(G; A)$. Furthermore, let $C^0(G; A)$ be the set of functions from $V(G)$ into $A$. We consider $C^0(G; A)$ and $C^1(G; A)$ as additive groups. The homomorphism $\delta : C^0(G; A) \to C^1(G; A)$ is defined by $(\delta s)(x, y) = s(x) - s(y)$ for $s \in C^0(G; A)$ and $(x, y) \in A(D)$. The 1-cohomology group $H^1(G; A)$ with coefficients in $A$ is defined by $H^1(G; A) = C^1(G; A)/\text{Im} \delta$. For each $\alpha \in C^1(G; A)$, let $[\alpha]$ be the element of $H^1(G; A)$ which contains $\alpha$.

The automorphism group $\text{Aut} A$ acts on $C^0(G; A)$ and $C^1(G; A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma \alpha)(x, y) = \sigma(\alpha(x, y)) \text{ for } (x, y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C^1(G; A)$ and $\sigma \in \text{Aut} A$. A finite group $B$ is said to have the isomorphism extension property (IEP), if every isomorphism between any two isomorphic subgroups $E_1$ and $E_2$ of $B$ can be extended to an automorphism of $B$ (see \cite{[9]}). For example, the cyclic group $\mathbb{Z}_n$ for $n \in \mathbb{N}$, the dihedral group $D_n$ for odd $n \geq 3$, and the direct sum of $m$ copies of $\mathbb{Z}_p (p: \text{ prime})$ have the IEP.

Mizuno and Sato \cite{[17]} gave a characterization for two cyclic $A$-covers of $D$ to be $\Gamma$-isomorphic.

**Theorem 1.** (17, Corollary 3) Let $D$ be a connected symmetric digraph, $G$ the underlying graph of $D$, $A$ a finite abelian group with the IEP, $g \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq \text{Aut} D$. Assume that the order of $g$ is odd. Then the following are equivalent:

1. \( D_g(\alpha) \cong \Gamma D_g(\beta) \).

2. There exist \( \gamma \in \Gamma \), \( \sigma \in \text{Aut} A \) and \( s \in C^0(G; A) \) such that

\[
\beta = \sigma \alpha^\gamma + \delta s \quad \text{and} \quad \sigma(g) = g.
\]

Let \( \text{Iso}(D, A, g, \Gamma) \) denote the number of \( \Gamma \)-isomorphism classes of \( g \)-cyclic \( A \)-covers of \( D \). The following result holds.

**Theorem 2.** (17, Theorem 3) Let \( D \) be a connected symmetric digraph, \( A \) a finite abelian group with the IEP, \( g, h \in A \) and \( \Gamma \leq \text{Aut} D \). Assume that the orders of \( g \) and \( h \) are equal and odd, and \( \rho(g) = h \) for some \( \rho \in \text{Aut} A \). Then

\[
\text{Iso}(D, A, g, \Gamma) = \text{Iso}(D, A, h, \Gamma).
\]

Let \( p \) be odd prime and \( Z_p \) the cyclic group of order \( p \). Then \( Z_p^2 = Z_p \times Z_p \) has the IEP. Since \( Z_p \) is the 2-dimensional vector space over \( Z_p \), the general linear group \( GL_2(Z_p) \) is the automorphism group of \( Z_p^2 \). Furthermore, \( GL_2(Z_p) \) acts transitively on \( Z_p^2 \setminus \{0\} \). Set \( e = e_1 = (1 \ 0) \in Z_p^2 \). By Theorem 2, we have \( \text{Iso}(D, A, g, \Gamma) = \text{Iso}(D, A, e, \Gamma) \) for any element \( g \in Z_p^2 \setminus \{0\} \). Thus we consider the number of \( \Gamma \)-isomorphism classes of \( e \)-cyclic \( Z_p^2 \)-covers of \( D \).

Let \( \Gamma \leq \text{Aut} D \) and \( \Pi = GL_2(Z_p) \). Furthermore, set

\[
\Pi_e = \{ \sigma \in \Pi | \sigma(e) = e \}.
\]

An action of \( \Pi_e \times \Gamma \) on \( H^1(G; Z_p^2) \) is defined as follows:

\[
(A, \gamma)[\alpha] = [A \alpha^\gamma] = \{ A \alpha^\gamma + \delta s | s \in C^0(G; Z_p^2) \},
\]

where \( A \in \Pi_e \), \( \gamma \in \Gamma \) and \( \alpha \in C^1(G; Z_p^2) \). By Theorem 1, the number of \( \Gamma \)-isomorphism classes of \( e \)-cyclic \( Z_p^2 \)-covers of \( D \) is equal to that of \( \Pi_e \times \Gamma \)-orbits on \( H^1(G; Z_p^2) \).

Let \( D \) be a connected symmetric digraph, \( G \) the underlying graph of \( D \), \( \Gamma \leq \text{Aut} D \), \( \gamma \in \Gamma \) and \( \lambda \in Z_p^* \). A \( \langle \gamma \rangle \)-orbit \( \sigma \) of length \( k \) on \( E(G) \) is called **diagonal** if \( \sigma = \langle \gamma \rangle \{x, \gamma^k(x)\} \) for some \( x \in V(G) \). The vertex orbit \( \langle \gamma \rangle x \) and the arc orbit \( \langle \gamma \rangle(x, \gamma^k(x)) \) are also called **diagonal**. A diagonal arc orbit of length \( 2k \) (the corresponding edge orbit of length \( k \) and the corresponding vertex orbit of length \( 2k \)) is called **type-1** if \( k^2 = -1 \) (or \( m = 2k \)), and **type-2** otherwise, where \( m \) is the order \( ord(\lambda) \) of \( \lambda \).

For \( \gamma \in \Gamma \), let \( G(\gamma) \) be a simple graph whose vertices are the \( \langle \gamma \rangle \)-orbits on \( V(G) \), with two vertices adjacent in \( G(\gamma) \) if and only if some two of their representatives are adjacent in \( G \). The \( k \)th \( p \)-level of \( G(\gamma) \) is the induced subgraph of \( G(\gamma) \) on the vertices \( \omega \) such that \( \theta(|\omega|) = p^k \), where \( \theta(i) \) is the largest power
of $p$ dividing $i$. A $p$-level component $H$ of $G(\gamma)$ is a connected component of some $p$-level of $G(\gamma)$, where $H$ is considered as a subset of $V(G(\gamma))$. A $p$-level component $H$ is called minimal if there exists no vertex $\sigma$ of $H$ which is adjacent in $G(\gamma)$ to a vertex $\omega$ such that $\theta(|\sigma|) > \theta(|\omega|)$ (see [12]).

Let $k \in N$. Then a $\langle \gamma \rangle$-orbit $\sigma$ on $V(G)$, $E(G)$ or $A(D)$ is called $k$-divisible if $|\sigma| \equiv 0 \pmod{k}$. A vertex orbit $\sigma$ is called edge-induced if there exists an orbit $\langle \gamma \rangle \{x, y\}$ on $E(G)$ with $x, y \in \sigma$. A $k$-divisible $\langle \gamma \rangle$-orbit $\sigma$ on $V(G)$ is called strongly $k$-divisible if $\sigma$ is edge-induced and satisfies the following condition:

$$\text{If } \Omega = \langle \gamma \rangle(x, y) \text{ is any } \langle \gamma \rangle \text{-orbit on } A(D), \text{ and }$$
$$y = \gamma^j(x), \ x, y \in \sigma, \text{ then } j \equiv 0 \pmod{k}.$$ 

For $k \geq 1$, let $H$ be a $k$ th $p$-level component of $G(\gamma)$. Then $H$ is called $p$-favorable if $H$ is minimal and there exists a $\sigma \in H$ which is not strongly $p$-divisible. Furthermore, $H$ is called $p$-defective if $H$ is minimal and each vertex $\sigma$ of $H$ is strongly $p$-divisible.

Let $\lambda \in Z_p^*$ and $\text{ord} (\lambda) = m$. Then, let $G_{\lambda}(\gamma)$ be the subgraph of $G(\gamma)$ induced by the set of $m$-divisible $\langle \gamma \rangle$-orbits on $V(G)$. The $k$th $p$-level and $p$-level components of $G_{\lambda}(\gamma)$ are defined similarly to the case of $G(\gamma)$. A $p$-level component $H$ of $G_{\lambda}(\gamma)$ is called defective if each vertex $\sigma$ of $H$ is strongly $m$-divisible, not type-1 diagonal, and $H$ is minimal. Note that, if $\sigma = \langle \gamma \rangle x$ is strongly $m$-divisible, $|\sigma| = t$ and there exists a diagonal $\langle \gamma \rangle$-orbit $\Omega = \langle \gamma \rangle(x, \gamma^{t/2}(x))$ on $A(D)$, then $\Omega$ is type-2.

**THEOREM 3.** Let $D$ be a connected symmetric digraph, $G$ its underlying graph, $p$ odd prime, $g \in Z^2_p \setminus \{0\}$, and $\Gamma \leq \text{Aut} G$. For $\gamma \in \Gamma$, let $\epsilon(\gamma)$, $\rho(\gamma)$ and $\epsilon_1(\gamma)$ be the number of $\langle \gamma \rangle$-orbits, diagonal $\langle \gamma \rangle$-orbits and, not diagonal and $p$-divisible $\langle \gamma \rangle$-orbits on $E(G)$, respectively. Let $\nu(\gamma)$ and $\nu_0(\gamma)$ be the number of $\langle \gamma \rangle$-orbits and not $p$-divisible $\langle \gamma \rangle$-orbits on $V(G)$, respectively. Moreover, let $c(\gamma)$, $\xi(\gamma)$, $d(\gamma)$ and $d_1(\gamma)$ be the number of $p$-level components, minimal $p$-level components, $p$-defective $p$-level components and not minimal $p$-level components with $p$-divisible orbits in $G(\gamma)$, respectively. For $\gamma \in \Gamma$ and $\lambda \in Z_p^*$, let $\nu_0(\gamma, \lambda)$, $\mu(\gamma, \lambda)$ and $d(\gamma, \lambda)$ be the number of not $m$-divisible $\langle \gamma \rangle$-orbits on $V(G)$, type-2 diagonal $\langle \gamma \rangle$-orbits on $E(G)$ and defective $p$-level components in $G_{\lambda}(\gamma)$, respectively, where $m = \text{ord} (\lambda)$. Furthermore, let $\kappa(\gamma, \lambda)$ be the number of not $m$-divisible $\langle \gamma \rangle$-orbits on $E(G)$ which are not diagonal. Then the number of $\Gamma$-isomorphism classes of $g$-cyclic $Z_p^2$-covers of $D$ is

$$\text{Iso} (D, Z^2_p, g, \Gamma) = \frac{1}{p(p-1)|\Gamma|} \sum_{\gamma \in \Gamma} \left\{p^{2(\epsilon(\gamma) - \nu(\gamma) + \xi(\gamma) - \rho(\gamma))} + (p - 1)p^{\epsilon(\gamma) - 2\nu(\gamma) + \nu_0(\gamma) - \rho(\gamma) + \epsilon_1(\gamma) + \epsilon(\gamma) - d_1(\gamma) + d(\gamma)} \right\}$$
\[ + p \sum_{\lambda \in \mathbb{Z}_p^* \setminus \{1\}} p^{2(\epsilon(\gamma) - \nu(\gamma)) + \xi(\gamma) + \nu_0(\gamma, \lambda) - \epsilon(\gamma, \lambda) - \mu(\gamma, \lambda) + d(\gamma, \lambda)}. \]

**Proof.** By the preceding remark and Burnside' Lemma, the number of $\Gamma$-isomorphism classes of $e$-cyclic $\mathbb{Z}_p^2$-covers of $D$ is

\[ \frac{1}{|\Pi_e| \cdot |\Gamma|} \sum_{(A, \gamma) \in \Pi_e \times \Gamma} |H^1(G; \mathbb{Z}_p^2)^{(A, \gamma)}|, \]

where $U^{(A, \gamma)}$ is the set consisting of the elements of $U$ fixed by $(A, \gamma)$.

Now, we have

\[ \Pi_e = \left\{ \begin{bmatrix} 1 & \mu \\ 0 & \lambda \end{bmatrix} \mid \lambda = 1, 2, \ldots, p - 1; \mu = 0, 1, \ldots, p - 1 \right\}. \]

Then there exist $p + 1$ conjugacy classes of $\Pi_e$:

\[ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & p - 1 \\ 0 & 1 \end{bmatrix} \right\}, \]

\[ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix}, \ldots, \begin{bmatrix} 1 & p - 1 \\ 0 & \lambda \end{bmatrix} \right\} (\lambda = 2, \ldots, p - 1). \]

Let $A, B \in \Pi_e$ be conjugate. Then there exists an element $C \in \Pi_e$ such that $CAC^{-1} = B$. Thus $[\alpha] \in H^1(G; \mathbb{Z}_p^2)^{(A, \gamma)}$ if and only if $A\alpha^\gamma = \alpha + \delta s$ for some $s \in \mathcal{C}^0(G; \mathbb{Z}_p^2)$. But $A\alpha^\gamma = \alpha + \delta s$ if and only if $B(C\alpha)^\gamma = C\alpha + \delta(Cs)$, i.e., $[C\alpha] \in H^1(G; \mathbb{Z}_p^2)^{(B, \gamma)}$. By the fact that a mapping $[\alpha] \longmapsto [C\alpha]$ is bijective, we have

\[ |H^1(G; \mathbb{Z}_p^2)^{(A, \gamma)}| = |H^1(G; \mathbb{Z}_p^2)^{(B, \gamma)}|. \]

Therefore the number of $\Gamma$-isomorphism classes of $e$-cyclic $\mathbb{Z}_p^2$-covers of $D$ is

\[ \frac{1}{|\Pi_e| \cdot |\Gamma|} \sum_{\gamma \in \Gamma} \left( |H^1(G; \mathbb{Z}_p^2)^{(I, \gamma)}| + (p - 1)|H^1(G; \mathbb{Z}_p^2)^{(A_1, \gamma)}| \right) + p \sum_{\lambda = 2}^{p-1} |H^1(G; \mathbb{Z}_p^2)^{(A_\lambda, \gamma)}|, \]

where

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}. \]

Let $(A, \gamma) \in \Pi_e \times \Gamma$.

**Case 1:** $A = I$. 

Then \([a] \in H^1(G; \mathbb{Z}_p^2)^{(I, \gamma)}\) if and only if \(\alpha' = I\alpha' = \alpha + \delta s\) for some \(s \in C^0(G; \mathbb{Z}_p^2)\).

Now, let \(\alpha = ae_1 + be_2, a, b \in C^1(G; \mathbb{Z}_p)\), where \(e_1 = t(10)\) and \(e_2 = \ell(01)\). Furthermore, let \(s = we_1 + ze_2, w, z \in C^0(G; \mathbb{Z}_p)\). Then \(\alpha' = \alpha + \delta s\) if and only if \(a' = a + \delta w\) and \(b' = b + \delta z\), i.e., \([a]' = [a]\) and \([b]' = [b]\). Note that \([a], [b] \in H^1(G; \mathbb{Z}_p)\). Since \([ae_1 + be_2] = [a]e_1 + [b]e_2\), we have

\[|H^1(G; \mathbb{Z}_p^2)^{(I, \gamma)}| = |H^1(G; \mathbb{Z}_p)\|^2.\]

By Theorem 5 of [8], it follows that

\[|H^1(G; \mathbb{Z}_p^2)^{(I, \gamma)}| = (p^{\epsilon(\gamma) - \nu(\gamma) + \xi(\gamma) - \rho(\gamma) + \nu_0(\gamma, \lambda) - \kappa(\gamma, \lambda) - \mu(\gamma, \lambda) + d(\gamma)}).\]

Case 2: \(A = A_\lambda\).

Then \([a] \in H^1(G; \mathbb{Z}_p^2)^{(A_\lambda, \gamma)}\) if and only if \(A_\lambda \alpha' = \alpha + \delta s\) for some \(s \in C^0(G; \mathbb{Z}_p^2)\). Let \(\alpha = ae_1 + be_2, a, b \in C^1(G; \mathbb{Z}_p)\) and \(s = we_1 + ze_2, w, z \in C^0(G; \mathbb{Z}_p)\). Then \(A_\lambda \alpha' = \alpha + \delta s\) if and only if \(a' = a + \delta w\) and \(b' = b + \delta z\), i.e., \(a' = a + \delta w\) and \(b' = b + \delta z\). Thus \((A_\lambda, \gamma)[a] = [a]\) if and only if \([a]' = [a]\) and \(\lambda[b]' = [b]\). Therefore, we have

\[|H^1(G; \mathbb{Z}_p^2)^{(A_\lambda, \gamma)}| = |H^1(G; \mathbb{Z}_p)^\gamma| \cdot |H^1(G; \mathbb{Z}_p)^{(\gamma, \lambda)}|.\]

By Theorem 5 of [8] and Theorem 3.3 of [18], it follows that

\[|H^1(G; \mathbb{Z}_p^2)^{(A_\lambda, \gamma)}| = p^{\epsilon(\gamma) - 2\nu(\gamma) + \nu_0(\gamma, \lambda) - \rho(\gamma) - \kappa(\gamma, \lambda) - \mu(\gamma, \lambda) + d(\gamma)}\]

Case 3: \(A = A_1\).

Then we have

\[|H^1(G; \mathbb{Z}_p^2)^{(A_1, \gamma)}| = p^{\epsilon(\gamma) - 2\nu(\gamma) - \rho(\gamma) - \kappa(\gamma, \lambda) - \mu(\gamma, \lambda) + d(\gamma)}\]

The detail is developed in Section 3.

By cases 1,2 and 3, the result follows. \(\blacksquare\)

**Corollary 1.** (15, Corollary 4.6) Let \(D\) be a connected symmetric digraph and \(p\) odd prime. Then the number of \(I\)-isomorphism classes of \(g\)-cyclic \(\mathbb{Z}_p^2\)-covers of \(D\) is

\[\text{Iso}(D, \mathbb{Z}_p^2, g, I) = p^{B(D)} + p^{B(D)-1}(p^{B(D)} - 1).\]

where \(B(D) = \frac{1}{2}|A(D)| - |V(G)| + 1\) is the Betti-number of \(D\).

**Proof.** Since \(I = \{1\}\), we have \(\epsilon(1) = \kappa(1, \lambda) = |E(G)|\), \(\rho(1) = \epsilon_1(1) = \mu(1, \lambda) = 0\), \(\nu(1) = \nu_0(1) = \nu_0(1, \lambda) = |V(G)|\), \(c(1) = \xi(1) = 1\) and \(d_1(1) = d(1) = d(1, \lambda) = 0\). \(\blacksquare\)
3. The elements of $H^1(G;\mathbb{Z}_p^2)$ fixed by $(A_1, \gamma)$

Let $D$ be a connected symmetric digraph, $G$ its underlying graph and $\Gamma \leq \text{Aut} \ G$. We present the number of elements on $H^1(G;\mathbb{Z}_p^2)$ fixed by $(A_1, \gamma)$ for each $\gamma \in \Gamma$. The argument is an analogue of Hofmeister’s method \cite{5}.

Let

$$A = A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$ 

Note that the order $\text{ord}(A)$ of $A$ is $p$ and

$$A^j = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$$

for any $j$. Set $C^0 = C^0(G;\mathbb{Z}_p^2)$, $C^1 = C^1(G;\mathbb{Z}_p^2)$, and $H^1 = H^1(G;\mathbb{Z}_p^2)$. We consider the following exact sequence:

$$0 \longrightarrow \text{Ker} \delta \longrightarrow C^0 \xrightarrow{\delta^0} C^0 \longrightarrow C^1 \xrightarrow{\delta} H^1 \longrightarrow 0,$$

where $\delta^0$ is the canonical monomorphism and $\delta^1$ is the canonical epimorphism. For $\gamma \in \Gamma$, two endomorphisms $\mu_{\gamma}: C^1 \rightarrow C^1$ and $\nu_{\gamma}: H^1 \rightarrow H^1$ are defined as follows: $\mu_{\gamma}(\alpha) = A\alpha^\gamma - \alpha$ and $\nu_{\gamma}([\alpha]) = [A\alpha^\gamma - \alpha]$, where $\alpha \in C^1$. Then, note that $\nu_{\gamma}\delta^1 = \delta\mu_{\gamma}$ and $\text{Ker} \nu_{\gamma} = (H^1)^{(A, \gamma)}$.

Now, let $C^0_{\gamma} = \delta^{-1}(\text{Im} \mu_{\gamma})$ and $C^1_{\gamma} = \mu^{-1}(\text{Im} \delta)$.

Let $(x, y)$ be any arc of $A(D)$, $\gamma \in \Gamma$ and $t = |\langle \gamma \rangle(x, y)|$ the length of the arc $\langle \gamma \rangle$-orbit containing $(x, y)$. Furthermore, let $s = t(u, v) = ue_1 + ve_2 \in C^0$, $u, v \in C^0(G;\mathbb{Z}_p)$.

**Lemma 1.** Let $\gamma \in \Gamma$, $s \in C^0$. Then $s \in C^0_{\gamma}$ if and only if

$$v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x) = v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t-1}}(y) \quad \cdots (\ast)_1$$

for each $(x, y) \in A(D)$. Specially, if $p \mid t$, then

$$\sum_{j=0}^{t-1} A^j s^{\gamma^j}(x) = \sum_{j=0}^{t-1} A^j s^{\gamma^j}(y) \quad \cdots (\ast)_2.$$

**Proof.** Set $t = |\langle \gamma \rangle(x, y)|$.

Suppose that $s \in C^0_{\gamma}$. Then there exists $\alpha \in C^1$ such that $A\alpha^\gamma - \alpha = \delta s$.

Thus

$$A^i \alpha^\gamma - \alpha = \delta(A^{i-1}\alpha^{\gamma^{i-1}} + \cdots + A\alpha^\gamma + s), \ i \geq 1.$$
Let \((x, y) \in A(D)\). Then we have
\[
\sum_{j=0}^{t-1} A^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} A^j s^{\gamma^j}(y) = A^t \alpha^\gamma(x, y) - \alpha(x, y) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \alpha(x, y).
\]
For the \((2, 1)\)-array of the above equation, we have
\[
v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x) - \{v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t-1}}(y)\} = 0.
\]
Specially, if \(p \mid t\), then we have
\[
\sum_{j=0}^{t-1} A^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} A^j s^{\gamma^j}(y) = 0.
\]
Conversely, assume that \(s = ^i(u, v)\) satisfies \((*)_1\) and \((*)_2\) for each \((x, y) \in A(D)\). Let \(\Omega\) be any \(\langle \gamma \rangle\)-orbit on \(A(D)\), \(|\Omega| = t\) and \((x, y) \in \Omega\). If \(\Omega\) is not diagonal, then let \(\alpha(x, y) = 0 \cdot e_1 + b(x, y)e_2, b \in C^1(G; \mathbb{Z}_p)\) be defined as follows:
\[
\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b(x, y) \end{bmatrix} = \sum_{j=0}^{t-1} A^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} A^j s^{\gamma^j}(y).
\]
If \(p \mid t\), then we may set \(b(x, y) = 0\). Furthermore, if \(\Omega\) is diagonal, then let
\[
\alpha(x, y) = -(A^l + I)^{-1} \sum_{j=0}^{l-1} (A^j s^{\gamma^j}(x) - A^j s^{\gamma^j}(y)),
\]
where \(l = \frac{t}{2}\).
Now, let
\[
A^i \alpha^\gamma(x, y) = \alpha(x, y) + \sum_{j=0}^{i-1} (A^j s^{\gamma^j}(x) - A^j s^{\gamma^j}(y)), \quad i \geq 1 \quad \cdots (1).
\]
If \(\Omega\) is not diagonal and \(t \not\equiv 0 \pmod{p}\), then we have
\[
A^{pt} \alpha^{\gamma^t}(x, y)
\]
\[
= \alpha(x, y) + \sum_{j=0}^{pt-1} (A^j s^{\gamma^j}(x) - A^j s^{\gamma^j}(y))
\]
\[
= \alpha(x, y) + (I + A^t + \cdots + A^{(p-1)t}) \sum_{j=0}^{t-1} (A^j s^{\gamma^j}(x) - A^j s^{\gamma^j}(y)) = \alpha(x, y).
\]
Furthermore, let \(\Omega\) is not diagonal and \(p \mid t\). Then we have
\[
A^t \alpha^\gamma(x, y) = \alpha(x, y) + \sum_{j=0}^{t-1} (A^j s^{\gamma^j}(x) - A^j s^{\gamma^j}(y)) = \alpha(x, y).
\]
Next, let $\Omega$ be diagonal. Then we have
\[
\sum_{j=0}^{r-1}(A^{i} s^{\gamma^{i}}(x) - A^{i} s^{\gamma^{i}}(y)) = (I - A^{i}) \sum_{j=0}^{r-1}(A^{j} s^{\gamma^{j}}(x) - A^{j} s^{\gamma^{j}}(y))
\]
\[
= -(I - A^{i})(I + A^{i})\alpha(x, y)
\]
\[
= \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \alpha(x, y),
\]
where $t = 2l$. If $p \mid t$, then $A^{t} \alpha^{\gamma^{t}}(x, y) = \alpha(x, y)$ similarly to the case that $\Omega$ is not diagonal. Otherwise,
\[
A^{2pl} \alpha^{2pl}(x, y)
\]
\[
= \alpha(x, y) + (I - A^{l} + A^{2l} + \cdots - A^{2(p-1)l+l}) \sum_{j=0}^{l-1}(A^{j} s^{\gamma^{j}}(x) - A^{j} s^{\gamma^{j}}(y))
\]
\[
= \alpha(x, y).
\]
Therefore it follows that (1) is well-defined.

By (1), we have
\[
A^{r} \alpha^{\gamma^{r+i}}(x, y) = \alpha^{\gamma^{i}}(x, y) + \sum_{j=0}^{r-1}(A^{j} s^{\gamma^{j}}(\gamma^{i}(x)) - A^{j} s^{\gamma^{j}}(\gamma^{i}(y))), \quad r, i \geq 1.
\]
If $\Omega$ is not diagonal, then we define $\alpha(v, u) = -\alpha(u, v), (u, v) \in \Omega$.
In the case that $\Omega$ is diagonal, we have
\[
\alpha(y, x) = \alpha^{\gamma^{l}}(x, y)
\]
\[
= A^{-l}(\alpha(x, y) + \sum_{j=0}^{l-1} \delta A^{j} s^{\gamma^{j}}(x, y))
\]
\[
= A^{-l} \alpha(x, y) - A^{-l}(A^{l} + I)\alpha(x, y)
\]
\[
= -\alpha(x, y).
\]
Furthermore, we have
\[
A^{i+l} \alpha^{\gamma^{i}}(y, x) = A^{l}\{A^{i} \alpha^{\gamma^{i+l}}(x, y)\}
\]
\[
= A^{l}\{\alpha^{\gamma^{i}}(x, y) + \sum_{j=0}^{i-1} (A^{j} s^{\gamma^{j}}(\gamma^{i}(x)) - A^{j} s^{\gamma^{j}}(\gamma^{i}(y)))\}
\]
\[
= -A^{l}\{\alpha(x, y) + \sum_{j=0}^{i-1} (A^{j} s^{\gamma^{j}}(x) - A^{j} s^{\gamma^{j}}(y))\}
\]
\[
= -A^{i+l} \alpha^{\gamma^{i}}(x, y),
\]
i.e., \( \alpha^\gamma(y, x) = -\alpha^\gamma(x, y), \ i \geq 1. \)

Therefore we obtain an \( \alpha \in C^1 \) such that \( A\alpha^\gamma - \alpha = \delta s \), i.e., \( s \in C^0_\gamma. \)

**Lemma 2.** For \( \gamma \in \Gamma \),

\[
|C^0_\gamma| = p^{2n-2\nu(\gamma)+\nu_0(\gamma)+c(\gamma)-d(\gamma)+\nu_1(\gamma), \ n = |V(G)|.
\]

**Proof.** We count the number of \( s \in C^0_\gamma \) which satisfy both \((*)_1\) and \((*)_2\) for each \((x, y) \in A(D)\).

Case 1: \( x, y \) are in the same \( \langle \gamma \rangle \)-orbit \( \sigma \) on \( V(G) \).

Then \( \Omega \) is not diagonal and \( |\sigma| = t \). Let \( y = \gamma^j(x) \ (1 \leq j < t) \). By Lemma 1, we have

\[
v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x) = v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t-1}}(y).
\]

This is an identical equation.

Case 1.1: \( \sigma \) is \( p \)-divisible.

Then \( A^t = I \). By Lemma 1, we have

\[
s(x) + As^\gamma(x) + \cdots + A^{t-1}s^{\gamma^{t-1}}(x) = s(y) + As^\gamma(y) + \cdots + A^{t-1}s^{\gamma^{t-1}}(y),
\]

i.e.,

\[
(A^{t-j} - I)(s(x) + As^\gamma(x) + \cdots + A^{t-1}s^{\gamma^{t-1}}(x)) = 0.
\]

If \( \sigma \) is strongly \( p \)-divisible or \( p | j \), then there are \( p^{2t} \) possible choices for the \( s(w) \) with \( w \in \sigma \). If \( \sigma \) is not strongly \( p \)-divisible, then we have

\[
-j(v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x)) = 0.
\]

Since \( u(w) \) is any, there are \( p^{2t-1} \) possible choices for the \( s(w) \) with \( w \in \sigma \).

Case 1.2: \( \sigma \) is not \( p \)-divisible.

Since \( u(w) \) and \( v(w) \) are any, there \( p^{2t} \) possible choices for the \( s(w) \) with \( w \in \sigma \).

Case 2: \( x \) and \( y \) are in different vertex \( \langle \gamma \rangle \)-orbits \( \sigma_1, \sigma_2 \) of length \( t_1, t_2 \), respectively.

Then \( t \) is the least common multiple \( [t_1, t_2] \) of \( t_1 \) and \( t_2 \). Let \( t_i = p^{a_i}q_i \), \( (p, q_i) = 1 \ (i = 1, 2) \), and \( a = \max\{a_1, a_2\} \). Then \( t = p^a[q_1, q_2] \). Let \( t'_i = [q_1, q_2]/q_i \ (i = 1, 2) \). By Lemma 1, we have

\[
p^{a-a_1}t'_1(v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t_1-1}}(x)) = p^{a-a_2}t'_2(v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t_2-1}}(y)).
\]
If $p | t$, then we have
\[
(I + A^{t_1} + \cdots + A^{t_1(p^a - 1)}) (s(x) + A^s\gamma(x) + \cdots + A^{t_1-1}s^{\gamma t_1-1}(x))
\]
\[
= (I + A^{t_2} + \cdots + A^{t_2(p^a - 1)}) (s(y) + A^s\gamma(y) + \cdots + A^{t_2-1}s^{\gamma t_2-1}(y)).
\]
Case 2.1: $\sigma_1$ is $p$-divisible and $\sigma_2$ is not $p$-divisible.
Since $A^{t_1} = I$ and $A^{t_2} \neq I$, we have
\[
I + A^{t_2} + \cdots + A^{t_2(p^a - 1)} = 0.
\]
Thus
\[
p^{a-a_1}t_1'(s(x) + As^\gamma(x) + \cdots + A^{t_1-1}s^{\gamma t_1-1}(x)) = 0.
\]
Since $a = a_1, p^{a-a_1}t_1' = t_1' \neq 0$, and so
\[
s(x) + As^\gamma(x) + \cdots + A^{t_1-1}s^{\gamma t_1-1}(x) = 0.
\]
Case 2.2: Both $\sigma_1$ and $\sigma_2$ are $p$-divisible.
Then $A^{t_1} = A^{t_2} = I$. If $a_1 = a_2$, then we have
\[
t_1'(s(x) + As^\gamma(x) + \cdots + A^{t_1-1}s^{\gamma t_1-1}(x)) = t_2'(s(y) + As^\gamma(y) + \cdots + A^{t_2-1}s^{\gamma t_2-1}(y)).
\]
If $a_1 > a_2$, then we have
\[
s(x) + As^\gamma(x) + \cdots + A^{t_1-1}s^{\gamma t_1-1}(x) = 0.
\]
Case 2.3: Both $\sigma_1$ and $\sigma_2$ are not $p$-divisible.
Since $t \not\equiv 0 \pmod{p}$, we have
\[
t_1'(v(x) + v^\gamma(x) + \cdots + v^{\gamma t_1-1}(x)) = t_2'(v(y) + v^\gamma(y) + \cdots + v^{\gamma t_2-1}(y)).
\]
Let $H$ be a 0 th $p$-level component of $G(\gamma)$. Then some vertex $\sigma$ of $H$ admits $p^{2|\sigma|}$ choices according to Cases 1.2 and 2.1, while any other vertex $\omega$ of $H$ admits $p^{2|\omega|-1}$ choices by Case 2.3.
Let $H$ be a $k$ th $p$-level component of $G(\gamma)$ for $k \geq 1$. If $H$ is not minimal, then any vertex $\sigma$ of $H$ admits $p^{2|\sigma|-2}$ choices for the $s(\omega)$ with $\omega \in \sigma$ by cases 1.1, 2.1 and 2.2. If $H$ is $p$-favorable, then some vertex $\sigma$ of $H$ admits $p^{2|\sigma|-1}$ choices according to Case 1.1, while any other vertex $\omega$ of $H$ admits $p^{2|\omega|-2}$ choices by Case 2.2. If $H$ is $p$-defective, then some vertex $\sigma$ of $H$ admits $p^{2|\sigma|}$ choices according to Case 1.1, while any other vertex $\omega$ of $H$ admits $p^{2|\omega|-2}$ choices by Case 2.2.
Therefore it follows that
\[
|C_\gamma^0| = \prod_{H_1} \left( \prod_{\sigma_1 \in H_1} p^{2|\sigma_1|-1} \right) p \cdot \prod_{H_2} \left( \prod_{\sigma_2 \in H_2} p^{2|\sigma_2|-2} \right)
\]
\[
\times \prod_{H_3} \left( \prod_{\sigma_3 \in H_3} p^{2|\sigma_3|-2} \right) p \cdot \prod_{H_4} \left( \prod_{\sigma_4 \in H_4} p^{2|\sigma_4|-2} \right) p^2
\]
\[
= p^{2n-2\nu(\gamma)+\nu_0(\gamma)+2(\gamma-d_1(\gamma)+d(\gamma)),}
\]
where $H_1, H_2, H_3$ and $H_4$ runs over all 0th $p$-level components, nonzero th not minimal $p$-level components, nonzero th $p$-favorable $p$-level components and nonzero th $p$-defective $p$-level components of $G(\gamma)$, respectively. \[\blacksquare\]

Each $\langle \gamma \rangle$-orbit $\Omega'$ on $E(G)$ corresponds to two $\langle \gamma \rangle$-orbits on $A(D)$ if $\Omega'$ is not diagonal, and one $\langle \gamma \rangle$-orbit on $A(D)$ otherwise.

**Lemma 3.** For $\gamma \in \Gamma$, 

$$|\text{Ker} \mu_{\gamma}| = p^\epsilon(\gamma)-\rho(\gamma)+\epsilon_1(\gamma).$$

**Proof.** Let $\alpha \in \text{Ker} \mu_{\gamma}$. Then we have $\alpha = A\alpha^\gamma = A^2\alpha^\gamma = \cdots$.

Let $\Omega = \langle \gamma \rangle (x, y)$ be any $\langle \gamma \rangle$-orbit on $A(D)$ and $|\Omega| = t$.

Case 1: $x$ and $y$ are in the same $\langle \gamma \rangle$-orbit $\sigma$ on $V(G)$, and $\Omega$ is diagonal.

Let $t = 2k$. Then we have $\alpha^\gamma(x, y) = A^{i-1} \alpha(x, y)$ $(i \geq 1)$, $A^k \alpha(x, y) = -\alpha(x, y)$ and $A^{2k} \alpha(x, y) = \alpha(x, y)$. If $\Omega$ is $p$-divisible, then, since $A^k = I$, $\alpha(x, y) = 0$, i.e., $\alpha(u, v) = 0$ for each $(u, v) \in \Omega$. Otherwise we have

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

and so $\alpha(u, v) = 0$ for each $(u, v) \in \Omega$.

Case 2: $x$ and $y$ are not in the same $\langle \gamma \rangle$-orbit $\sigma$ on $V(G)$, or $\Omega$ is not diagonal.

Then we have $\alpha^\gamma(x, y) = A^{i-1} \alpha(x, y)$ $(i \geq 1)$ and $A^i \alpha(x, y) = \alpha(x, y)$. If $p | t$, then there are $p^2$ possible choices for $\alpha(x, y)$. Otherwise there are $p$ possible choices for $\alpha(x, y)$.

From the note preceding the lemma, it follows that

$$|\text{Ker} \mu_{\gamma}| = p^\epsilon(\gamma)-\rho(\gamma)+\epsilon_1(\gamma).$$

\[\blacksquare\]

**Theorem 4.** For $\gamma \in \Gamma$ and odd prime $p$, 

$$|H^1(G; \mathbb{Z}^2_p)(A_1, \gamma)| = p^\epsilon(\gamma)-2\nu(\gamma)+\nu_0(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma).$$

**Proof.** Let $\gamma \in \Gamma$. Set $\epsilon = \epsilon(\gamma)$, $\nu = \nu(\gamma)$, $\nu_0 = \nu_0(\gamma)$, $\cdots$.

Let $C_\gamma = \{(s, \alpha) \mid \delta \alpha = \mu_{\gamma}(\alpha) = A_1 \alpha^\gamma - \alpha\}$, and consider the two epimorphisms $\gamma^0 : C_\gamma \rightarrow C_{\gamma^0}$ and $\gamma^1 : C_\gamma \rightarrow C_{\gamma^1}$. By Lemmas 2 and 3 and the fact that $\text{Ker} \gamma^0 \cong \text{Ker} \mu_{\gamma}$, we have

$$|C_{\gamma}| = |C_{\gamma}^0| \cdot |\text{Ker} \gamma^0| = p^{2n-2\nu+\nu_0+c-d_1+d+\epsilon-\rho+\epsilon_1}. $$
Since $\text{Ker} \, \gamma^1 \cong \text{Ker} \, \delta$ and $|\text{Ker} \, \delta| = p^2$, it follows that
\[ |C_{\gamma}^1| = |C_{\gamma}|/|\text{Ker} \, \gamma^1| = p^{2n - 2\nu + \nu_0 + c - d_1 + d + c - \rho + \epsilon - 2}. \]

Set $\delta^1 = \delta^1 |C_{\gamma}^1$. Since $\text{Im} \, \delta \subset C_{\gamma}^1$, we have $\text{Ker} \, \delta^1 = \text{Ker} \, \delta^1 = \text{Im} \, \delta$. Thus $|\text{Ker} \, \delta^1| = p^{2n - 2}$. Furthermore, since $\text{Im} \, \delta^1 = \text{Ker} \, \nu_{\gamma}$, it follows that
\[ |\text{Ker} \, \nu_{\gamma}| = |C_{\gamma}^1|/|\text{Ker} \, \delta^1| = p^{\epsilon - 2\nu + \nu_0 - \rho + \epsilon_1 + c - d_1 + d}. \]

\[ \blacksquare \]

4. Cyclic $Z_p \times Z_p$-covers of special symmetric digraphs

At first, we consider cyclic $Z_p^2$-covers of a symmetric dipath and a symmetric dicycle. Let $PD_n$ and $CD_n$ be the symmetric dipath and the symmetric dicycle with $n$ vertices, respectively. We enumerate the number of the isomorphism classes of $g$-cyclic $Z_p^2$-covers of $PD_n$ and $CD_n$ with respect to its full automorphism group, respectively.

**THEOREM 5.** For $n \geq 3$ and odd prime $p$,
\[ \text{Iso}(CD_n, Z_p^2, g, \text{Aut} \, CD_n) = \frac{p + 3}{2} \]
and
\[ \text{Iso}(PD_n, Z_p^2, g, \text{Aut} \, PD_n) = 1. \]

**Proof.** Let $V(CD_n) = \{1, 2, \ldots, n\}$. Then the $n$-cycle is the underline graph of $CD_n$. Let $\Gamma = \text{Aut} \, CD_n$. Then we have $\Gamma = \langle \alpha, \beta \rangle$, where $\alpha = (12 \cdots n)$ and
\[ \beta = \begin{cases} (1 \ n)(2 \ n - 1) \cdots (\frac{n - 1}{2} \ \frac{n + 3}{2})(\frac{n + 1}{2}) & \text{if } n \text{ is odd}, \\ (1 \ n)(2 \ n - 1) \cdots (\frac{n}{2} \ \frac{n}{2} + 1) & \text{otherwise}. \end{cases} \]

For each $\gamma \in \Gamma$ and $\lambda \in Z_p^*$, all parameters $\epsilon(\gamma), \ldots, \kappa(\gamma, \lambda)$ are constant in each conjugacy class of $\Gamma$. The conjugacy classes of $\Gamma$ are given as follows:
\[ \{\alpha^i\} \ (1 \leq i \leq n), \ \{\beta \alpha^i \ | \ 1 \leq i \leq n\} \text{ if } n \text{ is odd}, \]
\[ \{\alpha^i\} \ (1 \leq i \leq n), \ \{\beta \alpha^{2j} \ | \ j = 1, 2, \ldots, \frac{n}{2}\}, \ \{\beta \alpha^{2j+1} \ | \ j = 1, 2, \ldots, \frac{n}{2} - 1\} \]
if $n$ is even.
Let $1 \leq i \leq n$ and $d = (n, i)$ the greatest common divisor of $n$ and $i$. Then we have $\text{ord}(\alpha^i) = \frac{n}{d}$. The cardinality of each $\langle \alpha^i \rangle$-orbit on $V(C_n)$ or $E(C_n)$ is $\frac{n}{d}$. For each $\gamma \in \Gamma$, $\nu(\gamma), \rho(\gamma), \nu_0(\gamma)$ and $\epsilon(\gamma)$ are given as follows:

$$
\epsilon(\gamma) = \begin{cases} 
    d & \text{if } \gamma = \alpha^i, \\
    \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma = \beta \alpha^{2j}, \\
    \frac{n}{2} + 1 & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}, \\
    \frac{n}{2} & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}.
\end{cases}
$$

$$
\nu(\gamma) = \begin{cases} 
    d & \text{if } \gamma = \alpha^i, \\
    \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma = \beta \alpha^{2j}, \\
    \frac{n}{2} & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}, \\
    \frac{n}{2} + 1 & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}.
\end{cases}
$$

and

$$
\rho(\gamma) = \begin{cases} 
    1 & \text{if } n \text{ is odd and } \gamma = \beta \alpha^{2j}, \\
    2 & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}, \\
    0 & \text{otherwise}.
\end{cases}
$$

Since $p \geq 3$ and the cardinality of each $\langle \beta \alpha^i \rangle$-orbit on $V(C_n)$ or $E(C_n)$ is at most two,

$$
\epsilon_1(\gamma) = \begin{cases} 
    d & \text{if } \gamma = \alpha^i, p \mid n \text{ and } \theta(n) > \theta(i), \\
    0 & \text{otherwise},
\end{cases}
$$

and

$$
\nu_0(\gamma) = \begin{cases} 
    0 & \text{if } \gamma = \alpha^i, p \mid n \text{ and } \theta(n) > \theta(i), \\
    \nu(\gamma) & \text{otherwise}.
\end{cases}
$$

There exists only one $p$-level component in $C_n(\gamma)$, and so $c(\gamma) = \xi(\gamma) = 1$. Each $\langle \gamma \rangle$-orbit on $V(C_n)$ which is not diagonal is not strongly $p$-divisible. Thus we have $d(\gamma) = d_1(\gamma) = 0$.

Let $\lambda \in \mathbb{Z}_p^*$ and $m = \text{ord}(\lambda)$. Then $m = 2$ if $\lambda = -1$, and $m > 2$ otherwise. Thus we have

$$
\nu(\gamma, \lambda) = \begin{cases} 
    \nu(\gamma) & \text{if } \gamma = \alpha^i, m \mid \frac{n}{d} \text{ or } n \text{ is odd}, \gamma = \beta \alpha^i, \lambda \neq -1, \\
    1 & \text{if } n \text{ is odd and } \gamma = \beta \alpha^i, \lambda = -1, \\
    2 & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j+1}, \lambda = -1, \\
    0 & \text{otherwise}.
\end{cases}
$$

In the case of $\gamma = \beta \alpha^i$, each diagonal $\langle \gamma \rangle$-orbit on $A(CD_n)$ is type-1 if $\lambda = -1$, and type-2 otherwise. Thus

$$
\mu(\gamma, \lambda) = \begin{cases} 
    1 & \text{if } n \text{ is odd and } \gamma = \beta \alpha^i, \lambda \neq -1, \\
    2 & \text{if } n \text{ is even and } \gamma = \beta \alpha^{2j}, \lambda \neq -1, \\
    0 & \text{otherwise}.
\end{cases}
$$
Furthermore,

$$
\kappa(\gamma, \lambda) = \begin{cases} 
0 & \text{if } \gamma = \alpha^i, \frac{n}{d} \not\equiv 0 \pmod{m} \text{ or } n \text{ is odd, } \gamma = \beta\alpha^i, \lambda = -1, \\
\epsilon(\gamma) - 1 & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \lambda \neq -1, \\
\epsilon(\gamma) - 2 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j}, \lambda \neq -1, \\
\epsilon(\gamma) & \text{otherwise.}
\end{cases}
$$

Therefore the result follows.

Similarly to the above, the second formula is obtained. \(\blacksquare\)

Finally, we shall give an example.

Let \(KD_n\) be the complete symmetric digraph of order \(n\), and \(\Gamma\) a subgroup of the symmetric group \(S_n\) on \(n\) elements. Set \(V(KD_n) = \{1, 2, \ldots, n\}\). For \(\gamma \in \Gamma\), let \((t_1, t_2, \ldots, t_n)\) be the cycle type of \(\gamma\). Then \(\nu(\gamma), \rho(\gamma), \nu_0(\gamma)\) and \(\epsilon(\gamma)\) are given as follows:

$$
\nu(\gamma) = \sum_{k=1}^{n} t_k, \quad \rho(\gamma) = \sum_{k: \text{even}} t_k, \quad \nu_0(\gamma) = \sum_{k \not\equiv 0 \pmod{p}} t_k
$$

and

$$
\epsilon(\gamma) = \sum_{k=1}^{n} \{t_k \lfloor \frac{k}{2} \rfloor + k \binom{j}{2} + \sum_{k=2}^{n-1} \sum_{l=1}^{k-1} t_k t_l(k,l),
$$

where \((k,l)\) is the greatest common divisor of \(k\) and \(l\).

Since the graph \(K_n(\gamma)\) is complete, we have \(\xi(\gamma) = 1\). Moreover, since any \(m\)-divisible vertex orbit is not strongly \(m\)-divisible, we have

$$
d(\gamma) = 0 \text{ and } c(\gamma) - d_1(\gamma) = 1.
$$

Let \(\lambda \in \mathbb{Z}_p^*\) and \(m = \text{ord } (\lambda)\). Then \(\nu_0(\gamma, \lambda)\) and \(\epsilon_1(\gamma)\) are given as follows:

$$
\nu_0(\gamma, \lambda) = \sum_{k \not\equiv 0 \pmod{m}} t_k
$$

and

$$
\epsilon_1(\gamma) = \sum_{k: \text{odd}} t_k \lfloor \frac{k}{2} \rfloor + \sum_{k: \text{even}} t_k \binom{j}{2} + \sum_{k=2}^{n-1} \sum_{l=1}^{k-1} t_k t_l(k,l),
$$

where \(k\) or \(l\) in five \(\sum\) runs over multiples of \(p\). Moreover, we have

$$
d(\gamma, \lambda) = 0, 1.
$$

Specially, \(d(\gamma, \lambda) = 1\) if and only if \(\lambda = 1\).
Let $\lambda = \zeta^i (1 \leq i \leq p-1)$, where $\zeta$ is a generator of $\mathbb{Z}_p^*$. Set $d = (i, p-1)$. Then we have $m = (p-1)/d$. Furthermore, we have

$$d(\gamma, \lambda) = \begin{cases} \rho(\gamma) - \sum_{p|k; k/m: \text{odd}} t_k & \text{if } i/d \text{ is odd and } m \text{ is even}, \\ \rho(\gamma) & \text{otherwise}. \end{cases}$$

It is clear that $\text{Iso} (KD_1, \mathbb{Z}_p^2, g, \Gamma_1) = \text{Iso} (KD_2, \mathbb{Z}_p^2, g, \Gamma_2) = 1$, where $\Gamma_i \leq S_i (i = 1, 2)$. In Table 1, we give some values of the number $\text{Iso} (KD_n, \mathbb{Z}_p^2, g, S_n)$ of $S_n$-isomorphism classes of $g$-cyclic $\mathbb{Z}_p^2$-covers of $KD_n (n \geq 3)$.

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Table 1.

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References


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