Isomorphisms of some graph coverings

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Received 8 February 1991
Revised 5 May 1992

Abstract

Let $G$ be a connected graph and $\Gamma$ a group of automorphisms of $G$. We enumerate the number of $\Gamma$-isomorphism classes of derived graph coverings of $G$ with voltages in a finite field of prime order $p (> 2)$.

1. Introduction

All graphs appearing here are simple. Let $G$ be a graph, $p$ prime, and $F = \mathbb{GF}(p)$. Let $A(G)$ be the arc set of the corresponding symmetric digraph to $G$. An ordinary voltage assignment $\alpha$ on $G$ in $F$ is a function from $A(G)$ into $F$ such that $\alpha(y, x) = -\alpha(x, y)$ for each $(x, y) \in A(G)$. The pair $(G, \alpha)$ is called an ordinary voltage graph of $G$ with voltages in $F$. For such an ordinary voltage graph $(G, \alpha)$, the derived graph $G^\alpha$ is defined as follows: $V(G^\alpha) = V(G) \times F$ and $((x, i), (y, j)) \in A(G^\alpha)$ if and only if $(x, y) \in A(G)$ and $j = \alpha(x, y) + i$. The natural projection $p : G^\alpha \to G$ is the function from $V(G^\alpha)$ onto $V(G)$ which erases the second coordinate. Then $p$ is a (topological) covering projection (see [3]).

Any voltage $i \in F$ determines a permutation $\rho(i)$ of the symmetric group $S_F$ on $F$ which is given by $\rho(i)(j) = i + j$ for $j \in F$. Thus each ordinary voltage graph of $G$ with voltages in $F$ can be viewed as a permutation voltage graph of $G$ with voltages in $S_F$ (see [4]).

Let $\alpha$ and $\beta$ be two ordinary voltage assignments on $G$ in $F$, and let $\Gamma$ be a subgroup of the automorphism group $\text{Aut} G$ of $G$, denoted $\Gamma \leq \text{Aut} G$. Two natural projections $p_\alpha : G^\alpha \to G$ and $p_\beta : G^\beta \to G$ are called $\Gamma$-isomorphic, denoted $G^\alpha \cong \Gamma G^\beta$, if there exist an isomorphism $\psi : G^\alpha \to G^\beta$ and a $g \in \Gamma$ such that $p_\beta \psi = gp_\alpha$.

A general theory of graph coverings is developed in [S]. Furthermore, double covers of graphs were dealt with in [6, 11].

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SSDI 0012-365X(92)00494-7
Counting formula of graph coverings are only known in the following four cases: 
\( r \)-isomorphism classes of derived graph coverings of a graph \( G \) with voltages in \( \mathbb{GF}(2) \) [6]; \( I \)-isomorphism classes of derived graph coverings of \( G \) with voltages in \( S_r \) [7, 9]; isomorphism classes of concrete derived graph coverings of \( G \) with voltages in \( S_r \) [8]; strong isomorphism classes of derived graph coverings of \( G \) with voltages in \( \mathbb{GF}(p^n) \) [2]. Here \( r \leq \text{Aut} \ G \), and \( I \) is the trivial subgroup of \( \text{Aut} \ G \). The first result for \( r = \text{Aut} \ G \) agrees with [12, Theorem 2.21]. In Section 2, we enumerate the number of \( I \)-isomorphism classes of derived graph coverings of a graph with voltages in \( \mathbb{GF}(p) \). In Section 3, we give an enumeration of the \( r \)-isomorphism classes of derived graph coverings of a connected graph with voltages in \( \mathbb{GF}(p) \) \( (p > 2) \).

2. \( I \)-isomorphism classes

Let \( G \) be a graph, \( T \) a spanning forest of \( G \), \( p \) prime, and \( F = \mathbb{GF}(p) \). Let \( H_1, \ldots, H_c \) be the components of \( G \), \( x_i \in V(H_i) \) for \( i = 1, \ldots, c \), and \( X = (x_1, \ldots, x_c) \). Let \( \alpha \) be any ordinary voltage assignment on \( G \) in \( F \) and \( W \) any walk in \( G \). The net voltage of \( W \), denoted \( \alpha(W) \), is the sum of the voltages of the edges of \( W \). Then the \((T, X)\)-voltage \( \alpha_T \) of \( \alpha \) is defined as follows:

\[ \alpha_T(u, v) = \alpha(P_u) + \alpha(u, v) - \alpha(P_v) \]

for each \((u, v) \in A(G)\), where \( P_u \) and \( P_v \) denote the unique walks from \( x_i \) to \( u \) and \( v \) in \( T \cap H_i \), respectively, if \( H_i \) is a component of \( G \) containing \((u, v)\).

**Theorem 2.1.** Let \( G \) be a graph, \( T \) a spanning forest of \( G \), \( p \) prime, and \( F = \mathbb{GF}(p) \). Let \( \alpha, \beta \) be two ordinary voltage assignments on \( G \) in \( F \). Then \( G^\alpha \cong I G^\beta \) if and only if, for each component \( H \) of \( G \), there is an element \( \lambda_H \) of \( F^* = F \setminus \{0\} \) such that \( \beta_T = \lambda_H \alpha_T \) on \( H \).

**Proof.** At first, suppose that \( G^\alpha \cong I G^\beta \). By [7, Theorem 7], we have \( G^\alpha \cong I G^{\alpha_T} \) and \( G^\beta \cong I G^{\beta_T} \), i.e. \( G^{\alpha_T} \cong I G^{\beta_T} \). However, two ordinary voltage graphs \((G, \alpha_T)\) and \((G, \beta_T)\) can be viewed as permutation voltage graphs of \( G \) with voltages in \( S_F \). Thus, by [7, Theorem 6], there exists a family \( \Pi = (\pi_i)_{i \in V(G)} \) of permutations in \( S_F \) such that

\[ \rho(\beta_T(x, y)) = \pi_x^{-1} \rho(\alpha_T(x, y)) \pi_x \]

for each \((x, y) \in A(G)\), where \( \rho(i) (i \in F) \) is an element of \( S_F \) such that \( \rho(i)(j) = i + j \), \( j \in F \) and the multiplication of permutations is carried out from right to left.

Let \( H \) be any component of \( G \). Then \( \alpha_T(u, v) = \beta_T(u, v) = 0 \) for each \((u, v) \in A(H) \cap A(T)\). Thus \( \pi_x = \pi_x \) for any \( x \neq y \in V(H) \). That is, we have \( \rho(\beta_T(u, v)) = \pi_x^{-1} \rho(\alpha_T(u, v)) \pi_x \) for each \((u, v) \in A(H)\), where \( x \in V(H) \). If \( \alpha_T = 0 \) on \( H \), then \( \beta_T = 0 \) on \( H \), i.e. \( \lambda_H \) is arbitrarily element of \( F^* \).

Now, suppose that \( \alpha_T \neq 0 \) and \( \beta_T \neq 0 \) on \( H \). Let \((u, v)\) be an element of \( A(H) \setminus A(T) \) such that \( \alpha_T(u, v) = l \neq 0 \) and \( \beta_T(u, v) = k \neq 0 \). Set \( \pi_x^{-1}(i) = \sigma_i \), \( i \in F \). Then we have \( \sigma_i = jk + \sigma_0 \) for \( j \in F \).
If \((z, w) \neq (u, v)\) is another element of \(A(H) \setminus A(T)\) such that \(\alpha_T(z, w) \neq 0\) and \(\beta_T(z, w) \neq 0\). Set \(\alpha_T(z, w) = r\) and \(\beta_T(z, w) = s\). Then \(\sigma_j = js + \sigma_0\) for \(j \in F\). Thus we obtain \(\sigma_0 = -kr + \sigma_0\). Set \(\lambda = kl^{-1} = sr^{-1}\). Then \(k = 1\lambda\) and \(s = 1\lambda\). It follows that \(\beta_T = \lambda \alpha_T\) on \(H\).

If there is not such element \((z, w)\), then the result is trivial.

By [7, Theorem 6] and the fact that \(G^x \cong 1 G^x\), the converse follows. \(\square\)

As a generalization of [6, Theorem 2.2], we obtain the following result.

**Corollary 2.2.** Let \(G\) be a graph, \(p\) prime, and \(F = GF(p)\). Then the number of \(\Gamma\)-isomorphism classes of derived graph coverings of \(G\) with voltages in \(F\) is

\[
\prod_H \left( \frac{(p^{m(H)} - \pi(H) + 1)}{(p-1) + 1} \right).
\]

where \(H\) ranges over all components of \(G\), and \(m(H) = |E(H)|\), and \(n(H) = |V(H)|\).

### 3. \(\Gamma\)-isomorphism classes

Let \(\mathcal{F}_r\) denote the set of permutation voltage assignments on \(G\) in \(S_r\), and \(\Gamma \triangleleft \text{Aut } G\). Then \(\Gamma\) acts on \(\mathcal{F}_r\) as follows: \(\alpha^g(u, v) = \alpha(g(u), g(v))\) for each \((u, v) \in A(G)\), where \(\alpha \in \mathcal{F}_r\) and \(g \in \Gamma\).

**Lemma 3.1.** Let \(\alpha, \beta \in \mathcal{F}_r\). Then \(G^\alpha \cong \Gamma G^\beta\) if and only if there is a \(g \in \Gamma\) such that \(G^\alpha \cong \Gamma G^\beta\).

**Proof.** By [7, Theorem 6], \(G^\alpha \cong \Gamma G^\beta\) if and only if there are a \(\Pi = (\pi_x)_{x \in V(G)} \in S_r^{\Gamma}(G)\) and a \(g \in \Gamma\) such that \(\alpha(x, y) = \pi_x^{-1} \beta(g(x), g(y)) \pi_x\) for each \((x, y) \in A(G)\), i.e. if and only if there is a \(g \in \Gamma\) such that \(G^\alpha \cong \Gamma G^\beta\). \(\square\)

Let \(G\) be a connected graph, \(p > 2\) prime, and \(F = GF(p)\). Let \(C^1\) be the set of ordinary voltage assignments on \(G\) in \(F\) and \(C^0\) the set of functions from \(V(G)\) into \(F\). The coboundary operator \(\delta: C^0 \rightarrow C^1\) is the linear operator defined by \((\delta_s)(x, y) = s(x) - s(y)\) for \(s \in C^0\) and \((x, y) \in A(G)\). For each \(\alpha \in C^1\), let \([\alpha]\) be the element of \(C^1/Im \delta\) which contains \(\alpha\).

**Lemma 3.2.** Let \(G\) be a connected graph, \(p > 2\) prime, \(F = GF(p)\), and \(\alpha \in C^1\). Then the following four conditions are equivalent:

1. \(\alpha_T = 0\) for a spanning tree \(T\) of \(G\),
2. \(\alpha_T = 0\) for any spanning tree \(T\) of \(G\),
3. \(\alpha(C) = 0\) for all cycle \(C\) in \(G\),
4. \(\alpha \in Im \delta\).
Proof. (1)⇒(2): By Theorem 2.1, we have $G^* \cong G^0$, where $o(x, y) = 0$ for any $(x, y) \in A(G)$. Furthermore, by Theorem 2.1, the result follows.

(2)⇒(3): Let $C$ be any cycle of $G$ and $e \in E(C)$. Then there exists a spanning tree $T$ of $G$ such that $E(C) \setminus E(T) = \{e\}$. By (2), we have $\alpha(C) = 0$.

(3)⇒(1), (4)⇒(3): Clear.

(3)⇒(4): Let $T$ be a spanning tree of $G$. Then, by a similar argument to the proof of \cite[(2.1)]{1}, there is a unique element $\beta$ of $\text{Im} \delta$ such that $\beta(x, y) = \alpha(x, y)$ for all $(x, y) \in A(T)$.

Let $(u, v)$ be any arc of $A(G) \setminus A(T)$, and let $C$ be a unique cycle of $G$ such that $E(C) \setminus E(T) = \{u\}$. Since $\alpha(C) = \beta(C) = 0$, we have $\alpha(u, v) = \beta(u, v)$, i.e. $\alpha = \beta$. □

Let $G$ be a connected graph, $p > 2$ prime, $F = GF(p)$, and $\Gamma \leq \text{Aut} G$. Let $\alpha, \beta \in C^1$. Then, by Theorem 2.1 and Lemmata 3.1 and 3.2, $G^* \cong \Gamma G^0$ if and only if $\beta = \lambda \alpha + \delta s$ for some $g \in \Gamma$, some $\lambda \in F^*$ and some $s \in C^0$. Let the group $\Gamma \times F^*$ act on $C^1/\text{Im} \delta$ as follows:

$$[\alpha]^{(g, \lambda)} = \lambda \alpha \delta^s \in \{\lambda \alpha \delta^s | s \in C^0\},$$

where $\alpha \in C^1, \lambda \in F^*$ and $g \in \Gamma$. Thus, the number of $\Gamma$-isomorphism classes of derived graph coverings of $G$ with voltages in $F$ is equal to that of $\Gamma \times F^*$-orbits on $C^1/\text{Im} \delta$.

By Burnside's Lemma, that number is equal to

$$\frac{1}{|\Gamma|(p-1)} \sum_{(g, \lambda) \in \Gamma \times F^*} |(C^1/\text{Im} \delta)^{(g, \lambda)}|,$$

where $U^{(g, \lambda)}$ is the set consisting of the elements of $U$ fixed by $(g, \lambda)$.

Let $g \in \Gamma, \lambda \in F^*$ and $\text{ord}(\lambda) = m$ the order of $\lambda$. A $(g)$-orbit $\sigma$ of length $k$ on $E(G)$ is called diagonal if $\sigma = (g) \{x, g^k(x)\}$ for some $x \in V(G)$. The vertex orbit $(g)x$ and the arc orbit $(g') = (g)(x, g^k(x))$ are also called diagonal. A diagonal arc orbit $(g')$ of length $2k$ (the corresponding edge orbit of length $k$ and the corresponding vertex orbit of length $2k$) is called type-1 if $\lambda^k = -1$ (or $m = 2k$), and type-2 otherwise.

For $g \in \Gamma$, let $G(g)$ be a simple graph whose vertices are the $(g)$-orbits on $V(G)$, with two vertices adjacent in $G(g)$ if and only if some two of their representatives are adjacent in $G$. Let $\lambda \in F^*$ and $\text{ord}(\lambda) = m$. A $(g)$-orbit $\sigma$ on $V(G), E(G)$ or $A(G)$ is called $m$-divisible if $|\sigma| \equiv 0 \pmod{m}$. An $m$-divisible $(g)$-orbit $\sigma$ on $V(G)$ is called strongly $m$-divisible if $\sigma$ satisfies the following condition: If $\Omega = (g)(x, y)$ is any not diagonal $(g)$-orbit on $A(G)$, and $y = g^j(x), x, y \in \sigma$, then $j \equiv 0 \pmod{m}$.

Let $G_2(g)$ be the subgraph of $G(g)$ induced by the set of $m$-divisible $(g)$-orbits on $V(G)$. The $k$th $p$-level of $G_2(g)$ is the induced subgraph of $G_2(g)$ on the vertices $\omega$ such that $\theta_p(|\omega|) = p^k$, where $\theta_p(i)$ is the largest power of $p$ dividing $i$. A $p$-level component of $G_2(g)$ is a connected component of some $p$-level of $G_2(g)$.

A $p$-level component $H$ is called defective if each vertex $\sigma$ of $H$ is strongly $m$-divisible, not type-1 diagonal, and satisfies $\theta_p(|\sigma|) > \theta_p(|\omega|)$ whenever $\omega \not\in V(H)$ and $\omega \in E(G(g))$. Otherwise $H$ is called favorable.
Theorem 3.3. Let $G$ be a connected graph, $p (> 2)$ prime, $F = GF(p)$ and $\Gamma \leq Aut G$. For $g \in \Gamma$, let $\nu(g)$ and $\pi(g)$ be the number of $\langle g \rangle$-orbits on $E(G)$, $V(G)$, respectively. For $g \in \Gamma$ and $\lambda \in F^\times$, let $v_0(g, \lambda)$, $\mu(g, \lambda)$ and $d(g, \lambda)$ be the number of not $m$-divisible $\langle g \rangle$-orbits on $V(G)$, type-2 diagonal $\langle g \rangle$-orbits on $E(G)$ and defective $p$-level components in $G$, respectively, where $m = ord(\lambda)$. Furthermore, let $x(g, \lambda)$ be the number of not $m$-divisible $\langle g \rangle$-orbits on $E(G)$ which are not diagonal. Then the number of $\Gamma$-isomorphism classes of derived graph coverings of $G$ with voltages in $F$ is

$$
\frac{1}{|\Gamma|(|p - 1|)} \sum_{g \in \Gamma} \sum_{\lambda \in F^\times} \frac{\nu(g) - \pi(g) - \nu_0(g, \lambda) - \mu(g, \lambda) - d(g, \lambda)}{\nu_0(g, \lambda) \pi(g) - \mu(g, \lambda) d(g, \lambda)}.
$$

The proof of Theorem 3.3 uses an analogue of Hofmeister's method [6]. At first, we consider the following exact sequence:

$$
0 \rightarrow \ker \delta \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^1/Im \delta \rightarrow 0,
$$

where $\delta^0$ is the canonical monomorphism, and $\delta^1$ is the canonical epimorphism. For $(g, \lambda) \in \Gamma \times F^\times$, two endomorphisms $\mu_{\lambda}: C^1 \rightarrow C^1$ and $\nu_{\lambda}: C^1/Im \delta \rightarrow C^1/Im \delta$ are defined as follows: $\mu_{\lambda}(x) = \lambda x - x$ and $\nu_{\lambda}([x]) = [\lambda x - x]$, where $x \in C^1$. Then, note that $\nu_{\lambda} \delta^1 = \delta^1 \mu_{\lambda}$, and $\ker \nu_{\lambda} = (C^1/Im \delta)(\lambda)$.

Now, let $C_{g, \lambda}^0 = \delta^{-1}(Im \mu_{\lambda})$ and $C_{g, \lambda}^1 = \mu_{\lambda}^{-1}(Im \delta)$.

Let $g \in \Gamma$, $\lambda \in F^\times$. For any arc $(x, y)$ of $A(G)$, let $l(x, y)$ be defined by

$$
l(x, y) = \begin{cases} 
\frac{t}{2} & \text{if the arc orbit } \langle g \rangle (x, y) \text{ is type-1 diagonal,} \\
[t, m] & \text{otherwise,}
\end{cases}
$$

where $t = |\langle g \rangle (x, y)|$, $m = ord(\lambda)$, and $[a, b]$ denotes the least common multiple of $a$ and $b$.

Lemma 3.4. Let $g \in \Gamma$, $\lambda \in F^\times$, $s \in C^0$. Then $s \in C_{g, \lambda}^0$ if and only if, for each $(x, y) \in A(G)$,

$$
\sum_{i=0}^{l(x, y) - 1} \lambda^i s^g(x) = \sum_{i=0}^{l(x, y) - 1} \lambda^i s^g(y).
$$

Proof. Set $m = ord(\lambda)$, $t = |\langle g \rangle (x, y)|$, $c = [t, m]$ and $l = l(x, y)$.

Suppose that $s \in C_{g, \lambda}^0$. Then there is a $x \in C^1$ such that $\lambda x = \alpha = \delta s$. Thus

$$
\lambda^i x^g = \alpha = \delta \sum_{i=1}^{c-1} \lambda^i s^g + \lambda^c s^g.
$$

For each $(x, y) \in A(G)$, we have

$$
\sum_{i=0}^{c-1} \lambda^i s^g(y) = \sum_{i=0}^{c-1} \lambda^i s^g(x) = \lambda^c \alpha (x, y).
$$
If \( \langle g \rangle (x, y) \) is type-1 diagonal, then we have
\[
\sum_{i=0}^{t-1} \lambda^i s^\theta(x) - \sum_{i=0}^{t-1} \lambda^i s^\theta(y) = \lambda^k \alpha^\theta(x, y) - \alpha(x, y)
\]
\[
= -\alpha(y, x) - \alpha(x, y) = 0, \quad \text{where } t = 2k.
\]

Conversely, suppose that \( s \) satisfies \((*)\) for each \((x, y) \in \mathbb{A}(G)\). Let \( \Omega \) be any \( \langle g \rangle \)-orbit on \( \mathbb{A}(G) \), \(|\Omega| = t\), and \((x, y) \in \Omega\). Then, let
\[
\alpha(x, y) = \begin{cases} 
(\lambda^t - 1)^{-1} \sum_{i=0}^{t-1} (\lambda^i s^\theta(x) - \lambda^i s^\theta(y)) & \text{if } \Omega \text{ is not diagonal and } t \not| m, \\
-(\lambda^k + 1)^{-1} \sum_{i=0}^{k-1} (\lambda^i s^\theta(x) - \lambda^i s^\theta(y)) & \text{if } \Omega \text{ is type-2 diagonal}, \\
0 & \text{otherwise},
\end{cases}
\]
where \( t = 2k \) in the case that \( \Omega \) is diagonal. Furthermore, let
\[
\lambda^i \alpha^\theta(x, y) = \alpha(x, y) + \sum_{j=0}^{i-1} (\lambda^j s^\theta(x) - \lambda^j s^\theta(y)) \quad \text{for } i \geq 1.
\]

Then we have
\[
\lambda^r \alpha^\theta + i = \alpha^\theta(x, y) + \sum_{j=0}^{r-1} (\lambda^j s^\theta(g^i(x)) - \lambda^j s^\theta(g^i(y))) \quad \text{for } r, i \geq 1.
\]

If \( \Omega \) is not diagonal, then we define \( \alpha(v, u) = -\alpha(u, v) \) for \((u, v) \in \Omega\). If \( \Omega \) is type-1 diagonal, then we have
\[
\lambda^{i+k} \alpha^\theta(y, x) = \lambda^i \alpha^\theta(x, y) + \lambda^k \sum_{j=0}^{k-1} (\lambda^j s^\theta(g^i(x)) - \lambda^j s^\theta(g^i(y))) = \lambda^{i+k} \alpha^\theta(x, y),
\]
i.e.
\[
\alpha^\theta(y, x) = -\alpha^\theta(x, y) \quad \text{for } i \geq 1.
\]

In the case that \( \Omega \) is type-2 diagonal, we have \( \alpha(y, x) = -\alpha(x, y) \) by the definition of \( \alpha(x, y) \). Thus, we have
\[
\lambda^{i+k} \alpha^\theta(y, x) = \lambda^k \{ \lambda^i \alpha^\theta(x, y) \}
\]
\[
= \lambda^k \{ \alpha^\theta(x, y) + \sum_{j=0}^{i-1} (\lambda^j s^\theta(g^k(x)) - \lambda^j s^\theta(g^k(y))) \}
\]
\[
= -\lambda^k \{ \alpha(x, y) + \sum_{j=0}^{i-1} (\lambda^j s^\theta(x) - \lambda^j s^\theta(y)) \} = -\lambda^{i+k} \alpha^\theta(x, y),
\]
i.e. \( \alpha^\theta(y, x) = -\alpha^\theta(x, y) \) \((i \geq 1)\).

Therefore, we obtain an \( \alpha \in C^1 \) such that \( \lambda \alpha^\theta - \alpha = \delta s \), i.e. \( s \in C^0_{\theta, \lambda} \). \( \Box \)
Lemma 3.5. For \( g \in \Gamma \) and \( \lambda \in \mathbb{F}^* \),
\[
|C_{g, \lambda}^0| = n^{\gamma(g)} + v_3(g, \lambda) + 2^{\gamma(g, \lambda)}, \text{ where } n = |V(G)|.
\]

Proof. We enumerate the number of \( s \in C_{g, \lambda}^0 \) which satisfy (*') for each \((x, y) \in A(G)\).

Let \((x, y) \in A(G)\), \( \Omega = \langle g \rangle (x, y) \) the arc \( \langle g \rangle \)-orbit containing \((x, y)\), \( |\Omega| = t \), \( \text{ord}(\lambda) = m \) and \( l = l(x, y) \).

Case 1: \( x, y \) are in the same \( \langle g \rangle \)-orbit on \( V(G) \), and \( \sigma \) is not diagonal. Then \( \Omega \) is not diagonal, \( |\sigma| = t \) and \( l = [m, t] \). Let \( y - g^j(x) (1 < j < t) \) and \( m' = l/t \). By Lemma 3.4, we have
\[
(1 + \lambda^t + \cdots + \lambda^{t(m'-1)})(s(x) + \lambda s^\theta(x) + \cdots + \lambda^{t-1} s^{\theta t-1}(x)) = (1 + \lambda^t + \cdots + \lambda^{t(m'-1)})(s(y) + \lambda s^\theta(y) + \cdots + \lambda^{t-1} s^{\theta t-1}(y)).
\]

Case 1.1: \( \sigma \) is \( m \)-divisible. Then \( \lambda = 1 \). Since \( m' = 1 \), we have
\[
(\lambda^{t-1} - 1)(s(x) + \lambda s^\theta(x) + \cdots + \lambda^{t-1} s^{\theta t-1}(x)) = 0.
\]
If \( \sigma \) is strongly \( m \)-divisible, then \( \lambda^{t-1} = 1 \), i.e. there are \( p \) possible choices for the \( s(w) \) with \( w \in \sigma \). If \( \sigma \) is not strongly \( m \)-divisible, then \( s(x) + \lambda s^\theta(x) + \cdots + \lambda^{t-1} s^{\theta t-1}(x) = 0 \).

Case 1.2: \( \sigma \) is not \( m \)-divisible.
Since \( \lambda \neq 1 \), we have \( 1 + \lambda^t + \cdots + \lambda^{t(m'-1)} = (1 - \lambda^t)/(1 - \lambda^t) = 0 \). Thus there are \( p \) possible choices for the \( s(w) \) with \( w \in \sigma \).

Case 2: \( x, y \) are in the same \( \langle g \rangle \)-orbit on \( V(G) \), and \( \sigma \) is diagonal. Let \( \Omega \) be diagonal. Then \( t = 2k, |\sigma| = t \) and \( y = g^k(x) \).

Case 2.1: \( \Omega \) is type-2. Then \( l = [m, t] \). Let \( m' = l/t \). By Lemma 3.4, we have
\[
(1 - \lambda^k + \lambda^{2k} - \cdots - \lambda^{k(2m'-1)}) (s(x) + \lambda s^\theta(x) + \cdots + \lambda^{k-1} s^{\theta k-1}(x)) = (1 - \lambda^k + \lambda^{2k} - \cdots - \lambda^{k(2m'-1)}) (s(y) + \lambda s^\theta(y) + \cdots + \lambda^{k-1} s^{\theta k-1}(y)).
\]
Since \( \lambda \neq 1 \), \( 1 - \lambda^k + \lambda^{2k} - \cdots - \lambda^{k(2m'-1)} = (1 - \lambda^k)/(1 + \lambda^k) = 0 \).

We consider not diagonal \( \langle g \rangle \)-orbits \( \langle g \rangle (x, z) \) on \( A(G) \) such that \( z \in \sigma \). Since \( \Omega \) is type-2, we have either \( m|k \) or \( m'k = 2k \). If \( \sigma \) is either strongly \( m \)-divisible or not \( m \)-divisible, then there are \( p \) possible choices for the \( s(w) \) with \( w \in \sigma \) by case 1. If \( \sigma \) is \( m \)-divisible but not strongly \( m \)-divisible, then
\[
s(x) + \lambda s^\theta(x) + \cdots + \lambda^{t-1} s^{\theta t-1}(x) = 0
\]
according to case 1.1.

Case 2.2: \( \Omega \) is type-1. Then \( \lambda^k = 1 \) and \( l = m = t \). By Lemma 3.4, we have
\[
s(x) + \lambda s^\theta(x) + \cdots + \lambda^{t-1} s^{\theta t-1}(x) = 0.
\]

Case 3: \( x \) and \( y \) are in different vertex \( \langle g \rangle \)-orbits \( \sigma_1, \sigma_2 \) of length \( t_1, t_2 \). Then \( t = [t_1, t_2] \). Let \( t_i = p^a q_i \), \( (p, q_i) = 1 \) \((i = 1, 2)\), and \( a = \max \{q_1, q_2\} \). Since \( m \nmid p \), \( t = p^{a}[q_1, q_2] \) and \( l = p^{a}[q_1, q_2, m] \). Let \( t_i' = [q_1, q_2, m]/q_i \) \((i = 1, 2)\). By Lemma 3.4,
we have
\[(1 + \lambda^{t_1} + \cdots + \lambda^{t_m(p^\sigma - 1)}) (s(x) + \lambda s^\sigma(x) + \cdots + \lambda^{t_m - 1} s^{p^\sigma - 1}(x)) \]
\[= (1 + \lambda^{t_1} + \cdots + \lambda^{t_m(p^\sigma - 1)}) (s(y) + \lambda s^\sigma(y) + \cdots + \lambda^{t_m - 1} s^{p^\sigma - 1}(y)).\]

**Case 3.1:** \(\sigma_1\) is \(m\)-divisible and \(\sigma_2\) is not \(m\)-divisible. Then
\[(1 + \lambda^{t_1} + \cdots + \lambda^{t_m(p^\sigma - 1)}) = (1 - \lambda^1)/(1 - \lambda^{t_1}) = 0.\]

Thus we have
\[p^\sigma - \sigma_1 t_1 (s(x) + \lambda s^\sigma(x) + \cdots + \lambda^{t_1 - 1} s^{p^\sigma - 1}(x)) = 0.\]

If \(a_1 < a_2\), then \(p^\sigma - \sigma_1 t_1 = 0\). In the case that \(a_1 \geq a_2\), then \(p^\sigma - \sigma_1 t_1 = t_1 \neq 0\), and so
\[s(x) + \lambda s^\sigma(x) + \cdots + \lambda^{t_1 - 1} s^{p^\sigma - 1}(x) = 0.\]

**Case 3.2:** Both \(\sigma_1\) and \(\sigma_2\) are \(m\)-divisible.

Then \(\lambda^{t_1} = \lambda^{t_2} = 1\). If \(a_1 = a_2\), then we have
\[t_1 (s(x) + \lambda s^\sigma(x) + \cdots + \lambda^{t_1 - 1} s^{p^\sigma - 1}(x)) = t_2 (s(y) + \lambda s^\sigma(y) + \cdots + \lambda^{t_1 - 1} s^{p^\sigma - 1}(y)).\]

If \(a_1 > a_2\), then we have
\[s(x) + \lambda s^\sigma(x) + \cdots + \lambda^{t_1 - 1} s^{p^\sigma - 1}(x) = 0.\]

**Case 3.3:** Both \(\sigma_1\) and \(\sigma_2\) are not \(m\)-divisible. Since \(\lambda^{t_1} \neq 1\) and \(\lambda^{t_2} \neq 1\), we have
\[1 + \lambda^{t_1} + \cdots + \lambda^{t_m(p^\sigma - 1)} = (1 - \lambda^1)/(1 - \lambda^{t_1}) = 0\]
and
\[1 + \lambda^{t_2} + \cdots + \lambda^{t_m(p^\sigma - 1)} = (1 - \lambda^1)/(1 - \lambda^{t_2}) = 0.\]

Let \(\sigma\) be not \(m\)-divisible \(\langle g \rangle\)-orbit on \(V(G)\). In view of cases 1.2, 2.1, 3.1 and 3.3, there are \(p^{1|\sigma|}\) choices for the \(s(w)\) with \(w \in \sigma\).

If \(H\) is a favorable \(p\)-level component of \(G_\lambda(g)\), then any vertex \(\sigma\) of \(H\) admits \(p^{1|\sigma|-1}\) choices for the \(s(w)\) with \(w \in \sigma\) by cases 1.1, 2.1, 2.2, 3.1 and 3.2. However, if \(H\) is defective, then some vertex \(\sigma\) of \(H\) admits \(p^{1|\sigma|}\) choices according to cases 1.1, 2.1, 3.1 and 3.2, while any other vertex \(\sigma\) of \(H\) admits \(p^{1|\sigma|-1}\) choices according to case 3.2.

Therefore, it follows that
\[|C_{g, \lambda}^0| = \prod_{\sigma} p^{1|\sigma|} \left( \prod_{H_1} \left( \prod_{\sigma_1 \in H_1} p^{1|\sigma_1| - 1} \right) \right) \left( \prod_{H_2} \left( \prod_{\sigma_2 \in H_2} p^{1|\sigma_2| - 1} \right) \right) \]
\[= p^n - (v(g) - v_0(g, \lambda)) + d(g, \lambda),\]

where \(\sigma, H_1, H_2\) runs over all not \(m\)-divisible \(\langle g \rangle\)-orbits on \(V(G)\), favorable \(p\)-level components of \(G_\lambda(g)\) and defective \(p\)-level components of \(G_\lambda(g)\), respectively.

Each \(\langle g \rangle\)-orbit \(\Omega\) on \(E(G)\) corresponds to two \(\langle g \rangle\)-orbits on \(A(G)\) if \(\Omega\) is not diagonal, and one \(\langle g \rangle\)-orbit on \(A(G)\) otherwise.
Lemma 3.6. For $g \in \Gamma$ and $\lambda \in F^*$, $|\text{Ker} \mu_{g, \lambda}| = p^{e(g) - \kappa(g, \lambda) - \mu(g, \lambda)}$.

Proof. Let $\alpha \in \text{Ker} \mu_{g, \lambda}$. Then we have $\alpha = \lambda \alpha' = \lambda^2 \alpha'' = \cdots$.

Let $\Omega = \langle g \rangle(x, y)$ be any $\langle g \rangle$-orbit on $A(G)$, $|\Omega| = t$ and $m = \text{ord}(\lambda)$.

Case 1: $x$ and $y$ are in the same $\langle g \rangle$-orbit on $V(G)$, and $\Omega$ is diagonal. Let $t = 2k$.

Then we have $\lambda^i x(y, x) = \lambda^{i-1} x(y, x)$ ($i \geq 1$), $\lambda^k x(y, x) = -x(y, x)$ and $x(y, x) = \lambda^k x(y, x)$. If $\Omega$ is type-1, then there are $p$ possible choices for $x(y, x)$. Otherwise $x(u, v) = 0$ for each $(u, v) \in \Omega$.

Case 2: $x$ and $y$ are not in the same $\langle g \rangle$-orbit on $V(G)$, or $\Omega$ is not diagonal. Then we have $\lambda^i x(y, x) = \lambda^{i-1} x(y, x)$ ($i \geq 1$) and $x(y, x) = \lambda^i x(y, x)$. If $m/t$, then there are $p$ possible choices for $x(y, x)$. Otherwise $x(u, v) = 0$ for each $(u, v) \in \Omega$.

From the note preceding the lemma, it follows that

$$\frac{|\text{Ker} \mu_{g, \lambda}|}{p^{e(g) - \kappa(g, \lambda) - \mu(g, \lambda)}} = \square.$$

Proof of Theorem 3.3. Let $g \in \Gamma$ and $\lambda \in F^*$. Set $\varepsilon = e(g)$, $\nu = v(g)$, $\nu_0 = v_0(g, \lambda)$, ....

Let $C_{g, \lambda} = \{(s, \alpha) \in C \times C^1 \mid \delta s = \mu_{g, \lambda}(\alpha) = \lambda \alpha' - \alpha\}$, and consider the two canonical epimorphisms $\gamma^0 : C_{g, \lambda} \rightarrow C_{g, \lambda}^0$ and $\gamma^1 : C_{g, \lambda} \rightarrow C_{g, \lambda}^1$. By Lemmata 3.5 and 3.6 and the fact that $\text{Ker} \gamma^0 \cong \text{Ker} \mu_{g, \lambda}$, we have $|C_{g, \lambda}^0| = |C_{g, \lambda}^1| = |\text{Ker} \gamma^0| = p^{n - v_0 + v_0 - \nu} - \kappa - \mu$. Since $\text{Ker} \gamma^0 \cong \text{Ker} \delta$ and $|\text{Ker} \delta| = p^j$, it follows that

$$|C_{g, \lambda}^1| = |\text{Ker} \gamma^1| = p^{n - v_0 + v_0 - \nu + \kappa - \mu - 1}.$$ 

Set $\delta^1 = \delta^1 |C_{g, \lambda}^1$. Since $\text{Im} \delta \subset C_{g, \lambda}^1$, we have $\text{Ker} \delta^1 = \text{Ker} \delta^1 = \text{Im} \delta$. Thus $|\text{Ker} \delta^1| = p^{n - v_0 - \nu + \kappa - \mu + 1}$. Furthermore, since $\text{Im} \delta^1 = \text{Ker} \nu_{g, \lambda}$, it follows that $|\text{Ker} \nu_{g, \lambda}| = |C_{g, \lambda}^1| = |\text{Ker} \delta^1| = p^{n - v_0 + v_0 - \nu + \kappa - \mu + 1}$. By Burnside’s Lemma, the result follows. $\square$

Theorem 3.3 holds for $p = 2$ or $\Gamma = I$, and so it is a generalization of [6, Theorem 3.4] and [10, Theorem 5].

Now we give some examples. Let $G = K_n$, $\Gamma \leq S_n$ and $F = GF(p)$. Set $V(G) = \{1, 2, \ldots, n\}$. Let $e(n, p, \Gamma)$ be the number of $\Gamma$-isomorphism classes of derived graph coverings of $K_n$ with voltages in $F$. It is clear that $e(1, p, \Gamma_1) = e(2, p, \Gamma_2) = 1$ for any prime $p$, where $\Gamma_i \leq S_i$ ($i = 1, 2$). Let $\lambda \in F^*$. Then, note that each of $d(g, \lambda)$, $v_0(g, \lambda)$, $\kappa(g, \lambda)$ and $\mu(g, \lambda)$ is constant on each conjugacy class of $S_n$.

For $n = p = 3$, we obtain Table 1. By Theorem 3.3, we have $e(3, 3, S_3)_1 = (3 + 1 + 3 + 9 + 6 + 2)/12 = 2$.

Next, let $n = 4$, $p = 3$ and $\Gamma = \{1, (1234), (13)(24), (1432)\}$. Then we have $e(4, 3, S_4) = 4$ and $e(4, 3, \Gamma) = 6$.

In Table 2, we give some values of $e(n, p, S_n)$ the number $e(n, p, S_n)$ of $S_n$-isomorphism classes of derived graph coverings of the complete graph $K_n$ with voltages in $GF(p)$.
Table 1

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<th>Class representative $g$</th>
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<th>(123)</th>
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<td>-1</td>
<td>3</td>
</tr>
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<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>$\nu(g)$</td>
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<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\nu_0(g, \lambda)$</td>
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<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$\kappa(g, \lambda)$</td>
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<td>0</td>
</tr>
<tr>
<td>$\mu(g, \lambda)$</td>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$d(g, \lambda)$</td>
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<td>0</td>
<td>1</td>
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Table 2

<table>
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<tr>
<th>$n \setminus p$</th>
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<th>7</th>
<th>11</th>
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</tbody>
</table>

Acknowledgment

The author would like to thank Professor M. Hofmeister for sending him three papers [2, 7, 8] in references and the referees for many helpful comments and suggestions.

References